the limiting value of the integral of $g(\xi-\eta)\, N_1'(\xi)$ approaches zero. Hence Theorem II[1] is a simple special case of Theorem III.

References.

Bochner and Hardy, 1, "Notes on two theorems of Norbert Wiener", *Journal London Math. Soc.*, 1 (1926), 240–4.

N. Wiener, 1, "On a theorem of Bochner and Hardy", *Journal London Math. Soc.*, 2 (1927), 118–123.

N. Wiener, 2, "A new method in Tauberian theorems", *Journal of Math. and Phys., Mass. Inst. Tech.*, 7 (1928), 161–184.

M. Jacob, 1, "Über ein Theorem von Bochner-Hardy-Wiener", *Journal London Math. Soc.*, 3 (1928), 182–7.

M. Plancherel, 1, "Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies", *Rend. di Palermo*, 30 (1910), 289–335.

THEOREMS STATED BY RAMANUJAN (IX): TWO CONTINUED FRACTIONS

G. N. WATSON*.

In the previous papers of this series, Mr. Preece and I have investigated the theorems enunciated in Ramanujan's first letter to Hardy; we now intend to examine the theorems numbered (1)–(23) in his second letter, which was dated 27 February, 1913. These theorems are not classified like the theorems in his first letter, and, in order to make subsequent papers less fragmentary, it seems desirable to take the theorems in an order different from that in which they were originally stated. In the present paper, I discuss the theorems numbered (1) and (6), which are to be found on page xxviii of the *Collected Papers*. The former theorem is, in effect, a collection of corollaries of results which have already been discussed in Paper VII of this series†.

(1)

If $$F(x) = \frac{1}{1}+\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\frac{x^4}{1}+\frac{x^5}{1}+\dots,$$

then $$\left\{\frac{\sqrt{5}+1}{2}+e^{-2\alpha/5}F(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2}+e^{-2\beta/5}F(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2}$$

with the conditions $\alpha\beta=\pi^2$, ..., *e.g.*

$$\frac{1}{1}+\frac{e^{-2\pi\sqrt{5}}}{1}+\frac{e^{-4\pi\sqrt{5}}}{1}+\dots = e^{2\pi/\sqrt{5}}\left\{\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}}}-\frac{\sqrt{5}+1}{2}\right\}.$$

* Received 6 October, 1928; read 8 November, 1928.

† *Journal London Math. Soc.*, 4 (1929), 39–48.

*The above theorem is a particular case of a theorem ...**.

To establish the first part of this result, we use one of the lemmas required in proving Theorem (7) in Paper VII, namely that $e^{-2a/5}F(e^{-2a})$ is a root of the quadratic equation

$$\frac{1}{\mathbf{A}}-1-\mathbf{A}=\left(\frac{k_1k_1'K_1^3}{k_2k_2'K_2^3}\right)^{\frac{1}{4}},$$

where k_1 and k_2 are the moduli of elliptic functions whose quarter-periods satisfy the relations

$$\pi K_1'/K_1=\tfrac{1}{5}a,\quad \pi K_2'/K_2=5a.$$

Since $$\pi K_2/K_2'=\tfrac{1}{5}\beta,\quad \pi K_1/K_1'=5\beta,$$

in consequence of the relation $a\beta=\pi^2$, it is evident that $e^{-2\beta/5}F(e^{-2\beta})$ is a root of the quadratic equation

$$\frac{1}{\mathbf{B}}-1-\mathbf{B}=\left(\frac{k_2'k_2K_2'^3}{k_1'k_1K_1'^3}\right)^{\frac{1}{4}}.$$

It follows that

$$\left(\frac{1}{\mathbf{A}}-1-\mathbf{A}\right)\left(\frac{1}{\mathbf{B}}-1-\mathbf{B}\right)=\left(\frac{K_1K_2'}{K_1'K_2}\right)^{\frac{1}{2}}=\left(\frac{5\beta}{\pi}\cdot\frac{5a}{\pi}\right)^{\frac{1}{2}}=5,$$

and so $\mathbf{A}$ and $\mathbf{B}$ are connected by the quadri-quadric relation

$$(\mathbf{A}^2+\mathbf{A}-1)(\mathbf{B}^2+\mathbf{B}-1)=5\mathbf{AB}.$$

This relation may be rewritten in the form

$$\{\mathbf{AB}+\tfrac{1}{2}(\mathbf{A}+\mathbf{B})-1\}^2=\tfrac{5}{4}(\mathbf{A}+\mathbf{B})^2.$$

