### 7. Electromagnetic relations

Revised September 2005 by H.G. Spieler (LBNL).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Gaussian CGS</th>
<th>SI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charge:</td>
<td>2.997 921 58 x 10^9 esu</td>
<td>1 C = 1 A s</td>
</tr>
<tr>
<td>Potential:</td>
<td>(1/299.792 458) statvolt (ergs/esu)</td>
<td>1 V = 1 J C^{-1}</td>
</tr>
<tr>
<td>Magnetic field:</td>
<td>10^4 gauss = 10^4 dyne/esu</td>
<td>1 T = 1 N A^{-1} m^{-1}</td>
</tr>
</tbody>
</table>

\[ \mathbf{F} = q \left( \mathbf{E} + \frac{\nabla}{c} \times \mathbf{B} \right) \]

\[ \nabla \cdot \mathbf{D} = 4 \pi \rho \]
\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4 \pi}{c} \mathbf{J} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]

Constitutive relations:
\[ \mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu \]
\[ \mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu \]

**Linear media:**
\[ \varepsilon = 8.854 \times 10^{-12} \text{ F m}^{-1} \]
\[ \mu_0 = 4 \pi \times 10^{-7} \text{ N A}^{-2} \]

\[ \mathbf{D} = \frac{1}{4 \pi \varepsilon_0} \sum \frac{q_i}{|\mathbf{r} - \mathbf{r}'|} \mathbf{r}' d^3 x' \]
\[ \mathbf{A} = \frac{1}{4 \pi} \sum \frac{q_i}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}(\mathbf{r}') d^3 x' \]

\[ \mathbf{E}_0^* = \mathbf{E}_0 \]
\[ \mathbf{E}_0^* = \gamma (\mathbf{E}_0 + \frac{1}{c} \mathbf{v} \times \mathbf{B}) \]
\[ \mathbf{B}_0^* = \mathbf{B}_0 \]
\[ \mathbf{B}_0^* = \gamma (\mathbf{B}_0 - \frac{1}{c} \mathbf{v} \times \mathbf{E}) \]

\[ \frac{1}{4 \pi \varepsilon_0} = c^2 \times 10^{-7} \text{ N A}^{-2} = 8.987 55 \ldots \times 10^9 \text{ m}^{-1} \]
\[ \frac{\mu_0}{4 \pi} = 10^{-7} \text{ N A}^{-2} \]
\[ c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} = 2.997 924 \times 10^8 \text{ m s}^{-1} \]
7.1. Impedances (SI units)

\[ \rho = \text{resistivity at room temperature in } 10^{-8} \Omega \text{ m:} \]

\[ \begin{align*}
\sim 1.7 \text{ for Cu} & \sim 5.5 \text{ for W} \\
\sim 2.4 \text{ for Au} & \sim 73 \text{ for SS 304} \\
\sim 2.8 \text{ for Al} & \sim 100 \text{ for Nichrome} \\
\end{align*} \]

(Al alloys may have double the Al value.)

For alternating currents, instantaneous current \( I \), voltage \( V \), angular frequency \( \omega \):

\[ V = V_0 \, e^{j \omega t} = ZI. \quad (7.1) \]

Impedance of self-inductance \( L \): \( Z = j \omega L \).

Impedance of capacitance \( C \): \( Z = 1/j \omega C \).

Impedance of free space: \( Z = \sqrt{\mu_0 / \epsilon_0} = 376.7 \Omega \).

High-frequency surface impedance of a good conductor:

\[ Z = (1 + j) \frac{\rho}{\delta}, \quad \text{where } \delta = \text{skin depth} ; \]

\[ \delta = \frac{\rho}{\pi \nu \mu} \approx 6.6 \text{ cm} \quad \sqrt{\nu/(\text{Hz})} \quad \text{for Cu}. \quad (7.2) \]

7.2. Capacitors, inductors, and transmission lines

The capacitance between two parallel plates of area \( A \) spaced by the distance \( d \) and enclosing a medium with the dielectric constant \( \epsilon \) is

\[ C = K \epsilon A/d, \quad (7.4) \]

where the correction factor \( K \) depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor \( K \approx 0.8 \) for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length \( l \) is much greater than the wire diameter \( d \) is

\[ L \approx 2.0 \left[ \frac{\text{mH}}{\text{cm}} \right] \cdot l \left( \ln \left( \frac{4l}{d} \right) - 1 \right). \quad (7.5) \]

For very short wires, representative of vias in a printed circuit board, the inductance is

\[ L(\text{in mH}) \approx \ell/d. \quad (7.6) \]

A transmission line is a pair of conductors with inductance \( L \) and capacitance \( C \). The characteristic impedance \( Z = \sqrt{L/C} \) and the phase velocity \( v_p = 1/\sqrt{LC} = 1/\sqrt{\mu \epsilon} \), which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm.

