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ORDER $\alpha^4 R_\infty$ CORRECTIONS TO POSITRONIUM P LEVELS

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Order $\alpha^4 R_\infty$ corrections to positronium P levels

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Abstract

The order $\alpha^4 R_\infty$ corrections to positronium P levels are found. The calculation is reduced to the ordinary perturbation theory for the nonrelativistic Schrödinger equation. The perturbation operators have the Breit-type structure and are obtained by calculating on-mass-shell diagrams. Found energy corrections constitute numerically $\delta E(2^1P_1) = 0.06$ MHz, $\delta E(2^3P_2) = 0.08$ MHz, $\delta E(2^3P_1) = 0.025$ MHz, $\delta E(2^3P_2) = -0.58$ MHz.

1 Introduction

Precision measurements of positronium structure provide a unique test of quantum electrodynamics. The typical accuracy reached in the measurements of the positronium $2^3S_1 - 2^3P_J (J = 0, 1, 2)$, $1^3S_1 - 2^3S_1$ and $2^3S_1 - 2^1P_1$ intervals constitutes few MHz [1, 2, 3, 4, 5, 6, 7]. The order $\alpha^3 \log(1/\alpha) R_\infty$ and $\alpha^3 R_\infty$ corrections ($R_\infty = 109\,737.315\,682\,7(48)$ cm^{-1} is the Rydberg constant) [8, 9, 10] are insufficient now for the comparison of quantum electrodynamics with those experimental data.

The two-body bound state QED problem is by itself of independent theoretical interest. The generally accepted theoretical approach to it goes back to Refs [11, 12, 13]. This approach starts from the introduction of a relativistic two-body wave equation, which can be solved exactly and in the nonrelativistic limit reduces to the Schrödinger equation. Then a perturbation series is developed about the exact solution.

Our approach is different and is described, as applied to the corrections logarithmic in α, in Refs [14, 15, 16]. The corrections discussed in Refs [14, 15, 16] and here, are of relativistic origin and can be found as follows. We construct effective perturbation operators by calculating on-mass-shell diagrams and then use those operators in the standard perturbation theory for the nonrelativistic Schrödinger equation.

The order $\alpha^4 \log(1/\alpha) R_\infty$ corrections to the positronium levels were calculated recently by Fell [17] and then by us [16] (we have corrected in Ref [16] a numerical error made in Ref [15]). This shift exists in the S states only

and scales with the principal quantum number n as n^{-3}. The logarithmic structure of this correction allows one to treat the relativistic effects as a perturbation when deriving the result. However, if one tries to go beyond the logarithmic approximation, the logarithmic integrals which are cut off at the electron mass m should be treated exactly, and the problem becomes extremely tedious.

Fortunately, for states of higher angular momenta, $L > 0$, the situation is better since their nonrelativistic wave functions fall off at small distances. Therefore, the integrals arising in the perturbation theory, converge in the nonrelativistic region which makes the problem quite tractable. The main complication (underestimated by us at the beginning of the work) is of the "book-keeping" nature.

Here we present the results of our analytical calculations for the order $\alpha^4 R_\infty$ corrections to positronium P levels. In the case $n = 2$ the results for the fine-splitting of P levels can be directly compared with the data extracted from the experimental results of Refs [2, 5, 6, 7].

Analogous corrections for the electron-electron interaction in helium were obtained numerically in Ref [18].

2 Contributions of irreducible operators

2.1 Relativistic correction to the dispersion law

Let us start with the kinematic correction generated by the v^4/c^4 term in the dispersion law for electron and positron,

$$\sqrt{m^2+p^2}-m = \frac{p^2}{2m}-\frac{p^4}{8m^3}+\frac{p^6}{16m^5}+\dots, \tag{1}$$

$$E^{(1)}_{kin} = 2\left\langle\frac{p^6}{16m^5}\right\rangle. \tag{2}$$

By means of motion equations we transform the expectation value to

$$\begin{aligned} E^{(1)}_{kin} &= \frac{m\alpha^6}{64}\left\langle(\mathcal{E}_n+\tfrac{1}{r})\frac{p^2}{2}(\mathcal{E}_n+\tfrac{1}{r})\right\rangle \\ &= \frac{m\alpha^6}{64}\left\langle {\mathcal{E}_n}^3+\frac{3{\mathcal{E}_n}^2}{r}+\frac{3\mathcal{E}_n}{r^2}+\frac{1}{r^3}+\frac{1}{2r^4}\right\rangle; \\ \mathcal{E}_n &\equiv \frac{2E_n}{m\alpha^2}=-\frac{1}{2n^2}. \end{aligned} \tag{3}$$