Now $\mathbf{A}$ and $\mathbf{B}$ are both positive when a is positive; and then $\mathbf{A}^2+\mathbf{A}-1$ and $\mathbf{B}^2+\mathbf{B}-1$ are both negative, so that $\mathbf{A}$ and $\mathbf{B}$ are less than $\tfrac{1}{2}(\sqrt{5}-1)$, and therefore $\mathbf{AB}+\tfrac{1}{2}(\mathbf{A}+\mathbf{B})-1$ is negative. Consequently, on taking the square root of each side of the last equation, we have

$$\mathbf{AB}+\tfrac{1}{2}(\mathbf{A}+\mathbf{B})-1=-\tfrac{1}{2}(\mathbf{A}+\mathbf{B})\sqrt{5}.$$

From this equation the homographic relation

$$(2\mathbf{A}+1+\sqrt{5})(2\mathbf{B}+1+\sqrt{5})=10+2\sqrt{5}$$

follows at once, and this establishes the first part of the theorem.

* Ramanujan proceeds to indicate the existence of theorems, which he does not enunciate, concerning more general types of continued fractions. The investigation of such theorems is a more substantial piece of work than would be appropriate here, and I accordingly withhold it until some future occasion.

The second part of the theorem is a little more recondite; we need the formulae, proved in connection with Theorem (7) of Paper VII,

$$\frac{1}{\mathbf{A}^5}-11-\mathbf{A}^5=\frac{k_0k_0'K_0^3}{k_2k_2'K_2^3},\quad \frac{1}{\mathbf{B}^5}-11-\mathbf{B}^5=\frac{k_0'k_0K_0'^3}{k_1'k_1K_1'^3},$$

where k_0 is the modulus of elliptic functions whose quarter-periods satisfy the relation

$$\pi K_0'/K_0=\alpha.$$

It follows from these formulae that

$$\frac{\mathbf{B}^{-5}-11-\mathbf{B}^5}{\mathbf{A}^{-5}-11-\mathbf{A}^5}=\left(\frac{K_1K_0'}{K_0K_1'}\right)^3\frac{1}{(\mathbf{A}^{-1}-1-\mathbf{A})^6}=\frac{125}{(\mathbf{A}^{-1}-1-\mathbf{A})^6},$$

a result which could have been deduced by elementary algebra from the quadri-quadric relation, with much labour.

Now take $\alpha=\pi\sqrt{5}$ throughout the work, so that $\alpha=5\beta$; it then follows from Theorem (4) of Paper VII that

$$\mathbf{B}^{-5}-11-\mathbf{B}^5=\frac{(\mathbf{A}^{-1}-1-\mathbf{A})^6}{\mathbf{A}^{-5}-11-\mathbf{A}^5}.$$

If we eliminate $\mathbf{A}$ between this equation and the preceding equation, we see that, *when α has the special value $\pi\sqrt{5}$*, $\mathbf{B}$ satisfies the equation

$$(\mathbf{B}^{-5}-11-\mathbf{B}^5)^2=125,$$

so that then
$$\mathbf{B}^{-5}-11-\mathbf{B}^5=5\sqrt{5},$$

and hence
$$\begin{aligned}2\mathbf{B}^5&=-(11+5\sqrt{5})+\sqrt{(250+110\sqrt{5})}\\&=-\frac{(1+\sqrt{5})^5}{16}+\frac{5^{\frac{3}{4}}(1+\sqrt{5})^{\frac{5}{2}}}{2\sqrt{2}}\\&=\frac{(1+\sqrt{5})^5}{16}\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}.\end{aligned}$$

Consequently
$$\mathbf{B}=\frac{1+\sqrt{5}}{2}\sqrt[5]{\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}},$$

and therefore, from the homographic relation (the first part of the present theorem), it follows that

$$\mathbf{A}=\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right\}}}-\frac{1+\sqrt{5}}{2},$$

when $\alpha=\pi\sqrt{5}$; and this is the second part of the theorem.

