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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

On Ramanujan's continued fraction

by

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1. In his first letter to Hardy written in 1913, Ramanujan stated a number of results concerning the continued fraction

$$R(q) = \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\cdots, \quad |q| < 1. \tag{1}$$

In particular he evaluated $R(q)$ for a number of values of q. These were subsequently proved by Watson ([15], [16]). In his Notebooks and more particularly in the so-called "Lost" notebook, Ramanujan gave a large number of results about $R(q)$ some of which have been discussed recently by G. E. Andrews ([1], [2]). There are also a number of results stated in the unpublished papers to be found in the Trinity College [7].

Our object in this note is to show that essentially there are two fundamental results, given as Theorems 1 and 3 below, from which many of the results stated by Ramanujan on the values of $R(q)$ follow. The formulae required for proving these are also to be found in the manuscripts.

Ramanujan gives other continued fractions which have been encountered, independently, by Atle Selberg [12]. These are special cases of a general continued fraction of Ramanujan's, namely:

$$\frac{1}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\cdots \tag{2}$$

which has been discussed by Andrews [1]. Special cases of (2) are to be found also in the Notebooks. In the Trinity papers Ramanujan states:

$$\frac{e^{-\frac{\pi}{3}\sqrt{10}}}{1+}\,\frac{e^{-\pi\sqrt{10}}+e^{-2\pi\sqrt{10}}}{1+}\,\frac{e^{-2\pi\sqrt{10}}+e^{-4\pi\sqrt{10}}}{1+}\cdots = \frac{\sqrt{9+3\sqrt{6}}-\sqrt{7+3\sqrt{6}}}{(1+\sqrt{5})\sqrt{\sqrt{5}+\sqrt{6}}}. \tag{3}$$

We make some remarks on (2) and prove (3). We shall deal with other results of Ramanujan on continued fractions in a subsequent note.

2. Let $\tau = \xi + i\eta$, $\eta > 0$, be a parameter in the upper half τ-plane and let

$$q = e^{\pi i\tau}$$

so that $|q| < 1$. The continued fraction

$$R(q) = \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\cdots \tag{4}$$

converges. This continued fraction was encountered for the first time by L. J. Rogers in his beautiful work [10]. Ramanujan, who rediscovered it before 1913 ([8], page 204), seems to be the first mathematician to recognize and appreciate its importance; he further applied the methods of complex multiplication to it and stated many interesting results relating to it. It is known that

$$R(q^2) = A(2\tau) = q^{2/5}\,\frac{\prod_1^\infty (1-q^{2(5n-1)})(1-q^{2(5n-4)})}{\prod_1^\infty (1-q^{2(5n-2)})(1-q^{2(5n-3)})} \tag{5}$$

and that further

$$\frac{1}{A(2\tau)} - 1 - A(2\tau) = \frac{\eta(\tau/5)}{\eta(5\tau)}, \tag{6}$$

$$\frac{1}{(A(2\tau))^5} - 11 - (A(2\tau))^5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6, \tag{7}$$

where $\eta(\tau)$ is Dedekind's modular form

$$\eta(\tau) = e^{\pi i\tau/12}\prod_1^\infty (1-e^{2n\pi i\tau}). \tag{8}$$

The result (5) was stated by Ramanujan in the Notebooks [8] and (6) and (7) were proved by him in a manuscript which is unpublished. Ramanujan's proofs of (6) and (7) were published by Watson [15] (see also [5]). The modular form $\eta(\tau)$ satisfies the functional equation

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{9}$$

where $(-i\tau)^{1/2}$ is that branch which is positive for purely imaginary $\tau = i\eta$, $\eta > 0$. It is interesting to note that formula (9) is to be found in Ramanujan's Notebooks ([8], p. 199, 27(iii)), in his own notation.

Let us put

$$\tau_0 = \tfrac{1}{2}(5+\tau).$$

Then

$$A(2\tau_0) = R(e^{\pi i(5+\tau)}) = -\left(\frac{q^{1/5}}{1-}\,\frac{q}{1+}\,\frac{q^2}{1-}\,\frac{q^3}{1+}\cdots\right). \tag{10}$$

