THE FORMAL POWER SERIES FOR LOG $e^x e^y$

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1. **Introduction.** That $\log e^x e^y$ can be expressed as a sum of x, y, and the commutators ($xy - yx$, and so on) of x and y was first shown by J. E. Campbell [2] in 1898, H. F. Baker [1] in 1905, and F. Hausdorff [7] in 1906. The expansion in terms of the commutators has been used in such fields as group theory [8] and differential equations [9].

To this day the determination of the coefficients of the commutators is difficult; only scattered results are available when the degree of the commutator is greater than six.

It might be useful, therefore, to investigate $\log e^x e^y$ as a formal power series in the non-commuting variables x and y, with the hope that the information gained in such an investigation will be of use in the problem of the commutator coefficients. An algorithm due to E. B. Dynkin [3] may prove useful for this purpose. See also D. Finkelstein [4] who gives an expression for $\log e^x e^y$, in terms of certain symbolic operators, which may be regarded either as a commutator or a power series expansion.

The major result of this paper is a formula for the coefficients in the formal power series stated in two different ways. Theorem 1 gives these coefficients in terms of certain fixed polynomials, and any specified coefficient may be computed easily from this form. Theorem 2 gives a generating function for these coefficients from which certain coefficients are obtained as a sum of Bernoulli numbers.

2. **Statement of the Theorems.** In a formal power series in non-commutative variables x and y the general term beginning with a power of x is

$$W_x \equiv W_x(s_1, s_2, \cdots, s_m) = x^{s_1} y^{s_2} \cdots (x \vee y)^{s_m} \tag{1}$$

where $s_1 s_2 \cdots s_m \neq 0$ and $(x \vee y)^{s_m}$ denotes x^{s_m} if m is odd and y^{s_m} if m is even. W_y is similarly defined.

In the formal power series for $\log e^x e^y$ we denote the coefficient of $W_x(s_1, \cdots, s_m)$ by $c_x \equiv c_x(s_1, \cdots, s_m)$, and that of W_y by c_y.

The main theorems of this paper can now be stated:

THEOREM 1.

$$c_x(s_1, \cdots, s_m) = (-1)^{n-1} c_y = \int_0^1 t^{m'}(t-1)^{m''} G_{s_1}(t) \cdots G_{s_m}(t)\, dt$$

where $n = \sum_{i=1}^m s_i$, $m' = [m/2]$, $m'' = [(m-1)/2]$ and the polynomials $G_s(t)$ are defined recursively by $G_1(t) = 1$ and $sG_s(t) = d/dt\, t(t-1)G_{s-1}(t)$ for $s = 2, 3, \cdots$. Also $c_x = c_y$ if m is odd.

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The polynomials $G_s(t)$ are related to the homogeneous "Euler Polynomials" $R_s(x, y)$ by the relation $s!\, G_s(t) = R_s(t, t-1)$. The properties of these polynomials have been developed at length by G. Frobenius [6] and J. Worpitzky [12].

THEOREM 2. *The generating function for c_x is given by*

$$\sum_{i=1}^{m} \sum_{s_i=1}^{\infty} c_x(s_1, \cdots, s_m) z_1^{s_1} \cdots z_m^{s_m} = \sum_{i=1}^{m} z_i e^{m' z_i} \prod_{j \neq i} (e^{z_i} - 1)(e^{z_i} - e^{z_j})^{-1}.$$

An interesting fact which these theorems make evident is that the coefficient $c_x(s_1, \cdots, s_m)$ is left invariant by any permutation of the s_i. Furthermore if m is odd $c_x = c_y$, and if n is even $c_x = -c_y$. Therefore $c_x(s_1, \cdots, s_m) = 0$ when m is odd and $\sum_{i=1}^{m} s_i$ is even.

To illustrate the consequences of these theorems we shall prove

THEOREM 3.

$$c_x(s_1, s_2) = \frac{(-1)^{s_1}}{s_1! s_2!} \sum_{i=1}^{s_2} \binom{s_2}{i} B_{s_1+s_2-i}$$

where B_k is the k^{th} Bernoulli number in the usual notation ($B_1 = -\frac{1}{2}$).

3. **A General Lemma.** The usual definitions hold for the formal power series of e^x and $\log(1+u)$ so that $\log e^x e^y = \sum_{q=1}^{\infty} (-1)^{q-1} u^q / q$ where $u = e^x e^y - 1 = \sum_i \sum_j x^i y^j / i! j!$ with $i, j = 0, 1, \cdots$ but $i + j \neq 0$.

