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PARTICULARITIES OF THE NNLLA BFKL

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# Particularities of the NNLLA BFKL \*

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## Abstract

Peculiar properties of the BFKL approach in the next-to-next-to-leading logarithmic approximation (NNLLA) are discussed. In this approximation the scheme of derivation of the BFKL equation must be changed because of violation of the simple factorized form of amplitudes with multi-Reggeon exchanges and necessity to take into account imaginary parts of amplitudes in the unitarity relations.

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# 1 Introduction

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) approach [1, 2, 3, 4] provides a general way for the theoretical description of processes at high c.m.s. energy  $\sqrt{s}$  and fixed (not growing with  $s$ ) momentum transfers. It is based on a remarkable property of QCD – gluon Reggeization [5], which gives a very powerful tool for the description of such processes. This property makes it possible to derive the equation of evolution of the amplitudes of processes with energy (the BFKL equation). Now the BFKL approach is well developed in the leading logarithmic approximation (LLA), which summarizes the terms  $(\alpha_s \ln s)^n$  and in the next after it (NLLA), giving the opportunity to summarize also the terms  $\alpha_s(\alpha_s \ln s)^n$  (see e.g. [6] and references therein). Development of the next (NNLLA) approximation is a long-standing problem, actual both in terms of phenomenology and theory. There is no doubt that such a development is possible. It turns out, however, that the scheme of derivation of the BFKL equation must be changed in the NNLLA.

## 2 The scheme of derivation of the BFKL equation

The BFKL equation is derived by analyzing the  $s$ -channel discontinuities of elastic amplitudes calculated using unitarity. The main contributions to the discontinuities are given by the multi-Regge kinematics (MRK). Due to the Reggeization, the amplitudes used in the unitarity conditions have a simple factorized form in the NLLA as well as in the LLA. The Reggeization allows to express an infinite number of amplitudes through several effective vertices and gluon trajectory. For elastic scattering processes  $A + B \rightarrow A' + B'$  the Reggeization means that scattering amplitudes with gluon quantum numbers in the  $t$ -channel in the Regge kinematic region  $s \simeq -u \rightarrow \infty$ ,  $t$  fixed, can be presented as

$$\mathcal{A}_{AB}^{A'B'} = \Gamma_{A'A}^R \left[ \left( \frac{-s}{-t} \right)^{\omega(t)} - \left( \frac{s}{-t} \right)^{\omega(t)} \right] \Gamma_{B'B}^R, \quad (1)$$

where  $\Gamma_{P',P}^R$  are energy-independent particle-particle-Reggeon (PPR) vertices (or scattering vertices),  $j(t) = 1 + \omega(t)$  is the Reggeized gluon trajectory. The Reggeization means also definite (multi-Regge) form of production amplitudes in the multi-Regge kinematics (MRK). MRK is the kinematics where all particles have limited (not growing with  $s$ ) transverse momenta and are combined into jets with limited invariant mass of each jet and large (growing with  $s$ ) invariant masses of any pair of the jets. This kinematics gives dominant contributions to cross sections of QCD processes at high energy  $\sqrt{s}$ .

For the amplitude  $\mathcal{A}_{2 \rightarrow n+2}$  of the process  $A+B \rightarrow A'+G_1+\dots+G_n+B'$  of production of  $n$  gluons with momenta  $k_1, k_2, \dots, k_n$  the MRK means

$$s \gg s_i \gg |t_i| \simeq \vec{q}_i^2, \quad s \simeq \frac{\prod_{i=1}^{n+1} s_i}{\prod_{i=1}^n \vec{k}_i^2}, \quad (2)$$

where

$$s = (p_A + p_B)^2, \quad s_i = (k_{i-1} + k_i)^2, \quad i = 1, \dots, n+1, \quad k_0 \equiv p_{A'}, \quad k_{n+1} \equiv p_{B'}, \quad (3)$$

$$q_1 = p_A - p_{A'}, \quad q_{j+1} = q_j - k_j, \quad j = 1, \dots, n, \quad q_{n+1} = p_{B'} - p_B, \quad (4)$$