The impedance of a coaxial cable with outer diameter \( D \) and inner diameter \( d \) is

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{D}{d}, \quad (7.7) \]

where the relative dielectric constant \( \epsilon_r = \epsilon / \epsilon_0 \). A pair of parallel wires of diameter \( d \) and spacing \( a \geq 2.5d \) has the impedance

\[ Z = 120 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{2a}{d}. \quad (7.8) \]

This yields the impedance of a wire at a spacing \( h \) above a ground plane,

\[ Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{4h}{d}. \quad (7.9) \]

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*


7.3. Synchrotron radiation (CGS units)

For a particle of charge \( e \), velocity \( \mathbf{v} = \beta \mathbf{c} \), and energy \( E = \gamma mc^2 \), traveling in a circular orbit of radius \( R \), the classical energy loss per revolution \( \delta E \) is

\[ \delta E = \frac{4 \pi \epsilon^2}{3} \frac{\beta^2 \gamma^4}{R}. \quad (7.10) \]

For high-energy electrons or positrons (\( \beta \approx 1 \)), this becomes

\[ \delta E \text{ (in MeV)} \approx 0.0885 \left( E \text{ (in GeV)} \right)^2 / R \text{ (in m)} . \quad (7.11) \]

For \( \gamma \gg 1 \), the energy radiated per revolution into the photon energy interval \( d(\hbar \omega) \) is

\[ dI = \frac{8 \pi}{9} \alpha \gamma F(\omega / \omega_c) d(\hbar \omega), \quad (7.12) \]

where \( \alpha = e^2 / \hbar c \) is the fine-structure constant and

\[ \omega_c = \frac{3 \gamma^3 c}{2 R}. \quad (7.13) \]

is the critical frequency. The normalized function \( F(y) \) is

\[ F(y) = \frac{9}{8 \pi} \sqrt{3} y \int_y^\infty K_{5/3} \left( x \right) dx, \quad (7.14) \]

where \( K_{5/3} \left( x \right) \) is a modified Bessel function of the third kind. For electrons or positrons,

\[ \hbar \omega_c \text{ (in keV)} \approx 2.22 \frac{\left( E \text{ (in GeV)} \right)^2}{R \text{ (in m)}} . \quad (7.15) \]

Fig. 7.1 shows \( F(y) \) over the important range of \( y \).

![Figure 7.1: The normalized synchrotron radiation spectrum \( F(y) \).](image)

For \( \gamma \gg 1 \) and \( \omega \ll \omega_c \),

\[ \frac{dI}{d(\hbar \omega)} \approx 3.3 \alpha \left( \omega R / c \right)^{1/3}, \quad (7.16) \]

whereas for

\[ \gamma \gg 1 \text{ and } \omega \geq 3 \omega_c, \]

\[ \frac{dI}{d(\hbar \omega)} \approx \frac{\sqrt{9 \pi}}{2} \alpha \gamma \frac{\omega}{\omega_c} \frac{1}{2} e^{-\omega / \omega_c} \left[ 1 + \frac{55 \omega_c}{72 \omega} + \ldots \right]. \quad (7.17) \]

The radiation is confined to angles \( \lesssim 1 / \gamma \) relative to the instantaneous direction of motion. For \( \gamma \gg 1 \), where Eq. (7.12) applies, the mean number of photons emitted per revolution is

\[ N_\gamma = \frac{5 \pi}{16 \sqrt{3}} \gamma, \quad (7.18) \]

and the mean energy per photon is

\[ \langle \hbar \omega \rangle = \frac{8}{15 \sqrt{3}} \hbar \omega_c. \]

When \( \langle \hbar \omega \rangle \gtrsim O(E) \), quantum corrections are important.

See J.D. Jackson, Classical Electrodynamics, 3rd edition (John Wiley & Sons, New York, 1998) for more formulas and details. (Note that earlier editions had \( \omega_c \) twice as large as Eq. (7.13)).