To determine the coefficient c_x in $\log e^x e^y$ we must find the coefficient of W_x in $(e^x e^y - 1)^q$, multiply by $(-1)^{q-1}/q$, and sum over all positive integers q.

It is clear that the main problem is that of finding the coefficient of W_x in $(e^x e^y - 1)^q$, and we proceed to attack this problem in a somewhat generalized fashion.

To be precise we may consider all our operations performed in the free associative ring over the rationals generated by the non-commuting variables x and y.

Let

$$u = f(x)f(y) - 1 \quad \text{where} \quad f(x) = 1 + \sum_{i=1}^{\infty} a_i x^i, \qquad a_i \text{ rational}$$

and let $U_q(W_x)$ denote the coefficient of W_x (as defined in (1)) in u^q.

To determine $U_q(W_x)$ we shall find all possible decompositions of W_x into products of q elements of the type $x^i y^j$ with $i, j \geq 0$ but $i + j \neq 0$ and sum their contributions (products of $a_i a_j$) to $U_q(W_x)$. Clearly all products enter into the computation since all elements of this type are found in u.

Consider the general decomposition of W_x:

$$\begin{aligned} (2) \quad & W_x(s_1, \cdots, s_m) \\ & = x^{d_{11}} \cdots x^{d_{1,r_1-1}} \cdot w_1 \cdot y^{d_{22}} \cdots y^{d_{2,r_2}} \cdot x^{d_{31}} \cdots x^{d_{3,r_3-1}} \cdot w_2 \cdot y^{d_{42}} \cdots (x \vee y)^{d_{m,r_m}} \end{aligned}$$

where $d_{ij} \geq 1$ and w_i is either the element $x^{d_{ir_i}} y^{d_{i+1,1}}$ or the product of two elements: $x^{d_{ir_i}} \cdot y^{d_{i+1,1}}$.

If p of the w_i are single elements rather than the product of two elements then the total number of elements in the decomposition is

$$q = r_1 + \cdots + r_m - p$$

where $r_1, \cdots, r_m$ are the lengths of the partitions

$$s_i = d_{i1} + \cdots + d_{ir_i} \quad \text{with} \quad d_{ij} \geq 1, \quad \text{for} \quad i = 1, \cdots, m. \tag{3}$$

This decomposition gives a contribution of $\prod_{i=1}^{m} \prod_{j=1}^{r_i} a_{d_{ij}}$ to $U_q(W_x)$. The sum of such contributions over all decompositions of $W_x(s_1, \cdots, s_m)$ with a fixed q will yield $U_q(W_x)$ exactly.

We begin by fixing $r_1, \cdots, r_m$ and letting the d_{ij} range over all values such that (3) holds. Define

$$\alpha_s^{(r)} = \sum a_{d_1} \cdots a_{d_r} \tag{4}$$

summed over all partitions $s = d_1 + \cdots + d_r$ with $d_i \geq 1$.

Then $\prod_{i=1}^{m} \alpha_{s_i}^{(r_i)}$ is the total of all possible contributions to $U_q(W_x)$ of decompositions (2) with fixed $r_1, \cdots, r_m, p$ of the w_i single elements and $\sum r_i - p = q$.

Now fix p, define $r = p + q$, and allow $r_1, \cdots, r_m$ to range over all values such that

$$r = r_1 + \cdots + r_m \quad \text{with} \quad r_i \geq 1 \tag{5}$$

and define

$$A_r \equiv A_r(s_1, \cdots, s_m) = \sum \alpha_{s_1}^{(r_1)} \cdots \alpha_{s_m}^{(r_m)} \tag{6}$$

summed over all partitions $r = r_1 + \cdots + r_m$, $r_i \geq 1$.

This is the total contribution to $U_q(W_x)$ of all decompositions (2) restricted by (5) with a fixed number, $p = r - q$, of single elements among the w_i.

Finally we must find what values r can take and how many ways it can take each value.

In the first place since each d_{ij} is at least 1 we have

$$n = \sum_{i=1}^{m} s_i = \sum_{i=1}^{m} \sum_{j=1}^{r_i} d_{ij} \geq \sum_{i=1}^{m} r_i = r$$

with equality when $d_{ij} = 1$ for all i, j. Furthermore since each r_i is at least 1 we have

$$r = \sum_{i=1}^{m} r_i \geq m$$

with equality when $r_i = 1$ for all i.

