The constructible reals can be (almost) anything

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Theorem There are models of ZFC in which the set of constructible reals is, respectively, exactly the following set of reals $\Delta^1_3, \Delta^1_4, \ldots, \Delta^1_\omega =$ projective, Δ^m_n, $1 \le n \le \omega$, $2 \le m \le \omega$.

We will give the proof for Δ^1_3 and then indicate a few of the modifications needed for the full result.

We will use the notation of [1]. Our model will be a substructure of a model constructed in that paper.

Working inside L, Jensen and Solovay in [1] construct a sequence of posets Q_i, $1 \le i < \omega$, and a Σ^1_3 predicate $\varphi(j)$ such that:

the Q_i's are homogeneous,

if Q_0 is the poset which collapses $\aleph_1^L$ to ω, and if $\bigoplus_{i\in\omega} G_i$ is $\bigoplus_{i\in\omega} Q_i$ generic over L, then for $j \geq 1$,

$$L[G_0, G_j] \models \varphi(j), \text{ but}$$

$$L[\bigoplus_{j\neq i} G_i] \models \neg\varphi(j).$$

Q_0 can be defined in various equivalent ways. We define it by: $\langle m, f\rangle \in Q_0$ iff $m \in \omega$, f a <u>partial</u> function from m to $\aleph_1$; $\langle m, f\rangle \leq \langle m', f'\rangle$ iff $m \leq n'$, $f' \restriction n = f$.

G_0 can be identified with a partial map from ω onto $\aleph_1^L$.

Let h_α be the αth constructible real, Code triples of integers as one positive integer and let $Z(n,\alpha) = \{\langle n, k, \ell\rangle : h_\alpha(k) = \ell\}$.

$$\text{Let } Z = \bigcup\{Z(n,\alpha) : n \in \text{dom of } G_0 \text{ and } G_0(n) = \alpha\}$$

$$\text{Let } N = L[G_0 \oplus (\bigoplus_{i\in Z} G_i)].$$

$N \models \varphi(j)$ iff $j \in Z$, so each h_α is Δ_3^1 in N. For the converse:

Let $\mathcal{P}(n,\alpha) = \bigoplus_{i \in Z(n,\alpha)} Q_i$.

Let $\mathcal{R}$ be the poset: $r \in \mathcal{R}$ iff $r = \langle m, f, g \rangle$, $\langle m, f \rangle \in Q_0$, domain of g = dom of f, for $n \in \operatorname{dom} f$ $g(n) \in \mathcal{P}(n, f(n))$.

$G_0 \oplus (\bigoplus_{i \in Z} G_i)$ is $\mathcal{R}$-generic over L.

Since the Q_i's are homogeneous, for φ a parameterless sentence of set theory, $\langle m, f, g \rangle \Vdash \varphi \implies \langle m, f, 0 \rangle \Vdash \varphi$.

For $r = \langle m, f, 0 \rangle$, $r' = \langle m', f', 0 \rangle$, let $r \subseteq r'$ mean: $f \subseteq f'$.

If $r \subseteq r'$ then $r' \Vdash$ "in N there is a generic filter on $\mathcal{R}$ which extends r".

So if θ is a Σ^1_3 sentence, and if $r \subseteq r'$, then $r \Vdash \theta \implies r' \Vdash \theta$.

Let $\theta_0(n)$, $\theta_1(n)$ be Σ^1_3 formulas. Let $r = \langle m, f, 0 \rangle \in \mathcal{R}$, and assume $r \Vdash$ "$\exists n\, (\theta_0(n) \vee \theta_1(n))$".

Suppose we have $n \ni r \Vdash \theta_0(n)$ and $r \Vdash \theta_1(n)$. (If there is no such n, then the Δ^1_3 real θ_0, θ_1 define – if they do define a real – must be constructible.)

Pick $r' = \langle m', f', 0 \rangle \geq r \ni r' \Vdash \theta_0(n)$. Let $r^0 = \langle m', f, 0 \rangle$. Since $r^0 \leq r$ we can find $\langle m'', f'', 0 \rangle = r'' \geq r^0 \ni r'' \Vdash \theta_1(n)$. Let $r''' = \langle m'', f' \cup f'', 0 \rangle$. Then $r''' \geq r$ and $r', r'' \leq r'''$, so $r''' \Vdash$ "$\theta_0(n) \wedge \theta_1(n)$". $\square$

The only real obstacle to generalizing this argument is that it used Shoenfield Absoluteness (Σ^1_3 goes up). This is not an insuperable obstacle since Shoen. Abs. was only used between N and N with a few of the G_i's deleted.