the vector sign means transverse to the  $p_A, p_B$  plane components. In this region, one has for the amplitude  $\mathcal{A}_{2 \rightarrow n+2}$

$$\begin{aligned} \Re \mathcal{A}_{2 \rightarrow n+2} &= 2s \Gamma_{A'A}^{R_1} \left( \prod_{i=1}^n \frac{1}{t_i} \left( \frac{s_i}{|\vec{k}_{i-1}| |\vec{k}_i|} \right)^{\omega(t_i)} \gamma_{R_i R_{i+1}}^{G_i} \right) \\ &\times \frac{1}{t_{n+1}} \left( \frac{s_{n+1}}{|k_n| |\vec{q}_{n+1}|} \right)^{\omega(t_{n+1})} \Gamma_{B'B}^{R_{n+1}}. \end{aligned} \quad (5)$$

Here  $\Gamma_{A'A}^R$  and  $\Gamma_{B'B}^R$  are the same scattering vertices as in (1) and  $\gamma_{R_i R_{i+1}}^{G_i}$  are the Reggeon-Reggeon-Particle (RRP) vertices (or the production vertices).

In the LLA only gluons can be produced. In the NLA one has to consider not only the amplitudes (1), (5), but also amplitudes obtained from them by replacement of one of final particles by a jet containing a couple of particles with fixed (of order of transverse momenta) invariant mass.

The Reggeon vertices and the gluon trajectory are known in the next-to-leading order (NLO), that means the one-loop approximation for the vertices and the two-loop approximation for the trajectory. It is just the accuracy which is required for the derivation of the BFKL equation in the NLLA. Validity of the forms (1) and (5) is proved now in all orders of perturbation

theory in the coupling constant  $g$  both in the LLA [7] and in the NLLA [8, 9, 10, 11, 12]. Note that the simple factorized form (5) is valid only for the real part of the amplitudes (the sign  $\Re$  in the Equation (5) means “real part”). Fortunately, the imaginary parts are not essential for the derivation of the BFKL equation in the NLLA.

The Reggeization provides a simple derivation of the BFKL equation both in the LLA and NLLA. Two-to-two scattering amplitudes with all possible quantum numbers in the  $t$ -channel are calculated using Equation (1) and (5) in the  $s$ -channel unitarity relations and analyticity. The  $s$ -channel discontinuities of the processes  $A + B \rightarrow A' + B'$  are presented as the convolutions  $\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$ , where the impact factors  $\Phi_{A'A}$  and  $\Phi_{B'B}$  describe transitions  $A \rightarrow A'$  and  $B \rightarrow B'$  due to interactions with Reggeized gluons,  $G$  is the Green's function for two interacting Reggeized gluons with an operator form  $\hat{G} = e^{Y\hat{K}}$ , where  $Y = \ln(s/s_0)$ ,  $s_0$  is an energy scale,  $\hat{K}$  is the BFKL kernel. The impact factors and the BFKL kernel are expressed in terms of the Reggeon vertices and trajectory. Energy dependence of scattering amplitudes is determined by the BFKL kernel, which is universal (process independent). The kernel  $\hat{K} = \omega_\infty + \hat{\omega}_2 + \hat{K}_r$  is expressed through the Regge trajectories  $\hat{\omega}_1$  and  $\hat{\omega}_2$  of two gluons and the “real part”  $\hat{K}_r$  describing production of particles in their interaction:  $\hat{K}_r = \hat{K}_G + \hat{K}_{Q\bar{Q}} + \hat{K}_{GG}$ . In the LLA only  $\hat{K}_G$  must be kept, because only gluons can be produced; in the NNLLA production of quark-antiquark ( $Q\bar{Q}$ ) and gluon ( $GG$ ) pairs is also possible.

One might think that this scheme is applicable in the NNLLA as well. In this case it would be sufficient to calculate three-loop corrections to the trajectory, two-loop corrections to  $\hat{K}_G$ , one-loop corrections to  $\hat{K}_{Q\bar{Q}}$  and  $\hat{K}_{GG}$  and to find in the Born approximation two new contributions,  $\hat{K}_{Q\bar{Q}G}$  and  $\hat{K}_{GGQ}$ , to  $\hat{K}_r$ .