On the other hand there are $m' = [m/2]$ of the w_i so that there are $\binom{m'}{p}$ different ways for there to be p single elements among the w_i. This means that $r = p + q$ in $\binom{m'}{r-q}$ distinct ways; or that A_r as defined in (6) contributes to $U_q(W_x)$ in $\binom{m'}{r-q}$ ways.

This completes our discussion of the determination of $U_q(W_x)$ and we sum up in

LEMMA 1.

$$U_q(W_x) = \sum_{r=m}^{n} \binom{m'}{r-q} A_r(s_1, \cdots, s_m)$$

where $m' = [m/2]$ *and* $A_r(s_1, \cdots, s_m)$ *is defined in* (6).

As an immediate corollary to Lemma 1 we have

COROLLARY 1.

$$U_q(W_y) = \sum_{r=m}^{n} \binom{m''}{r-q} A_r(s_1, \cdots, s_m)$$

where $m'' = [(m-1)/2]$, since our entire argument for the coefficient of W_x carries over for that of W_y except that in the decomposition of W_y there are only $[(m-1)/2]$ possible ambiguous elements.

Since $m' = m''$ when m is odd it follows that

COROLLARY 2. $U_q(W_x) = U_q(W_y)$ *if* m *is odd.*

From all of this it is clear that the arrangement of the s_i is immaterial to the coefficient of $W_x(s_1, \cdots, s_m)$ in $(f(x)f(y) - 1)^q$.

4. **Two More General Lemmas.** Lemma 1 is useful as an explicit starting point for the proof of our theorems, but it is not useful in itself since the function A_r is rarely known in closed form.

However the definition (6) of $A_r(s_1, \cdots, s_m)$ can be compared with the product of polynomials with coefficients $\alpha_{s_i}^{(r_i)}$ to obtain a generating function for $U_q(W_x)$ that will lead to our main theorems.

Define polynomials $G_s(t)$ by

$$G_s(t) = \sum_{j=1}^{s} (-1)^{s-j} \alpha_s^{(j)} t^{j-1}. \tag{7}$$

Since $\alpha_s^{(j)}$ as defined in (4) has the obvious generating function $\sum_{s=1}^{\infty} \alpha_s^{(j)} z^s = (f(z) - 1)^j$ we have as a generating function for $G_s(t)$

$$\sum_{s=1}^{\infty} G_s(t) z^s = \{(1 - f(-z))^{-1} - t\}^{-1} \tag{8}$$

which supplies an alternative definition for these polynomials. When defined in this manner the polynomials $G_s(t)$ are the derivatives of Faber polynomials. See M. Schiffer [10] and I. Schur [11] for the definitions and some properties of the Faber polynomials.

Using (7) the product of $G_{s_i}(t)$ for $i = 1, \cdots, m$ can be expressed in terms of $A_r(s_1, \cdots, s_m)$:

$$\prod_{=1}^{m} G_{s_i}(t) = \sum_{r=m}^{n} (-1)^{n-r} A_r(s_1, \cdots, s_m) t^{r-m} \quad \text{where} \quad n = \textstyle\sum_{i=1}^{m} s_i .$$

We shall express $\sum_{q=1}^{n} U_q(W_x)(-t)^{q-1}$ in terms of this product and then integrate with respect to t in the interval from 0 to 1 to get $\sum_{q=1}^{n}(-1)^{q-1}U_q(W_x)/q$. This is the coefficient, call it $\bar{c}_x(s_1, \cdots, s_m)$, of W_x in $\log f(x)f(y)$ since it is clear that $U_q(W_x) = 0$ for $q > n$.

Using the formula of Lemma 1 we get

$$\begin{aligned}
\sum_{q=1}^{n} U_q(W_x)(-t)^{q-1} &= \sum_{q=1}^{n}\sum_{r=m}^{n}\binom{m'}{r-q}A_r(s_1, \cdots, s_m)(-t)^{q-1} \\
&= \sum_{r=m}^{n} A_r(s_1, \cdots, s_m)(-t)^{r-1}\sum_{q=1}^{n}\binom{m'}{r-q}(-t)^{q-r} \\
&= \sum_{r=m}^{n} A_r(s_1, \cdots, s_m)(-t)^{r-1}(1-t^{-1})^{m'} \\
&= (-1)^{n-1}(t-1)^{m'}t^{m-m'-1}\sum_{r=m}^{n}(-1)^{n-r}A_r(s_1, \cdots, s_m)t^{r-m}.
\end{aligned}$$

The third step goes through if we change the index of summation in the sum at the far right from q to $q^* = r - q$ in which case the sum extends from $q^* = r - n \leq 0$ to $q^* = r - 1 \geq m - 1 \geq m'$ for $m \geq 1$. Clearly $m' \geq r - q \geq 0$ so that the sum actually extends from 0 to m' to give $(1 - t^{-1})^{m'}$.