The second type of continued fraction, which I shall now discuss, has some features in common with the first type, and in several respects it is even more fascinating. Ramanujan's enunciation of his theorems concerning it is as follows :—

(6)

If $$v = \frac{x}{1} + \frac{x^3+x^6}{1} + \frac{x^6+x^{12}}{1} + \frac{x^9+x^{18}}{1} + \ldots,$$

then $$\text{(i)}\quad x\left(1+\frac{1}{v}\right) = \frac{1+x+x^3+x^6+x^{10}+\ldots}{1+x^9+x^{27}+x^{54}+x^{90}+\ldots},$$

$$\text{(ii)*}\quad x^3\left(1+\frac{1}{v^3}\right) = \left(\frac{1+x^3+x^9+x^{18}+x^{30}+\ldots}{1+x^9+x^{27}+x^{54}+x^{90}+\ldots}\right)^4.$$

The material which I shall require to prove these results consists of the formula due to Gauss†

$$\prod_{n=1}^{\infty} [(1+q^{2n})^2(1-q^{2n})] = 1+\sum_{n=1}^{\infty} q^{n^2+n},$$

together with my generalization‡ of the Rogers-Ramanujan identities, that, *if*

$${}_r\Phi_s\left[\begin{matrix}\alpha, \beta, \gamma, \ldots; x\\ \delta, \epsilon, \ldots;\end{matrix}\right] = 1+\sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\left\{\frac{(1-\alpha q^m)(1-\beta q^m)(1-\gamma q^m)\ldots}{(1-q^{m+1})(1-\delta q^m)(1-\epsilon q^m)\ldots}\right\}x^n,$$

where r is the number of the symbols $\alpha, \beta, \gamma, \ldots$, *and s is the number of the symbols* $\delta, \epsilon, \ldots$, *then*

$${}_8\Phi_7\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & c, & d, & e, & f, & g; & \frac{a^2q^2}{cdefg}\\ & \sqrt{a}, & -\sqrt{a}, & aq/c, & aq/d, & aq/e, & aq/f, & aq/g; & \end{matrix}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{\{1-aq^n\}\{1-aq^n/(fg)\}\{1-aq^n/(ge)\}\{1-aq^n/(ef)\}}{\{1-aq^n/e\}\{1-aq^n/f\}\{1-aq^n/g\}\{1-aq^n/(efg)\}}\right]$$

$$\times\, {}_4\Phi_3\left[\begin{matrix}aq/(cd), & e, & f, & g; & q\\ & efg/a, & aq/c, & aq/d; & \end{matrix}\right]$$

provided that e, f *or* g *is of the form* q^{-N}, *where* $N = 1, 2, 3, \ldots$.

We now consider the continued fraction

$$1+\frac{aq+a^2q^2}{1}+\frac{aq^2+a^2q^4}{1}+\frac{aq^3+a^2q^6}{1}+\ldots,$$

* This formula, though given correctly in Ramanujan's note-books, has been miscopied in the various places where it has been published by replacing x throughout the right-hand side by $\sqrt[3]{x}$.

† *Comm. soc. reg. sci. Gottingensis rec.*, 1 (1811), 7–12.

‡ *Journal London Math. Soc.*, 4 (1929), 4–9.

in which $|q|<1$. If, *qua* function of a, it is expressible in the form $G(a)/G(aq)$, then $G(a)$ must satisfy the functional equation

$$\frac{G(a)}{G(aq)} = 1 + \frac{aq+a^2q^2}{G(aq)/G(aq^2)},$$

so that
$$G(a) = G(aq)+aq(1+aq)\,G(aq^2);$$

and if now $H(a)$ is defined by the formula

$$G(a) = H(a)\prod_{n=0}^{\infty}(1+aq^n),$$

then $H(a)$ must satisfy the functional equation

$$(1+a)\,H(a) = H(aq)+aqH(aq^2).$$

It is easy to verify that a solution of this last equation, proceeding in ascending powers of a, is

$$H(a) = 1+\sum_{n=1}^{\infty}(-)^n\frac{(1-q)(1-q^3)(1-q^5)\dots(1-q^{2n-1})}{(1-q)(1-q^2)(1-q^3)\dots(1-q^n)}a^n;$$

and this series converges when $|a|<1$.