Unfortunately, the scheme based on the forms (1) and (5) does not work in the NNLLA. The reason is the need to take account of the contributions of Regge cuts and the imaginary parts of the amplitudes in the unitarity conditions.

### 3 Contributions of the three-Reggeon cut

In the NLLA, two large logarithms can be lost in the product of two amplitudes in the unitarity condition used for derivation of the BFKL equation. It can be done losing either both logarithms in one of the amplitudes, or one logarithm in each of the amplitudes. In the first case one of the amplitudes is taken in the NNLLA and the other in the LLA. Since the amplitudes in

the LLA are real, only real parts of the NNLLA amplitudes are important in this case. But even for these parts the forms (1) and (5) become inapplicable because of the contributions of the three-Reggeon cut that appear in this approximation.

The first observation of the violation of the form (1) was made [13] in the consideration of the high-energy limit of the two-loop amplitudes for quark-quark ( $qq$ ), quark-gluon ( $qg$ ) and gluon-gluon ( $gg$ ) scattering. The interference of the tree- and two-loop amplitudes for each of the processes has been explicitly computed. The discrepancy appears in non-logarithmic two-loop terms. If the form (1) would be correct in the NNLLA, they should satisfy the certain relation, because in (1) three amplitudes are expressed in terms of only two vertices  $\Gamma_{QQ}^R$  and  $\Gamma_{GG}^R$ . The explicit calculation gives that this relation is violated by terms of  $\mathcal{O}(\pi^2/\epsilon^2)$ .

Detailed consideration of the terms responsible for the violation of the factorized form (1) in the case of two-loop and three-loop quark and gluon amplitudes was performed in [14, 15, 16]. In particular, the non-logarithmic double-pole contribution at two-loops obtained in [13] was recovered and all non-factorizing single-logarithmic singular contributions at three loops were found using the techniques of infrared factorization.

All these results are explained by the three-Regge cut contributions [17]. Since our Reggeons are the Reggeized gluons, the cut starts from the diagrams with three-gluon exchanges. In the Feynman gauge the contribution of these diagrams to the amplitudes  $A^a$  with the adjoint representation of the colour group in the  $t$ -channel has the form

$$A_{ij}^a = \langle A' | T^a | A \rangle \langle B' | T^a | B \rangle \left[ C_{ij} A^{(eik)} + \frac{N_c^2}{8} (A_{ij}^s + A_{ij}^u) + \delta_{i,q} \delta_{j,q} \frac{4 - N_c^2}{8} (A_{ij}^s - A_{ij}^u) \right], \quad (6)$$

where  $ij$  are  $qq$ ,  $qg$  and  $gg$  for quark-quark, quark-gluon and gluon-gluon scattering correspondingly,  $A_{ij}^s$  and  $A_{ij}^u$  are the contributions of the ladder diagrams in the  $s$  and  $u$  channels respectively with omitted colour group factors,  $A^{(eik)}$  is the sum of such contributions for all the diagrams, and  $C_{ij}$  are the colour group coefficients:

$$C_{qq} = \frac{1}{4} \left( -1 + \frac{3}{N_c^2} \right), \quad C_{qg} = \frac{1}{4}, \quad C_{gg} = \frac{3}{2}. \quad (7)$$

The last term in (6) is the contribution of the positive signature in the quark-quark scattering and is imaginary. The second term has the form (1) and

can be assigned to the Reggeized gluon contribution. On the contrary, this is not true for the first term, because

$$2C_{gq} \neq C_{qq} + C_{gg} . \quad (8)$$

Exactly this term is responsible for the violation of the factorized form (1) at the two-loop level discovered in [13] and confirmed in [15]. This follows from the explicit form  $A^{eik} = g^2(s/t)(-4\pi^2/3)g^4 \vec{q}^2 A_{\perp}^{(3)}$ ,

$$\begin{aligned} A_{\perp}^{(3)} &= \int \frac{d^{2+2\epsilon}l_1 d^{2+2\epsilon}l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^2 \vec{l}_2^2 (\vec{q} - \vec{l}_1 - \vec{l}_2)^2} = \\ &= 3C_{\Gamma}^2 \frac{4}{\epsilon^2} \frac{(\vec{q}^2)^{2\epsilon}}{\vec{q}^2} \frac{\Gamma^2(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+3\epsilon)} , \end{aligned} \quad (9)$$

$$C_{\Gamma} = \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{(4\pi)^{2+\epsilon}\Gamma(1+2\epsilon)} . \quad (10)$$