After extracting the overall factor $m\alpha^6$ everything else here and below is written in the usual atomic units. The substitution of the nonrelativistic Coulomb expectation values for $1/r^k$,

$$\left\langle r^{-1}\right\rangle = \frac{1}{n^2}, \qquad \left\langle r^{-2}\right\rangle = \frac{2}{3n^3}, \tag{4}$$

$$\left\langle r^{-3}\right\rangle = \frac{1}{3n^3}, \qquad \left\langle r^{-4}\right\rangle = \frac{2}{5n^3}\left(1-\frac{2}{3n^2}\right), \tag{5}$$

reduces this energy correction to

$$E^{(1)}_{kin} = \frac{\epsilon_n}{2^6\cdot 3\cdot 5}\left(8-\frac{17}{n^2}+\frac{75}{8n^3}\right), \tag{6}$$

where $\epsilon_n \equiv m\alpha^6/n^3$.

2.2 Relativistic corrections to the Coulomb interaction

This perturbation operator will be extracted from the scattering amplitude for free particles. It is convenient to consider the positron as electron of opposite charge. Then the scattering amplitude due to the single Coulomb exchange is

$$A_C = -\frac{4\pi\alpha}{q^2}\rho(\vec{p}\,',\vec{p})\rho(-\vec{p}\,',-\vec{p}), \tag{7}$$

where

$$\rho(\vec{p}\,',\vec{p}) = u^+(\vec{p}\,')u(\vec{p}), \quad \vec{q}=\vec{p}\,'-\vec{p}. \tag{8}$$

We define scattering amplitudes with the sign opposite to the standard one, so that it can be immediately identified with the matrix element of an interaction operator in the momentum representation. Let us substitute into (7) the solution of the free Dirac equation

$$u(\vec{p}) = \sqrt{\frac{2\omega_{\vec{p}}}{\omega_{\vec{p}}+m}}\lambda_+(\vec{p})w. \tag{9}$$

where w is a bispinor describing particle at rest,

$$\lambda_+(\vec{p}) = \frac{1}{2}\left(1+\frac{\vec{\alpha}\vec{p}+\beta m}{\omega_{\vec{p}}}\right), \quad \omega_{\vec{p}}=\sqrt{p^2+m^2}.$$

The corrections of fourth order in v/c are

$$V_C = -\frac{\alpha}{16m^4}\frac{4\pi}{q^2}\left\{(p'^2-p^2)^2-i(\vec{\mathcal{S}},\vec{p}\,'\times\vec{p})(q^2+3p'^2+3p^2)-2(\vec{\mathcal{S}},\vec{p}\,'\times\vec{p})^2\right\}. \tag{10}$$

We have neglected here the operator proportional to $p^2 + p'^2$ since its expectation value in the coordinate representation $\langle \Delta\delta(\vec{r}) + \delta(\vec{r})\Delta \rangle$ vanishes for P states. The expectation value of operator (10) is again conveniently calculated in the coordinate representation. Its spin-independent part is

$$\left\langle \frac{4\pi}{q^2}(p'^2 - p^2)^2 \right\rangle = \langle [p^2,[p^2,\tfrac{1}{r}]] \rangle \tag{11}$$

$$= 2\langle [\tfrac{1}{r},[p^2,\tfrac{1}{r}]] \rangle = 4\langle r^{-4} \rangle . \tag{12}$$

The treatment of the spin-dependent operators is somewhat more complicated. Here we pass over to the Fourier transforms of the operators:

$$\int \frac{d^3q}{(2\pi)^3}\frac{4\pi\vec{q}}{q^2}e^{i\vec{q}\vec{r}} = \frac{i\vec{n}}{r^2}, \tag{13}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{4\pi q_i q_j}{q^2}e^{i\vec{q}\vec{r}} = \frac{\delta_{ij} - 3n_i n_j}{r^3} + 4\pi n_i n_j \delta(\vec{r}), \tag{14}$$

then use motion equations and expectation values (4), as well as the value of the radial wave function derivative at the origin

$$|R'_{n1}(0)|^2 = \frac{4}{9n^2}\left(1 - \frac{1}{n^2}\right). \tag{15}$$

In this way we get

$$E_C^{(1)} = \frac{\epsilon_n}{2^6 \cdot 5^2}\left\{-5\left(1 - \frac{2}{3n^2}\right) - \vec{S}\vec{L}\left(19 - \frac{121}{6n^2}\right) + \frac{(\vec{S}\vec{L})^2 + \vec{S}^2}{3}\left(1 - \frac{3}{2n^2}\right)\right\}. \tag{16}$$