Now if we integrate as indicated above we get

LEMMA 2.

$$\bar{c}_x(s_1, \cdots, s_m) = (-1)^{n-1}\int_0^1 (t-1)^{m'}t^{m-m'-1}G_{s_1}(t)\cdots G_{s_m}(t)\,dt$$

where $\bar{c}_x$ is the coefficient of W_x in $\log f(x)f(y)$, $n = \sum_{i=1}^{m}s_i$, $m' = [m/2]$, *and the polynomials $G_s(t)$ are defined by (7) or (8).*

Also (from the corollaries to Lemma 1) we have

$$(9)\qquad \bar{c}_y(s_1, \cdots, s_m) = (-1)^{n-1}\int_0^1 (t-1)^{m''}t^{m-m''-1}G_{s_1}(t)\cdots G_{s_m}(t)\,dt$$

where $\bar{c}_y$ is the coefficient of W_y in $\log f(x)f(y)$ *and* $m'' = [(m-1)/2]$; *and* $\bar{c}_x = \bar{c}_y$ *if m is odd.*

Later the generating function described in (8), with $f(z) = e^z$, will yield the properties of the polynomials $G_s(t)$ that will translate Lemma 2 into Theorem 1. However, for the moment, we wish to retain our generality to prove

LEMMA 3. *The generating function for the coefficient $\bar{c}_x$ of W_x in* $\log f(x)f(y)$ *is*

$$\sum_{i=1}^{m}\sum_{s_i=1}^{\infty}\bar{c}_x(s_1, \cdots, s_m)z_1^{s_1}\cdots z_m^{s_m}$$
$$= \sum_{i=1}^{m}\log f(z_i)(f(z_i))^{m'}\prod_{j\neq i}(f(z_j)-1)(f(z_i)-f(z_j))^{-1}.$$

We prove Lemma 3 from Lemma 2 by multiplying $\bar{c}_x(s_1, \cdots, s_m)$ by $(-1)^n$ $(-z_1)^{s_1} \cdots (-z_m)^{s_m}$ and performing the sum indicated by the lemma to obtain the product of the generating functions of the polynomials $G_s(t)$ under the integral of Lemma 2. The generating function for $\bar{c}_x$ is then

$$\int_0^1 -(t-1)^{m'} t^{m-m'-1}\{(1-f(z_1))^{-1}-t\}^{-1} \cdots \{(1-f(z_m))^{-1}-t\}^{-1}\, dt.$$

Note that $\int_0^1 \{(1-f(z))^{-1}-t\}^{-1}\, dt = -\log f(z)$ so that it will be simple to integrate this expression by partial fractions.

Let the integrand equal $-\sum_{i=1}^m D_i \{(1-f(z_i))^{-1}-t\}^{-1}$ where D_i is independent of t. Then the integral is $\sum_{i=1}^m D_i \log f(z_i)$.

To find D_i we begin by equating our two expressions for the integrand and multiplying both sides by the product of the generating functions for the polynomials $G_s(t)$ to get

$$\sum_{i=1}^m D_i \prod_{j \neq i} \{(1-f(z_j))^{-1}-t\} = (t-1)^{m'} t^{m-m'-1}.$$

If we let $t = (1-f(z_i))^{-1}$ this equation reduces to

$$D_i \prod_{j \neq i} \{(1-f(z_j))^{-1}-(1-f(z_i))^{-1}\} = (f(z_i))^{m'}(1-f(z_i))^{1-m}$$

or

$$D_i = (f(z_i))^{m'} \prod_{j \neq i} (1-f(z_j))(f(z_j)-f(z_i))^{-1}$$

which proves Lemma 3.