If we put

$$\bar{A}(\tau) = -A(2\tau_0) = \frac{q^{1/5}}{1-}\,\frac{q}{1+}\,\frac{q^2}{1-}\,\frac{q^3}{1+}\cdots,$$

then we have from (6) and (7)

$$\frac{1}{\bar{A}(\tau)} + 1 - \bar{A}(\tau) = \frac{\eta(\tau/5)}{\eta(5\tau)}\cdot\frac{f(\tau/5)}{f(5\tau)}, \tag{11}$$

$$\frac{1}{(\bar{A}(\tau))^5} + 11 - (\bar{A}(\tau))^5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6\left(\frac{f(\tau)}{f(5\tau)}\right)^6 \tag{12}$$

where $f(\tau)$ is Schläfli's modular function

$$f(\tau) = e^{-\pi i/24}\,\frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}. \tag{13}$$

For $\alpha > 0$, $f(i\alpha)$ is substantially the same as Ramanujan's $G(\alpha)$ ([8], p. 294). The function $f(\tau)$ satisfies

$$f(-1/\tau) = f(\tau). \tag{14}$$

(See [18], p. 115.) The formulae (6), (7), (11) and (12) are important in the sequel.

3. Let us take $\tau = i\alpha$, $\alpha > 0$ so that $A(2i\alpha)$ and $\eta(i\alpha/5)/\eta(i5\alpha)$ are both real and positive. If we put

$$A = A(2i\alpha),$$

then

$$\left(\frac{\sqrt{5}-1}{2} - A\right)\left(\frac{\sqrt{5}+1}{2} + A\right) = \frac{\eta(i\alpha/5)}{\eta(5i\alpha)}\cdot A. \tag{15}$$

Notice that (15) is same as (6). Furthermore we have from (15)

$$0 < A(2i\alpha) < (\sqrt{5}-1)/2. \tag{16}$$

Let us now choose two positive real numbers α and β such that

$$\alpha\beta = 1. \tag{17}$$

If we put $B = A(2i\beta)$, we get from (15)

$$\left(\frac{\sqrt{5}-1}{2} - A\right)\left(\frac{\sqrt{5}-1}{2} - B\right)\left(\frac{\sqrt{5}+1}{2} + A\right)\left(\frac{\sqrt{5}+1}{2} + B\right) = \frac{\eta(i\alpha/5)\cdot\eta(i\beta/5)}{\eta(i5\alpha)\cdot\eta(i5\beta)}\cdot AB.$$

Using the functional equation (9) and (17) we get

$$(18)\quad \left(\frac{\sqrt{5}-1}{2}-A\right)\left(\frac{\sqrt{5}-1}{2}-B\right)\left(\frac{\sqrt{5}+1}{2}+A\right)\left(\frac{\sqrt{5}+1}{2}+B\right)=5AB.$$

If we set

$$a=A+\frac{\sqrt{5}+1}{2},\qquad b=B+\frac{\sqrt{5}+1}{2}$$

then (18) reduces to

$$ab\,(a-\sqrt{5})(b-\sqrt{5})=5\left(ab-\frac{\sqrt{5}+1}{2}\,(a+b)+\left(\frac{\sqrt{5}+1}{2}\right)^2\right);$$

and this leads to the equation

$$(19)\quad \left(ab-\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}\right)^2=\sqrt{5}\left(ab-\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}\right)\left(a+b-2\,\frac{\sqrt{5}+1}{2}\right).$$

We assert that

$$(20)\quad ab-\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}=0.$$

For, if not from (19) we get

$$ab-\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}=\sqrt{5}(a+b-\sqrt{5}-1)$$

which gives

$$(21)\quad \left(\frac{\sqrt{5}-1}{2}-A\right)\left(\frac{\sqrt{5}-1}{2}-B\right)=\sqrt{5}\cdot\frac{\sqrt{5}-1}{2}.$$

By (16) the left side is $<(\sqrt{5}-1/2)^2<(\sqrt{5}-1)/2$ whereas the right side of (21) is $>(\sqrt{5}-1)/2$. This contradiction proves (20).

We have thus the first of Ramanujan's results:

THEOREM 1. *If α and β are positive and $\alpha\beta=1$, then*

$$\left\{\frac{\sqrt{5}+1}{2}+\frac{e^{-2\pi\alpha/5}}{1+}\,\frac{e^{-2\pi\alpha}}{1+}\,\frac{e^{-4\pi\alpha}}{1+}\cdots\right\}\left\{\frac{\sqrt{5}+1}{2}+\frac{e^{-2\pi\beta/5}}{1+}\,\frac{e^{-2\pi\beta}}{1+}\,\frac{e^{-4\pi\beta}}{1+}\cdots\right\}=\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}.$$

A similar result can be obtained with $\bar{A}=A(2\tau_0)$. Let us put

$$\bar{A}=\bar{A}(i\alpha),\quad \alpha>0;\qquad \bar{B}=\bar{A}(i\beta),\quad \beta>0$$

with $\alpha\beta=1$.