To get Δ^1_{n+3}: by examining [1] we see that $Q_i = Q(X_i)$ where $X_i \subseteq \aleph_2$ and Q is a canonical procedure for constructing posets from subsets of $\aleph_2$.

Let X'_i, $i<\aleph_2$ be a sequence of subsets of $\aleph_2$ which is Δ_{n+1} definable and $\underset{\sim}{\Sigma}_n$ generic over $L_{\aleph_2}$. (By generic here we mean wrt the $\aleph_1$-closed poset which adds subsets of $\aleph_2$ – i.e a Cohen subset of $\aleph_2$.)

Let $Q'_i = Q(X'_i)$. Define R' as above and let N be generic over L via the poset $R' \oplus (\bigoplus_{\omega\le i<\aleph_2} Q'_i)$.

The Q'_i's, $i\ge\omega$, obscure things enough so that a submodel of N, gotten by deleting a few Q'_i's, $i<\omega$, from the above poset, will be a Σ^1_{n+3} correct substructure of N.

To get Δ^1_ω, just diagonalize the above arguments. This involves finding $X_{i,n}$, $i<\aleph_2$, $n<\omega$, $X_i\subseteq\aleph_2$, $\ni$

For each $m<\omega$. $\langle X_{i,m}\rangle_{i<\aleph_2}$ is

Δ_{m+1} definable over $L_{\aleph_2}$, and $\langle X_{i,n} \rangle_{i<\aleph_2,\, n \geq m}$ is $\underset{\sim}{\Sigma}_m$ generic over $L_{\aleph_2}[\langle X_{i,n} \rangle_{i<\aleph_2,\, n<m}]$.

Such a sequence can be found.

The Δ_n^m's follow by a straightforward generalization, except for the Δ_1^m's. Here a different argument is needed – use the forcing which adds on a κ-closed unbounded subset to a stationary subset of κ^+ (where every ordinal in the stationary set has cofinality κ). This takes place over $L[G_0]$, where $\kappa = \aleph_{m-2}$.

[1] Jensen, Solovay, Some applications of almost disjoint sets, in: Math. Logic and Found. of Set Theory, Y. Bar-Hillel, ed.

<u>Addendum</u>: Models where Separation principles fail.

Kechris has observed that in the above model (for Δ^1_n), $PWO(\Sigma^1_n)$ and $PWO(\Pi^1_n)$ both fail (since $(\omega_\omega)^L \in \Sigma^1_2$).

By decending to §4 of [1] we can produce, for $n \geq 3$, models in which $Sep(\underset{\sim}{\Sigma}^1_n)$ and $Sep(\underset{\sim}{\Pi}^1_n)$ both fail. In fact we get models of ZFC in which:

there are two disjoint lightface Σ^1_n sets of reals which cannot be separated by a $\underset{\sim}{\Delta}^1_n$ set of reals, and also there are 2 disjoint lightface Π^1_n sets of reals which cannot be separated by a $\underset{\sim}{\Delta}^1_n$ set of reals.

This is done as follows. Start with L. Break $(\omega_\omega)^L$ into 2 parts. Generically split (via the usual ω-closed posets) each part into 3 pieces. Then make (using §4 of [1] plus the modifications sketched above) the 1st 2 pieces of the 1st part Σ^1_n, and make the 1st 2 pieces of the 2nd part Π^1_n. It is possible to show that any $\underset{\sim}{\Delta}^1_n$ subset of $({}^\omega\omega)^L$ in this model is constructible from a real; and also that all 6 of the above pieces are generic over each real in this model.

By diagonalizing, as was done above for Δ^1_ω, one can show that there is a model in which the above failures of separation occur for all $n \geq 3$ simultaneously. In fact (we believe) there is a model of ZFC in which separation fails for all of the following at once:

$$\Sigma^1_n, \Pi^1_n, \ 3 \leq n < \omega, \quad \Sigma^m_n, \Pi^m_n, \ 1 \leq n < \omega, 2 \leq m < \omega$$