Now, due to the Coulomb interaction electron (positron) can go over into a negative-energy intermediate state. The corresponding contributions are described by Z-diagrams of the kind presented in Fig.1. Here the particle

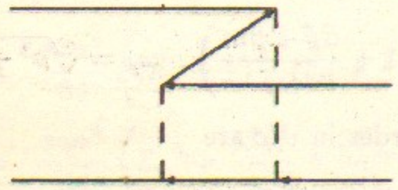

Fig.1. Z-type double-Coulomb exchange.

staying in a positive-energy state can be treated in the nonrelativistic approximation. The point is that the large energy denominator, equal approximately to $2m \gg E_n$, due to "heavy" intermediate states, and small matrix element of the Z-line provide the sufficient power of α in the perturbation. In this way we get for the perturbation operator

$$V_{C-} = \frac{(4\pi\alpha)^2}{2m}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2(\vec{k}-\vec{q})^2} \left(\frac{1}{2}\left(1 - \frac{m}{\omega_{\vec{p}+\vec{k}}}\right) + \frac{\vec{\sigma}\vec{p}'}{2m}\frac{\vec{\sigma}\vec{p}}{2m} - \frac{\vec{\sigma}\vec{p}'}{2m}\frac{\vec{\sigma}(\vec{p}+\vec{k})}{2m} - \frac{\vec{\sigma}(\vec{p}+\vec{k})}{2m}\frac{\vec{\sigma}\vec{p}}{2m}\right)$$

$$\to -\frac{(4\pi\alpha)^2}{(2m)^3}\int \frac{d^3k}{(2\pi)^3}\frac{\vec{k}(\vec{q}-\vec{k})}{k^2(\vec{q}-\vec{k})^2}. \tag{17}$$

Going over to the coordinate representation, we note that the last integral is in fact the convolution of the Fourier-transform of the operator $i\vec{n}/r^2$ with itself. So,

$$E_{C-}^{(1)} = \frac{m\alpha^6}{2^6}\langle r^{-4} \rangle = \frac{\epsilon_n}{2^5 \cdot 5}\left(1 - \frac{2}{3n^2}\right). \tag{18}$$

Let us note that actually the integral in (17) diverges at large k. It can be easily seen however that the divergent part is independent of the momentum transfer $\vec{q}$. Therefore in the coordinate representation the corresponding operator is just $\delta(\vec{r})$. Its expectation value does not vanish in S states only and leads there (via the accurate cut-off of the linear divergence at $k \sim m$) to a correction of the order $m\alpha^5$.

2.3 Single magnetic exchange

In the noncovariant perturbation theory the electron-positron scattering amplitude due to the exchange by one magnetic quantum is

$$A_M = -\frac{4\pi\alpha}{q}\frac{j_i(\vec{p}',\vec{p})j_j(-\vec{p}',-\vec{p})}{E_n - q - \frac{p^2+p'^2}{2m}}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right). \tag{19}$$

Here

$$\vec{j}(\vec{p}',\vec{p}) = u^+(\vec{p}')\vec{\alpha}u(\vec{p}) \tag{20}$$

is the matrix element of the current taken over the solutions (9) of the free Dirac equation. In the dispersion law for electron and positron it is sufficient here to confine to the nonrelativistic approximation.