Then $\bar{A}$ is positive and from (11)

$$\left(\bar{A}+\frac{\sqrt{5}-1}{2}\right)\left(\frac{\sqrt{5}+1}{2}-\bar{A}\right)=\frac{f(i\alpha/5)}{f(i5\alpha)}\cdot\frac{\eta(i\alpha/5)}{\eta(i5\alpha)}\,\bar{A}.$$

Since the right side is positive, we have

$$(22)\quad 0<\bar{A}<(\sqrt{5}+1)/2.$$

Using (9) and (14) we then have

$$\left(\bar{A}+\frac{\sqrt{5}-1}{2}\right)\left(\bar{B}+\frac{\sqrt{5}-1}{2}\right)\left(\bar{A}-\frac{\sqrt{5}+1}{2}\right)\left(\bar{B}-\frac{\sqrt{5}+1}{2}\right)=5\bar{A}\bar{B}.$$

As before if we put

$$\bar{a}=\bar{A}+\frac{\sqrt{5}-1}{2},\qquad \bar{b}=\bar{B}+\frac{\sqrt{5}-1}{2},$$

then

$$\left(\bar{a}\bar{b}-\sqrt{5}\cdot\frac{\sqrt{5}-1}{2}\right)^2=\sqrt{5}\left(\bar{a}\bar{b}-\sqrt{5}\cdot\frac{\sqrt{5}-1}{2}\right)(\bar{a}+\bar{b}-\sqrt{5}+1).$$

Arguing as before and using (22) we have

THEOREM 2. *If α and β are real and positive and $\alpha\beta=1$, then*

$$\left\{\frac{\sqrt{5}-1}{2}+\frac{e^{-\pi\alpha/5}}{1-}\,\frac{e^{-\pi\alpha}}{1+}\,\frac{e^{-2\pi\alpha}}{1-}\cdots\right\}\left\{\frac{\sqrt{5}-1}{2}+\frac{e^{-\pi\beta/5}}{1-}\,\frac{e^{-\pi\beta}}{1+}\,\frac{e^{-2\pi\beta}}{1-}\cdots\right\}=\sqrt{5}\cdot\frac{\sqrt{5}-1}{2}.$$

This result is stated by Ramanujan in his Notebooks ([8], p. 204).

4. We can now operate with the relations (7) and (12) in exactly a similar way.

As before take $\tau=i\alpha$, $\alpha>0$. If we put, as before $A=A(2i\alpha)$, then (7) gives

$$\left(\left(\frac{\sqrt{5}+1}{2}\right)^5+A^5\right)\left(\left(\frac{\sqrt{5}-1}{2}\right)^5-A^5\right)=\left(\frac{\eta(i\alpha)}{\eta(i5\alpha)}\right)^6\cdot A^5.$$

Let us now choose $\beta>0$ such that

$$(23)\quad \alpha\beta=1/5.$$

We then have

$$(24)\quad \left(A^5+\left(\frac{\sqrt{5}+1}{2}\right)^5\right)\left(B^5+\left(\frac{\sqrt{5}+1}{2}\right)^5\right)\left(A^5-\left(\frac{\sqrt{5}-1}{2}\right)^5\right)\left(B^5-\left(\frac{\sqrt{5}-1}{2}\right)^5\right) = 5^3A^5B^5.$$

If we put

$$a = A^5+\left(\frac{\sqrt{5}+1}{2}\right)^5, \quad b = B^5+\left(\frac{\sqrt{5}+1}{2}\right)^5,$$

then (24) reduces to

$$\left(ab-5\sqrt{5}\cdot\left(\frac{\sqrt{5}+1}{2}\right)^5\right)^2 = 5\sqrt{5}\left(ab-5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5\right)\left(a+b-2\left(\frac{\sqrt{5}+1}{2}\right)^5\right).$$

As before assuming $ab-5\sqrt{5}\cdot(\sqrt{5}+1/2)^5 \neq 0$ leads to a contradiction. We therefore have