If we adopt this series as the *definition* of $H(a)$, and retrace our steps, we find that, when $H(a)$ is so defined, then

$$\frac{(1+a)\,H(a)}{H(aq)} = 1+\frac{aq+a^2q^2}{(1+aq)\,H(aq)/H(aq^2)}.$$

By repeated substitutions it is now evident that

$$\frac{(1+a)\,H(a)}{H(aq)}$$
$$= 1+\frac{aq+a^2q^2}{1}+\frac{aq^2+a^2q^4}{1}+\dots+\frac{aq^n+a^2q^{2n}}{1}+\frac{aq^{n+1}}{H(aq^{n+1})/H(aq^{n+2})}.$$

Now, when $n\to\infty$, the nature of the convergence of this continued fraction is such that its last element may be omitted without affecting the value of its limit, since

$$\lim_{n\to\infty}\frac{aq^{n+1}}{H(aq^{n+1})/H(aq^{n+2})} = 0.$$

We have therefore proved that, when $|a|<1$,

$$1+\frac{aq+a^2q^2}{1}+\frac{aq^2+a^2q^4}{1}+\frac{aq^3+a^2q^6}{1}+\dots = \frac{(1+a)\,H(a)}{H(aq)},$$

where
$$H(a) = 1+\sum_{n=1}^{\infty}(-)^n\frac{(1-q)(1-q^3)(1-q^5)\dots(1-q^{2n-1})}{(1-q)(1-q^2)(1-q^3)\dots(1-q^n)}a^n.$$

We have next to transform $H(a)$. In my identity write

$$e=\sqrt{q}, \quad f=-\sqrt{q}, \quad g=q^{-N},$$

and make c, d and N tend to infinity, N, of course, passing through integral values only. We thus get

$$1+\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\dots(1-aq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$\times\frac{1-aq^{2n}}{1-a}\,\frac{(1-q)(1-q^3)\dots(1-q^{2n-1})}{(1-a^2q)(1-a^2q^3)\dots(1-a^2q^{2n-1})}\,q^{\frac{1}{2}n(3n-1)}a^{2n}$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1+aq^{n-1})}{(1-a^2q^{2n-1})}\right]H(a).$$

Consequently, when $|a|<1$,

$$1+\frac{aq+a^2q^2}{1}+\frac{aq^2+a^2q^4}{1}+\frac{aq^3+a^2q^6}{1}+\dots$$

$$=\frac{1-a^2q}{1-aq}\cdot\frac{1+\sum\limits_{n=1}^{\infty}\prod\limits_{m=1}^{n}\left[\frac{(1-aq^m)(1-q^{2m-1})}{(1-q^m)(1-a^2q^{2m-1})}\right]\frac{1-aq^{2n}}{1-aq^n}\,q^{\frac{1}{2}n(3n-1)}a^{2n}}{1+\sum\limits_{n=1}^{\infty}\prod\limits_{m=1}^{n}\left[\frac{(1-aq^{m+1})(1-q^{2m-1})}{(1-q^m)(1-a^2q^{2m+1})}\right]\frac{1-aq^{2n+1}}{1-aq^{n+1}}\,q^{\frac{1}{2}n(3n+3)}a^{2n}}.$$

In this result make $a\to 1$; there is now no theoretical difficulty in this passage to the limit, and we see at once that

$$1+\frac{q+q^2}{1}+\frac{q^2+q^4}{1}+\frac{q^3+q^6}{1}+\dots=\frac{1+\sum\limits_{n=1}^{\infty}(1+q^n)\,q^{\frac{1}{2}n(3n-1)}}{1+\sum\limits_{n=1}^{\infty}q^{\frac{1}{2}n(3n+3)}}.$$