Let us start from the contribution to the perturbation operator produced by the v^2/c^2 corrections to the currents:

$$V_{curr} = \frac{\alpha}{4m^4}\frac{4\pi}{q^2}\left\{4p'^2\left(p^2-\frac{(\vec{p}\vec{q})^2}{q^2}+\vec{p}\vec{q}\frac{i(\vec{q}\times\vec{p},\vec{S})}{q^2}+\frac{1}{2}(\vec{q}\vec{S})^2\right)\right. \quad (21)$$
$$\left. + \frac{p'^2-p^2}{2}\left((p'^2-p^2)\left(1+\frac{i(\vec{q}\times\vec{p},\vec{S})}{q^2}-\vec{S}^2\right)+(2\vec{p}+\vec{q})\vec{S}\,(\vec{q}\vec{S})\right)\right\}.$$

Here we have neglected again the terms with vanishing P state expectation values. Going over to the coordinate representation by means of (13), (14) and

$$\int\frac{d^3q}{(2\pi)^3}\frac{4\pi q_iq_j}{q^4}e^{i\vec{q}\vec{r}} = \frac{\delta_{ij}-n_in_j}{2r}, \quad (22)$$

$$\int\frac{d^3q}{(2\pi)^3}\frac{4\pi\vec{q}}{q^4}e^{i\vec{q}\vec{r}} = \frac{i\vec{n}}{2}, \quad (23)$$

we obtain

$$E^{(1)}_{curr} = \frac{\epsilon_n}{2^5\cdot 5}\left\{\frac{17}{3}-\frac{11}{n^2}+\frac{5}{n^3}\right. \quad (24)$$
$$\left. -\frac{7\vec{S}\vec{L}}{2}\left(1-\frac{1}{n^2}\right)+2(\vec{S}\vec{L})^2\left(1-\frac{7}{6n^2}\right)-2\vec{S}^2\left(1-\frac{1}{n^2}\right)\right\}.$$

Let us consider now the retardation effect. To this end the currents can be taken in the leading approximation:

$$\vec{j}(\vec{p}',\vec{p})\to\frac{1}{2m}(\vec{p}'+\vec{p}+i\vec{q}\times\vec{\sigma}). \quad (25)$$

At the atomic momentum transfer, $q\sim m\alpha$, the perturbation of interest originates from the second-order term of the expansion of the factor $[E_n-(p^2+p'^2)/2m-q]^{-1}$ in (19) in powers of $(E_n-(p^2+p'^2)/2m)/q$:

$$V_{ret} = -\frac{\alpha}{2m^2}\frac{4\pi}{q^2}\frac{\left(E_n-\frac{p^2+p'^2}{2m}\right)^2}{q^2}$$
$$\left\{q^2+2\vec{p}\vec{p}'-2\frac{(\vec{q}\vec{p}')(\vec{q}\vec{p})}{q^2}-2i(\vec{p}\times\vec{q},\vec{S})-q^2\vec{S}^2+(\vec{q}\vec{S})^2\right\}. \quad (26)$$

At first sight, the expectation value of this operator diverges linearly at small q. This divergence can be demonstrated however to be unrelated to the

order $m\alpha^6$ correction we are interested in. Indeed, let us split the region of integration over q into two, those from 0 to λ and from λ to ∞ where $m\alpha^2\ll\lambda\ll m\alpha$. In the second region our expansion is applicable and the result of integration contains a term proportional to $1/\lambda$. Since the initial integral is independent of λ, this term cancels in the sum with the integral over the first region calculated without the expansion. Meanwhile, this last integral has no contribution of the α^4R_∞ order independent of λ. Therefore, taking as the Fourier-transform of

$$\frac{4\pi}{q^4}\left(\vec{p}\vec{p}'-\frac{(\vec{q}\vec{p}')(\vec{q}\vec{p})}{q^2}\right), \quad (27)$$

the operator

$$-\frac{r}{8}(3p^2-(\vec{n}\vec{p})^2), \quad (28)$$

we get finally:

$$E^{(1)}_{ret} = \frac{\epsilon_n}{2^5\cdot 3\cdot 5}\left(14-\frac{15}{n}+\frac{13}{2n^2}\right). \quad (29)$$

Magnetic quantum propagates for a finite time and can cross arbitrary number of the Coulomb ones. Simple counting of the momenta powers demonstrates that it is sufficient to include the diagrams with one and two Coulomb quanta (dashed lines) crossed by the magnetic photon (wavy line).

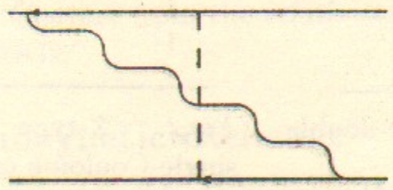

Fig.2. Single-magnetic-single-Coulomb exchange.