THEOREM 3. *If α and β are positive real numbers and $\alpha\beta = 1/5$, then*

$$\left\{\left(\frac{\sqrt{5}+1}{2}\right)^5+\left(\frac{e^{-2\pi\alpha/5}}{1+}\,\frac{e^{-2\pi\alpha}}{1+}\,\frac{e^{-4\pi\alpha}}{1+}\cdots\right)^5\right\}\times \left\{\left(\frac{\sqrt{5}+1}{2}\right)^5+\left(\frac{e^{-2\pi\beta/5}}{1+}\,\frac{e^{-2\pi\beta}}{1+}\,\frac{e^{-4\pi\beta}}{1+}\cdots\right)^5\right\} = 5\sqrt{5}\cdot\left(\frac{\sqrt{5}+1}{2}\right)^5.$$

It is now clear that we can obtain a similar theorem for $\bar{A} = A(2\tau_0)$ using (12). Thus

THEOREM 4. *If α and β are real and positive and $\alpha\beta = 1/5$, then*

$$\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5+\left(\frac{e^{-\pi\alpha/5}}{1-}\,\frac{e^{-\pi\alpha}}{1+}\,\frac{e^{-2\pi\alpha}}{1-}\cdots\right)^5\right\}\times \left\{\left(\frac{\sqrt{5}-1}{2}\right)^5+\left(\frac{e^{-\pi\beta/5}}{1-}\,\frac{e^{-\pi\beta}}{1+}\,\frac{e^{-2\pi\beta}}{1-}\cdots\right)^5\right\} = 5\sqrt{5}\cdot\left(\frac{\sqrt{5}-1}{2}\right)^5.$$

Theorem 1 was stated by Ramanujan in his first letter to Hardy (January 16, 1913) and proved by Watson [15] in a somewhat different way than ours. Theorems 1 and 2 are to be found stated by him in the unpublished Trinity papers. Theorems 2 and 3 do not seem to have been proved before. Theorem 3 is not given in his Notebooks. Theorem 4 which Ramanujan must certainly have known, is not stated anywhere.

5. We can now obtain many corollaries by specializing α and β. If we put $\alpha = \beta = 1$ in Theorems 1 and 2 we have at once

$$(24')\quad \begin{aligned} &\frac{e^{-2\pi/5}}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\cdots = \sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{2}}-\frac{\sqrt{5}+1}{2},\\ &\frac{e^{-\pi/5}}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+}\cdots = \sqrt{\frac{\sqrt{5}(\sqrt{5}-1)}{2}}-\frac{\sqrt{5}-1}{2}.\end{aligned}$$

These two results are stated in the Notebooks and were communicated to Hardy in Ramanujan's first letter.

We could put $\alpha = \beta = 1/\sqrt{5}$ in Theorems 2 and 4 and obtain two more results. Instead we shall proceed in the following way.

Let α, β and γ be three positive real numbers such that

$$(25)\quad \alpha\beta = 1, \quad \beta\gamma = 1/5.$$

We then have

$$(26)\quad \begin{aligned} &\left(A(2i\alpha)+\frac{\sqrt{5}+1}{2}\right)\left(A(2i\beta)+\frac{\sqrt{5}+1}{2}\right) = \sqrt{5}\cdot\frac{\sqrt{5}+1}{2},\\ &\left((A(2i\beta))^5+\left(\frac{\sqrt{5}+1}{2}\right)^5\right)\left((A(2i\gamma))^5+\left(\frac{\sqrt{5}+1}{2}\right)^5\right) = 5\sqrt{5}\cdot\left(\frac{\sqrt{5}+1}{2}\right)^5.\end{aligned}$$

An interesting property of (26) is that if one knows the value of one of $A(2i\alpha)$, $A(2i\beta)$ and $A(2i\gamma)$, the values of the other two, since they are real and positive, are determined uniquely.

Let us take

$$(27)\quad \alpha = \sqrt{5}, \quad \beta = 1/\sqrt{5}, \quad \gamma = 1/\sqrt{5}.$$

Since $\beta = \gamma$, the second relation in (26), which is Theorem 3, gives

$$(28)\quad \left(A\left(\frac{2i}{\sqrt{5}}\right)\right)^5 = \left(5\sqrt{5}\cdot\left(\frac{\sqrt{5}+1}{2}\right)^5\right)^{1/2}-\left(\frac{\sqrt{5}+1}{2}\right)^5.$$

Inserting this in the first equation we get

$$A(2i\cdot\sqrt{5}) = \frac{\sqrt{5}\cdot\frac{\sqrt{5}+1}{2}}{\frac{\sqrt{5}+1}{2}+\sqrt[5]{\left(5\sqrt{5}\cdot\left(\frac{\sqrt{5}+1}{2}\right)^5\right)^{1/2}-\left(\frac{\sqrt{5}+1}{2}\right)^5}}-\frac{\sqrt{5}+1}{2}.$$