We can achieve even greater generality by exactly the same procedure if we wish the generating function of the coefficients in $\log f(x)f^*(y)$ where $f^*(0) = 1$. The only change that is necessary in the formula of Lemma 3 is to replace $f(z_{2i})$ by $f^*(z_{2i})$ for $i = 1, \cdots, m'$. However such an expansion of the scope of this lemma is not useful for obtaining the main results of this paper.

5. **Proofs of Theorem 1 and Theorem 2.** Consider the formula of Lemma 3. If we set $f(z) = e^z$, we get Theorem 2 directly.

To prove Theorem 1 consider equation (9) in Lemma 2. Since $m = m' + m'' + 1$ we have $(-1)^{n-1} c_y = \int_0^1 t^{m'} (t-1)^{m-m'-1} G_{s_1}(t) \cdots G_{s_m}(t)\, dt$. Therefore it is sufficient to show that $c_x = (-1)^{n-1} c_y$ and that $sG_s(t) = d/dt\, t(t-1) G_{s-1}(t)$ with $G_1(t) = 1$ when $f(z) = e^z$ since every other statement in Theorem 1 carries over from Lemma 2.

That $c_x = (-1)^{n-1} c_y$ is easily proved. Since $(e^x e^y)(e^{-y}e^{-x}) = (e^{-y}e^{-x})(e^x e^y) = 1$ we have

$$\log e^x e^y = -\log e^{-y} e^{-x}.$$

Consider the coefficient of W_x in the expressions on each side of this equation. On the left it is c_x. On the right it is $-(-1)^{s_1} \cdots (-1)^{s_m} c_y$ or $(-1)^{n-1} c_y$.

To prove the recursion for $G_s(t)$ consider (8) with $f(z) = e^z$. Clearly $G_1(t) = 1$ as we can show by dividing both sides by z and then letting z go to 0.

On the other hand first take the derivative with respect to z to get

$$\sum_{s=1}^{\infty} sG_s(t)z^{s-1} = e^{-z}(1 - e^{-z})^{-2}\{(1 - e^{-z})^{-1} - t\}^{-2}.$$

Secondly multiply both sides of the original equation by $t(t - 1)$ and take the derivative with respect to t to get

$$\sum_{s=1}^{\infty} \frac{d}{dt} t(t - 1)G_s(t)z^s = -1 + e^{-z}(1 - e^{-z})^{-2}\{(1 - e^{-z}) - t\}^{-2}.$$

Comparing these results we get

$$\sum_{s=1}^{\infty} sG_s(t)z^{s-1} = 1 + \sum_{s=2}^{\infty} \frac{d}{dt} t(t - 1)G_{s-1}(t)z^{s-1}$$

which proves the recursion formula and, therefore, Theorem 1.

6. **Proof of Theorem 3.** For $m = 2$ the generating function given in Theorem 2 is

$$\begin{aligned} \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} c_x(s_1, s_2)z^{s_1}z^{s_2} &= z_1e^{z_1}(e^{z_2} - 1)(e^{z_1} - e^{z_2})^{-1} \\ &\qquad + z_2e^{z_2}(e^{z_1} - 1)(e^{z_2} - e^{z_1})^{-1} \\ &= -z_2 + (e^{z_2} - 1)(z_2 - z_1)(e^{z_2 - z_1} - 1)^{-1}. \end{aligned} \tag{10}$$

The generating function of the Bernoulli numbers is [5; 57]

$$\sum_{i=0}^{\infty} B_i \frac{z^i}{i!} = z(e^z - 1)^{-1}$$

so that

$$\begin{aligned} (z_2 - z_1)(e^{z_2 - z_1} - 1)^{-1} &= \sum_{i=0}^{\infty} B_i \frac{(z_2 - z_1)^i}{i!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j}B_i \frac{z_1^{i-j}z_2^j}{i!}. \end{aligned}$$

When multiplied by $e^{z_2} - 1 = \sum_{k=1}^{\infty} z_2^k/k!$ this equation becomes

$$\begin{aligned} (e^{z_2} - 1)(z_2 - z_1)(e^{z_2 - z_1} - 1)^{-1} &= \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^{i-j}B_i \frac{z_1^{s_1}z_2^{s_2}}{j!k!(i - j)!} \\ &= \sum_{s_1=0}^{\infty} \sum_{s_2=1}^{\infty} \frac{(-1)^{s_1}}{s_1!s_2!} \sum_{k=1}^{s_2} \binom{s_2}{k}B_{s_1+s_2-k}z_1^{s_1}z_2^{s_2} \end{aligned} \tag{11}$$

where we have let $s_1 = i - j$ and $s_2 = k + j$ so that $j = s_2 - k$ and $i = s_1 + s_2 - k$ in the second step.