In the first case, Fig.2, the perturbation operator arises as a product of the Pauli currents (25) and the first-order term in the expansion in $(E_n-(p^2+p'^2)/2m)/q$:

$$V_{MC} = \frac{\alpha^2}{m^2}\int\frac{d^3k}{(2\pi)^3}\frac{(4\pi)^2}{k^4(\vec{q}-\vec{k})^2}\left(\frac{p^2+p'^2-\vec{q}\vec{k}+k^2}{2m}-E_n\right) \quad (30)$$
$$\left\{k^2+2\vec{p}\vec{p}'-2\frac{(\vec{k}\vec{p}')(\vec{k}\vec{p})}{k^2}-2i(\vec{p}\times\vec{k},\vec{S})-k^2\vec{S}^2+(\vec{k}\vec{S})^2\right\}.$$

In the second case all the elements of diagram 3 should be taken to leading nonrelativistic approximation:

$$V_{MCC} = -\frac{\alpha^3}{2m^2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\frac{(4\pi)^3}{k^4(\vec{q}-\vec{k}')^2(\vec{k}'-\vec{k})^2} \tag{31}$$
$$\left\{k^2+2\vec{p}\vec{p}'-2\frac{(\vec{k}\vec{p}')(\vec{k}\vec{p})}{k^2}-2i(\vec{p}\times\vec{k},\vec{S})-k^2\vec{S}^2+(\vec{k}\vec{S})^2\right\}.$$

All the integrals in (30) and (31) constitute convolutions of the already known Fourier-transforms of powers of $\vec{r}$. In this way we get for the first operator:

$$E^{(1)}_{MC} = \frac{\epsilon_n}{2^5\cdot 3\cdot 5}\left\{-13+\frac{30}{n}-\frac{13}{n^2}\right.$$
$$\left.-\frac{3\vec{S}\vec{L}}{5}\left(29+\frac{2}{3n^2}\right)+2\frac{(\vec{S}\vec{L})^2-4\vec{S}^2}{5}\left(13-\frac{2}{n^2}\right)\right\}; \tag{32}$$

and for the second one:

$$E^{(1)}_{MCC} = \frac{\epsilon_n}{2^5\cdot 3\cdot 5}\left\{15\left(1-\frac{1}{n}\right)+9\vec{S}\vec{L}-2(\vec{S}\vec{L})^2+8\vec{S}^2\right\}. \tag{33}$$

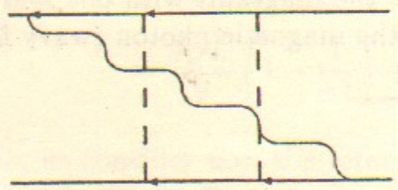

Fig.3. Single-magnetic-double-Coulomb exchange.

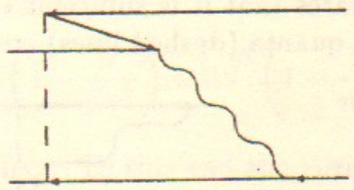

Fig.4. Z-type single-magnetic-single-Coulomb exchange.

One more energy correction of the $\alpha^4 R_\infty$ order at the single magnetic exchange is due to Z-type diagrams (see Fig.4). To leading approximation one gets easily

$$V_{M-} = \frac{\alpha^2}{m^3}\int\frac{d^3k}{(2\pi)^3}\frac{(4\pi)^2}{k^2(\vec{q}-\vec{k})^2}$$
$$\left\{\vec{k}(\vec{q}-\vec{k})(1-\vec{S}^2)+\vec{k}\vec{S}\,(\vec{q}-\vec{k})\vec{S}+\frac{1}{2}i(\vec{k}\times\vec{q},\vec{S})\right\}. \tag{34}$$

Standard calculations give now

$$E^{(1)}_{M-} = -\frac{\epsilon_n}{2^2\cdot 5^2}\left\{\frac{5}{2}+\frac{3}{4}\vec{S}\vec{L}-(\vec{S}\vec{L})^2-\vec{S}^2\right\}\left(1-\frac{2}{3n^2}\right). \tag{35}$$

2.4 True radiative corrections

Curiously enough, even the true radiative corrections of the $\alpha^4 R_\infty$ order to the P-levels energy can be presented in a simple form practically without special calculation. They can be easily demonstrated to be confined here to anomalous magnetic moment contribution to the single magnetic exchange, i.e. to a trivial modification of the usual Breit Hamiltonian. With this contribution the Pauli current becomes

$$\vec{j}(\vec{p}',\vec{p}) \to \frac{1}{2m}(\vec{p}'+\vec{p}+ig\vec{q}\times\vec{\sigma}), \tag{36}$$

where

$$g = 1+\frac{\alpha}{2\pi}-c\left(\frac{\alpha}{\pi}\right)^2+\ldots, \quad c\approx 0.328 \tag{37}$$