However, one can not affirm that this term is given entire by the three-Reggeon cut. Indeed, the coefficients  $C_{ij}$  in (7) can be presented as the sum

$$C_{ij} = C_{ij}^R + C_{ij}^C , \quad (11)$$

with the coefficients  $C_{ij}^R$  satisfying the equality

$$2C_{gq}^R = C_{qq}^R + C_{gg}^R . \quad (12)$$

The terms with  $C_{ij}^R$  have the form (1) and can be assigned to the Reggeized gluon contribution, so that the contribution of the three-Reggeon cut can be given by the terms with  $C_{ij}^C$  only. Since the coefficients  $C_{ij}^R$  obey only one condition, there is a great freedom in their choice. It occurs [17] that

$$C_{gg}^R = 3 , \quad C_{gg}^C = -\frac{3}{2} , \quad C_{gq}^R = \frac{7}{4} , \quad C_{gq}^C = -\frac{3}{2} , \quad C_{qq}^R = \frac{1}{2} , \quad C_{qq}^C = \frac{3(1-N_c^2)}{4N_c^2} . \quad (13)$$

The Reggeon and three-Reggeon cut contributions have different dependence on  $s$ . In the case of the Reggeized gluon it come solely from the Regge factor as in (1). In the case of the three-Reggeon cut, one has to take into account the Reggeization of each of the three gluons and the interaction between them. For the first logarithmic correction, the Reggeization gives  $\ln s$  with the coefficient  $3C_R$ , where

$$C_R = -g^2 N_c C_{\Gamma} \frac{4}{3\epsilon} (\vec{q}^2)^{\epsilon} \frac{\Gamma(1-3\epsilon)\Gamma(1+2\epsilon)\Gamma(1+3\epsilon)}{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)\Gamma(1+\epsilon)\Gamma(1+4\epsilon)} . \quad (14)$$

Interaction between two Reggeons with transverse momenta  $\vec{l}_1$  and  $\vec{l}_2$  and colour indices  $c_1$  and  $c_1$  is given by the real part of the BFKL kernel

$$\left[ \mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k}) \right]_{c_1 c_2}^{c'_1 c'_2} = T_{c_1 c'_1}^a T_{c_2 c'_2}^a \frac{g^2}{(2\pi)^{D-1}} \left[ \frac{\vec{q}_1^2 \vec{q}_2'^2 + \vec{q}_2^2 \vec{q}_1'^2}{\vec{k}^2} - \vec{q}^2 \right], \quad (15)$$

where  $\vec{k}$  is the momentum transferred from one Reggeon to another in the interaction,  $\vec{q}_1'$  and  $\vec{q}_2'$  ( $c'_1$  and  $c'_2$ ) are the Reggeon momenta (colour indices) after the interaction,  $\vec{q}_1' = \vec{q}_1 - \vec{k}$ ,  $\vec{q}_2' = \vec{q}_2 + \vec{k}$ , and  $\vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1' + \vec{q}_2'$ .