Comparing equations (10) and (11) we will get Theorem 3 if the right hand side of (11) when restricted by $s_1 = 0$ is equal to z_2.

Since

$$\sum_{k=1}^{s} \binom{s}{k} B_{s-k} = 0 \quad \text{for} \quad s > 1$$

(see [5; 28]) we have

$$\sum_{s_2=1}^{\infty} \sum_{k=1}^{s_2} \binom{s_2}{k} B_{s_2-k} \frac{z_2^{s_2}}{s_2!} = z_2$$

as we wished to show to complete the proof of Theorem 3.

This theorem is the principal example of the use of our main theorems that we have made to date. Other general formulas of some interest will be published elsewhere.

7. **Computation of the Coeffcients.** At times it is desirable to compute some coefficients directly. The form of Theorem 1 facilitates such computation.

First we make a list of the different powers (s_i) appearing among the elements whose coefficients are desired and compute the polynomials $G_{s_i}(t)$ corresponding to them. The computation of the polynomials can be made from the recursion formula since both polynomial multiplication by $t - 1$ and formal differentiation of a polynomial are simple operations to perform. An alternative method is to use a table of Stirling numbers of the second kind $S_s^{(j)}$ since these numbers are defined essentially in the same way as the $\alpha_s^{(j)}$ which are the coefficients in $G_s(t)$; they are related by $s!\, \alpha_s^{(j)} = j!\, S_s^{(j)}$.

Then for each coefficient $c_x(s_1, \cdots, s_m)$ we compute the polynomial $\prod_{i=1}^{m} G_{s_i}(t)$, multiply by $t^{m'}(t-1)^{m''}$ and integrate as indicated in Theorem 1 using the fact that $\int_0^1 t^p(1-t)^q dt = p!q!/(p+q+1)!$ when p and q are non-negative integers.

This process may be facilitated by considering $G_s(t)$ as a polynomial in $T = t(t-1)$ if s is odd or $t - \frac{1}{2}$ times a polynomial in T if s is even. We can show that this is possible directly from the generating function for $G_s(t)$:

$$\begin{aligned} 2 \sum_{s=1}^{\infty} G_{2s}(t) z^{2s} &= \{(1-e^{-z})^{-1} - t\}^{-1} + \{(1-e^{z})^{-1} - t\}^{-1} \\ &= (2t-1)\{e^z(e^z-1)^2 - t^2 + t\}^{-1} \end{aligned}$$

and

$$\begin{aligned} 2 \sum_{s=1}^{\infty} G_{2s-1}(t) z^{2s-1} &= \{(1-e^{-z})^{-1} - t\}^{-1} - \{(1-e^{z})^{-1} - t\}^{-1} \\ &= (e^z+1)(e^z-1)^{-1}\{e^z(e^z-1)^2 - t^2 + t\}^{-1}. \end{aligned}$$

Since $(t - \frac{1}{2})^2 = T + \frac{1}{4}$ and $t^{m'}(t-1)^{m''} = T^{m''} t^{m'-m''}$ where $m' - m''$ is either 0 or 1, there will be three types of integrals to evaluate in our calculations: (1) $\int_0^1 T^p dt = (-1)^p p!p!/(2p+1)!$; (2) $\int_0^1 tT^p dt = (-1)^p p!p!/2(2p+1)!$; and (3) $\int_0^1 t(t-\frac{1}{2})T^p dt = (-1)^p (p+1)!p!/2(2p+3)!$.

However the length or degree of the polynomials will be cut by a factor of two with a respective decrease in the difficulty of multiplying them together.

In this manner we calculated all the coefficients up to those for degree ten in less than three hours by hand.

For higher degrees it would be wise to use an electronic computer. The procedure we have described can be standardized and programmed for computer use; the chief difficulty is that computation with rationals is unavoidable until some idea of the factorization of the denominators of the coefficients is known. However for the small degrees, $n \leq 10$, all the denominators for the same degree n divide the denominator of $(B_{n-1} + B_{n-2})/n!$ and this may be the general case.

It may be of some interest that this paper is the result of generalizing the algorithms used in the original program written for the National Bureau of Standards Eastern Automatic Computer for computing the coefficients in the formal power series for $\log e^x e^y$.

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