Let us note here that the anomalous magnetic moment contributions to the first-order retardation effect and to diagram 2 cancel. It corresponds to the absence of the order v/c corrections to the Breit magnetic exchange. In this way we get the following correction to the P-level energy:

$$E^{(1)}_{rad} = -\frac{\epsilon_n}{2^4\cdot 3\cdot 5}\frac{1}{\pi^2}\left\{\left(\frac{3}{4}+4c\right)\vec{S}\vec{L}+(1-8c)\left(\frac{3}{2}(\vec{S}\vec{L})^2-\vec{S}^2\right)\right\}. \tag{38}$$

Two its features are noteworthy. First, it vanishes in singlet states which looks rather natural. Second, it has an extra factor $1/\pi^2$ which reflects its radiative origin as distinct from the relativistic origin of other order $\alpha^4 R_\infty$ corrections to P-levels.

2.5 Double magnetic exchange

Let us consider now irreducible diagrams with two magnetic quanta. To our approximation they are confined to the type presented in Fig.5. Their sum reduces to

$$V_{MM} = \frac{\alpha^2}{m^3}\int\frac{d^3k}{(2\pi)^3}\frac{(4\pi)^2}{k^2k'^2}\left\{\vec{p}\vec{p}'-2\frac{(\vec{k}\vec{p})(\vec{k}\vec{p}')}{k^2}+\frac{(\vec{k}\vec{p})(\vec{k}\vec{k}')(\vec{k}\vec{p}')}{k^2k'^2}\right.$$
$$\left.-\frac{\vec{k}\vec{k}'}{2}+i\vec{S}\left(\vec{q}\times\vec{p}-\vec{k}\times\vec{p}-\frac{\vec{q}\times\vec{k}\,(\vec{k}\vec{p}')}{k^2}\right)\right\}, \tag{39}$$

where $\vec{k}' = \vec{q}-\vec{k}$. Now we substitute the products of operators in the coordinate representation for the convolutions of their Fourier-transforms and take the corresponding expectation values.

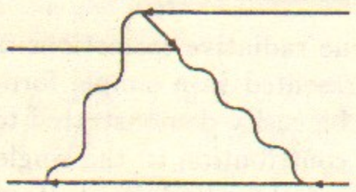

Fig.5. Double-magnetic exchange.

In this way we get

$$E^{(1)}_{MM} = \frac{\epsilon_n}{2^4 \cdot 3 \cdot 5}\left\{10\left(1-\frac{1}{n^2}\right) - 3\vec{\mathcal{S}}\vec{L}\left(1-\frac{2}{3n^2}\right)\right\}. \qquad (40)$$

3 Corrections of second order in the Breit Hamiltonian

Next class of the order $\alpha^4 R_\infty$ corrections originates from the iteration of the usual Breit Hamiltonian V of second order in v/c.

Before proceeding to the calculation, let us note that the Breit potential for the positronium (see, e.g., [19], §84),

$$\begin{aligned} V &= -\frac{p^4}{4m^3} + \frac{\pi\alpha}{m^2}\delta(\vec{r}) - \frac{\alpha}{2m^2 r}\left(p^2 + (\vec{n}\vec{p})^2\right) + \frac{3\alpha}{2m^2 r^3}\vec{L}\vec{\mathcal{S}} \\ &\quad + \frac{3\alpha}{2m^2 r^3}\left\{(\vec{\mathcal{S}}\vec{n})^2 - \frac{1}{3}\vec{\mathcal{S}}^2\right\} + \frac{\pi\alpha}{m^2}\left(\frac{7}{3}\vec{\mathcal{S}}^2 - 2\right)\delta(\vec{r}), \qquad (41) \end{aligned}$$

has nonvanishing matrix elements for $|\Delta L| = 0, 2$ only. A contribution of virtual F-levels will be found later. And now we average the angular dependence of (41) over a P-state wave function:

$$3\left(\vec{\mathcal{S}}\vec{n}\right)^2 - \vec{\mathcal{S}}^2 \to -\frac{3}{5}\left(2\left(\vec{\mathcal{S}}\vec{L}\right)^2 + \vec{\mathcal{S}}\vec{L} - \frac{4}{3}\vec{\mathcal{S}}^2\right). \qquad (42)$$