It occurs that, for the colour structure which we are interested in, account of interactions between all pairs of the Reggeons leads to the sum of the colour coefficients which differ from the coefficients  $C_{ij}$  (7) only by the common factor  $N_c$ . Therefore, the first order correction in the case of the three-Reggeon cut is presented as  $(-4C_R - C_3) \ln s$ , where the  $C_R$  and  $C_3$  come correspondingly from the first two terms and the last term in the square brackets in (15),

$$C_3 = g^2 N_c C_\Gamma \frac{32}{9\epsilon} (\vec{q}^2)^\epsilon \frac{\Gamma(1-3\epsilon)\Gamma(1-\epsilon)\Gamma^2(1+3\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)\Gamma(1+4\epsilon)}. \quad (16)$$

Thus, the first order correction in the case of the three-Reggeon cut is  $(-C_R - C_3) \ln s$ , where  $C_R$  and  $C_3$  are given by (14) and (16) respectively, and in the case of Reggeized gluon is  $\omega(t) \ln s$ , where

$$\omega(t) = -g^2 N_c \vec{q}^2 \int \frac{d^{2+2\epsilon} l}{2(2\pi)^{(3+2\epsilon)} \vec{l}^2 (\vec{q} - \vec{l})^2} = -g^2 N_c C_\Gamma \frac{2}{\epsilon} (\vec{q}^2)^\epsilon. \quad (17)$$

With the colour coefficients (11), (13), the terms singular in  $\epsilon$  of the total correction agree with the result obtained in [16].

Evidently, the three-Reggeon cut gives contributions to all  $2 \rightarrow n+2$  amplitudes in the MRK. They must be also found for further development of the BFKL approach.

## 4 Account of imaginary parts

The sign  $\Re$  in the Equation (5) means the ‘‘real part’’. It is important that the simple factorized form (5) is valid only for the real part of the amplitudes. Fortunately, the imaginary parts are not essential for the derivation of the BFKL equation in the NLLA. Indeed, they are suppressed by one power of  $\ln s_i$  in comparison with the real ones, and products of imaginary and real parts in the unitarity relations cancel due to summation of contributions

complex conjugated to each other. But their account becomes necessary in the NNLLA.

Fortunately, the imaginary parts are needed only in the main approximation, so that their calculation is not associated with large computational difficulties. However, it complicates the derivation of the BFKL equation and deprives its universality, because consideration of quark-quark, quark-gluon and gluon-gluon scattering becomes different. It is clear already from consideration of two-particle intermediate states in the unitarity condition. Remind that for two Reggeized gluons in the  $t$ -channel in QCD, that is for three colours, there are 6 irreducible representations:  $\underline{1}, \underline{8}_a, \underline{8}_s, \underline{10}, \overline{10}, \underline{27}$  (for  $N_c > 3$  there is one additional representation). The representations  $\underline{8}_a, \underline{10}, \overline{10}$  are anti-symmetric, while the representations  $\underline{1}, \underline{8}_s, \underline{27}$  (and the extra one for  $N_c > 3$ ) are symmetric. In real parts, with the NLLA accuracy, only the Reggeon channel,  $\underline{8}_a$ , is important. It provides universality of the NLLA:  $gg, qg$  and  $qq$  scattering can be considered in a unique way.

But account imaginary parts violate the universality, because  $gg$  scattering amplitudes can contain all the representations, while  $qg$  and  $qq$  amplitudes only  $\underline{1}, \underline{8}_a, \underline{8}_s$ .

Consideration of many-particle states in the unitarity condition is an even more complicated problem.

## 5 Summary

The basis of the BFKL approach is the remarkable property of QCD — gluon Reggeization. In this approach amplitudes of elastic scattering are restored analytically from the imaginary parts calculated using unitarity. The main contributions to the imaginary parts in the unitarity conditions come from the multi-Regge kinematics (MRK). In the leading (LLA) and next-to-leading (NLLA) logarithmic approximations the Reggeization provides with required accuracy a simple factorized form of QCD amplitudes used in the unitarity conditions. In the NNLLA such form is violated by the three-Reggeon cut and by imaginary parts of the amplitudes. For further development of the BFKL approach the Regge cut contributions must be found. Contributions of the three-Reggeon cut to elastic amplitudes were found in [17] and are presented here. Account of this cut in inelastic amplitudes is under consideration. As regards the imaginary parts required in the unitarity conditions, the way of their calculation is known. Unfortunately, the need to take them into account violates universality of the derivation of the BFKL equation for various processes, as well as the need to incorporate the Regge cuts.

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