If we now change to Ramanujan's notation by writing x^3 in place of q, we get

$$x\left(1+\frac{1}{v}\right)=\frac{1+x+\sum\limits_{n=1}^{\infty}(1+x^{3n})\,x^{\frac{3}{2}n(3n-1)}+\sum\limits_{n=1}^{\infty}x^{\frac{9}{2}n(n+1)+1}}{1+\sum\limits_{n=1}^{\infty}x^{\frac{9}{2}n(n+1)}}$$

$$=\frac{1+x+\sum\limits_{n=1}^{\infty}\left\{x^{\frac{1}{2}(3n-1)3n}+x^{\frac{1}{2}\cdot 3n(3n+1)}+x^{\frac{1}{2}(3n+1)(3n+2)}\right\}}{1+\sum\limits_{n=1}^{\infty}x^{\frac{9}{2}n(n+1)}}$$

$$=\frac{1+\sum\limits_{n=1}^{\infty}x^{\frac{1}{2}n(n+1)}}{1+\sum\limits_{n=1}^{\infty}x^{\frac{9}{2}n(n+1)}},$$

and this is Ramanujan's result (i).

To obtain (ii), express the fraction on the right as a product by using the formula ascribed to Gauss, so that

$$x\left(1+\frac{1}{v}\right) = \prod_{n=1}^{\infty}\left[\frac{(1+x^n)^2(1-x^n)}{(1+x^{9n})^2(1-x^{9n})}\right].$$

Now write x, ωx, $\omega^2 x$ in turn for x (and consequently v, ωv, $\omega^2 v$ for v), where ω is a complex cube root of unity, and multiply together the three equations so obtained. Since

$$(1\pm x^n)(1\pm\omega^n x^n)(1\pm\omega^{2n}x^n) = (1\pm x^n)^3 \quad \text{or} \quad (1\pm x^{3n}),$$

according as n is or is not a multiple of 3, we deduce that

$$x^3\left(1+\frac{1}{v^3}\right) = \prod_{n=1}^{\infty}\left[\frac{(1+x^{3n})^2(1-x^{3n})}{(1+x^{9n})^2(1-x^{9n})}\right]^4.$$

Transforming these products back into series by the same formula, we find immediately that

$$x^3\left(1+\frac{1}{v^3}\right) = \left(\frac{1+\sum\limits_{n=1}^{\infty} x^{\frac{3}{2}n(n+1)}}{1+\sum\limits_{n=1}^{\infty} x^{\frac{9}{2}n(n+1)}}\right)^4,$$

and this is Ramanujan's result (ii).

In conclusion, I might remark that I have no idea as to how Ramanujan carried out in his work the step equivalent to the transformation of $H(a)$; in view of the fact that he had discovered the Rogers-Ramanujan identities, but at that time had no proof of them, and of the apparent necessity of effecting this transformation of $H(a)$, it seems quite conceivable that he had discovered a limiting form of my identity (perhaps with $g = q^{-N}$, and N made infinite) which cannot be proved in any simple way except by deducing it from a form which is not limiting. It is possible also that he had discovered some different way of transforming $H(a)$; for example, it is possible to transform $H(q)$ and $H(q^2)$ by applying to the function

$$ {}_2\Phi_1\begin{pmatrix} b,\ c;\ a \\ 0; \end{pmatrix} \prod_{n=0}^{\infty}\frac{1}{(1-bq^n)(1-cq^n)}$$

the methods used by Rogers in his memoir "On a three-fold symmetry in the elements of Heine's series"* (whereby a transformation which is a limiting form of Rogers's transformation is obtained), and then writing $b=\sqrt{q}$, $c=-\sqrt{q}$, $a=q$, q^2. I do not give any details of this procedure because it seems less direct than the method which I have actually given for transforming $H(a)$ for general values of a.

---

* *Proc. London Math. Soc.* (1), 24 (1893), 171–179.