After this procedure the perturbation (41) can be presented as follows:

$$V \to \frac{m\alpha^4}{16} v, \qquad (43)$$

$$v = \left(a\left\{h, \tfrac{1}{r}\right\} + b[h, ip_r] + c\tfrac{1}{r^2}\right), \qquad (44)$$

where

$$a = -3, \quad b = 1+\kappa, \quad c = -4+\kappa; \qquad (45)$$

$$\kappa = \frac{6\vec{\mathcal{S}}\vec{L} - 3(\vec{\mathcal{S}}\vec{L})^2 + 2\vec{\mathcal{S}}^2}{5}; \qquad (46)$$

$p_r = -i(\partial_r + 1/r)$ is the radial momentum, while

$$h = \frac{p_r^2}{2} + \frac{1}{r^2} - \frac{1}{r}$$

is the unperturbed Hamiltonian for the radial motion with $L = 1$ in the Coulomb field.

According to the standard rules,

$$E^{(2)}_P = \langle VGV\rangle = \frac{m\alpha^6}{128}\langle vgv\rangle, \qquad (47)$$

$$G = \sum_k{}' \frac{|kP\rangle\langle kP|}{E_n - E_k} = \frac{2}{m\alpha^2} g, \qquad (48)$$

$$g = \sum_k{}' \frac{|kP\rangle\langle kP|}{\mathcal{E}_n - \mathcal{E}_k}. \qquad (49)$$

Since relativistic effects are included here into the perturbation, the intermediate states $|kP\rangle$ are merely eigenfunctions of the nonrelativistic Schrödinger equation in the Coulomb field.

The representation (44) enables us to find the energy correction (47) without recourse to the exact form of the Coulomb Green's function g. Indeed,

$$\begin{aligned} \langle vgv\rangle &= \frac{1}{2}\left\langle\left(2a\mathcal{E}_n\tfrac{1}{r} - a\left[h, \tfrac{1}{r}\right] + b[h, ip_r] + c\tfrac{1}{r^2}\right) gv\right. \\ &\quad \left. + vg\left(2a\mathcal{E}_n\tfrac{1}{r} + a\left[h, \tfrac{1}{r}\right] + b[h, ip_r] + c\tfrac{1}{r^2}\right)\right\rangle \qquad (50) \\ &= -a\mathcal{E}_n\left(2\langle v\rangle + \partial_\beta\langle v\rangle - \langle\partial_\beta v\rangle\right) - \frac{a}{2}\left\langle\left\{v, \tfrac{1}{r}\right\}\right\rangle \\ &\quad + \frac{b}{2}\langle[ip_r, v]\rangle + c\left(\partial_\gamma\langle v\rangle - \langle\partial_\gamma v\rangle\right). \qquad (51) \end{aligned}$$

When passing from (50) to (51) we used the equation of motion and presented the perturbations $1/r$ and $1/r^2$ as the result of β and γ variations in the modified Hamiltonian,

$$h(\beta,\gamma) = \frac{p_r^2}{2} + \frac{\gamma}{r^2} - \frac{\beta}{r}. \qquad (52)$$

Derivatives in (51) are taken at $\beta = \gamma = 1$.

Substituting (44) into (51) provides us with the expression containing just the mean values of $1/r^k$ (see (4)).

We collect now coefficients at the powers of $1/n$ to obtain:

$$E_P^{(2)} = \frac{m\alpha^6}{128n^3}\left\{-\frac{3a^2+14ab+13b^2}{15} - \frac{2c(2c+9a+9b)}{27} - \frac{2c^2}{3n} + \frac{2}{3n^2}\left(\frac{11a^2+13ab+6b^2}{5}+4ac\right) - \frac{5a^2}{2n^3}\right\}. \quad (53)$$

Finally, substituting the expressions for a, b, c (45), we get the contribution of intermediate P states to the iteration of the Breit Hamiltonian:

$$E_P^{(2)} = \epsilon_n\left\{-\frac{1022-844\kappa+227\kappa^2}{2^7\cdot 3^3\cdot 5} - \frac{(4-\kappa)^2}{2^6\cdot 3\cdot n} + \frac{102-29\kappa+2\kappa^2}{2^6\cdot 5\cdot n^2} - \frac{45}{2^8\cdot n^3}\right\}. \quad (54)$$

Transitions to intermediate F states are driven by the operator $(\vec{\mathcal{S}}\vec{n})^2$ contained in (41). Due to the conservation of the total angular momentum such transitions are possible in the states with $J = 2$ only. The matrix element squared of the operator $(\vec{\mathcal{S}}\vec{n})^2$ is easily calculated by means of closure:

$$|\langle P_2|(\vec{\mathcal{S}}\vec{n})^2|F_2\rangle|^2 = \langle P_2|(\vec{\mathcal{S}}\vec{n})^4|P_2\rangle - \langle P_2|(\vec{\mathcal{S}}\vec{n})^2|P_2\rangle^2 = \langle P_2|(\vec{\mathcal{S}}\vec{n})^2|P_2\rangle - \langle P_2|(\vec{\mathcal{S}}\vec{n})^2|P_2\rangle^2 = \frac{6}{25}. \quad (55)$$

We have used here the identity $(\vec{\mathcal{S}}\vec{n})^3|P\rangle = \vec{\mathcal{S}}\vec{n}|P\rangle$.

To calculate the radial part of the correction, proportional to

$$\left\langle \frac{1}{r^3}\sum_k \frac{|kF\rangle\langle kF|}{\mathcal{E}_n - \mathcal{E}_k}\frac{1}{r^3}\right\rangle, \quad (56)$$

we present the operator $1/r^3$ in the form

$$\frac{1}{r^3} = \frac{1}{18}\left[h\left(-ip_r - \frac{5}{r} + \frac{1}{5}\right) - \left(-ip_r - \frac{5}{r} + \frac{1}{5}\right)h_F\right] \quad (57)$$

$$= \frac{1}{18}\left[h_F\left(-ip_r + \frac{5}{r} - \frac{1}{5}\right) - \left(-ip_r + \frac{5}{r} - \frac{1}{5}\right)h\right], \quad (58)$$

where

$$h_F = \frac{p_r^2}{2} + \frac{6}{r^2} - \frac{1}{r} \quad (59)$$

is the unperturbed Hamiltonian for the radial motion with $L = 3$ in the Coulomb field. The energy correction is again expressed in terms of $\langle r^{-k}\rangle$:

$$E_F^{(2)} = m\alpha^6\frac{3}{2^8\cdot 5^2}\left\langle\frac{2}{5r^3} - \frac{7}{r^4}\right\rangle = -\frac{\epsilon_n}{2^6\cdot 5^3}\left(10 - \frac{7}{n^2}\right). \quad (60)$$

4 Numerical results

Let us summarize the numerical values of the $\alpha^4 R_\infty$ corrections to the energies of positronium $2P$ levels. They are

0.06 MHz for 2^1P_1,

0.08 MHz for 2^3P_2,

0.025 MHz for 2^3P_1,

– 0.58 MHz for 2^3P_0.

We wish to emphasize here that the last correction is quite comparable in magnitude to the corresponding logarithmic correction (of order $\alpha^4 \log\alpha\, R_\infty$) to the positronium $2S$-levels. The latter constitutes for instance 0.96 MHz for the 2^3S_1 state [17, 16]. Therefore, there is no special reasons to expect that the nonlogarithmic corrections (of order $\alpha^4 R_\infty$) to the positronium S levels are small. It makes their calculation quite an actual problem.

Including $\alpha^2 R_\infty$ and $\alpha^3 R_\infty$ terms we obtain the fine-structure intervals between $2^{2S+1}P_J$ levels. In the Table 1 these theoretical values are compared

Table 1.

Fine-structure intervals between $2^{2S+1}P_J$ levels (in MHz).

	Experiments		Theory	
	Michigan [2, 5]	Mainz [6, 7]	Total	$\mathcal{O}(\alpha^4 R_\infty)$
$E(^3P_2) - E(^3P_0)$	9884.5±10.5	9875.27 ±4.44	9871.54	0.66
$E(^3P_1) - E(^3P_0)$	5502.8±10.9	5487.23 ±4.50	5485.84	0.60
$E(^1P_1) - E(^3P_0)$	7323.1±16.5	7319.65 ±7.64	7312.88	0.64

with the transition frequencies extracted from the results of two recent experiments [2, 5, 6, 7] (all systematic and statistical errors are added quadratically when extracting the experimental numbers).

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