Cancelling $(\sqrt{5}+1)/2$ from numerator and denominator of first term on the right above, and using the fact that $((\sqrt{5}+1)/2)\cdot((\sqrt{5}-1)/2) = 1$, we get

$$(29)\qquad \frac{e^{-2\pi/\sqrt{5}}}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+}\cdots = \frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1}} - \frac{\sqrt{5}+1}{2}.$$

This is one of the results communicated to Hardy. It is not given in the Notebooks.

Under the conditions (25) we have results similar to (26) with $\bar{A}$ instead of A so that if one of $\bar{A}(i\alpha)$, $\bar{A}(i\beta)$, $\bar{A}(i\gamma)$ is known, then the other two are uniquely determined. In particular if α, β, γ are given by (27) then we have, analogous to (29)

$$(30)\qquad \frac{e^{-\pi\sqrt{5}/5}}{1-}\,\frac{e^{-\pi\sqrt{5}}}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1-}\cdots = \frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\left(\frac{\sqrt{5}+1}{2}\right)^{5/2}-1}} - \frac{\sqrt{5}-1}{2}.$$

Ramanujan considers a few other cases in the 'Lost' notebook.

Let us take $\alpha = \sqrt{2}$. Then

$$\frac{1}{A(2i\alpha)}-1-A(2i\alpha) = \frac{\eta(\sqrt{-2}/5)}{\eta(\sqrt{-50})}.$$

In order to evaluate the right side, we use (9) and so

$$\frac{\eta(\sqrt{-2}/5)}{\eta(\sqrt{-50})} = \frac{\sqrt{5}}{2^{1/4}}\,\frac{\eta(\sqrt{-50}/2)}{\eta(\sqrt{-50})}.$$

Now $\eta(\sqrt{-50}/2)/\eta(\sqrt{-50}) = f_1(\sqrt{-50})$ where f_1 is the function due to Weber (also equivalent to g_n of Ramanujan). Its value according to Weber is given by

$$f_1(\sqrt{-50}) = 2^{1/4}\lambda > 0$$

where

$$(\lambda^3-\lambda^2)/(\lambda+1) = (\sqrt{5}+1)/2.$$

Thus

$$\frac{e^{-\pi\sqrt{8}/5}}{1+}\,\frac{e^{-\pi\sqrt{8}}}{1+}\,\frac{e^{-2\pi\sqrt{8}}}{1+}\cdots = \frac{\sqrt{5(\lambda+1)+2\lambda\sqrt{5}}-(\sqrt{5}\lambda+1)}{2}.$$

In a similar way we obtain the value of

$$\frac{e^{-\pi\sqrt{2}/5}}{1+}\,\frac{e^{-\pi\sqrt{2}}}{1+}\,\frac{e^{-2\pi\sqrt{2}}}{1+}\cdots$$

in terms of λ. Notice that λ is a cubic irrationality. These results are stated by Ramanujan.

6. We remark that in the previous section we have been able to evaluate $\eta(it/5)/\eta(i5t)$ or $\eta(it)/\eta(i5t)$ for $t = 1, \sqrt{2}, \sqrt{5}$. We assert that we can evaluate

$$(31)\qquad \eta(i5^m t)/\eta(i5^l t)$$

for any two rational integers m and l. Let us notice that once we can evaluate $\eta(it/5)/\eta(i5t)$ or $\eta(it)/\eta(i5t)$ the other is easily evaluated. For if $\eta(it/5)/\eta(i5t)$ is known, let us take

$$\alpha = t,\quad \beta = 1/t,\quad \gamma = t/5$$

in (25). Then using (26) we can find $A(2i\gamma)$. But from (7) we have

$$(32)\qquad \left(\frac{1}{A(2i\gamma)}\right)^5 - 11 - (A(2i\gamma))^5 = \left(\frac{\eta(it/5)}{\eta(it)}\right)^6.$$

Since $\eta(it/5)/\eta(it)$ is positive, (32) determines $\eta(it/5)/\eta(it)$ uniquely and hence

$$(33)\qquad \frac{\eta(it)}{\eta(i5t)} = \left(\frac{\eta(it)}{\eta(it/5)}\right)\cdot\left(\frac{\eta(it/5)}{\eta(i5t)}\right).$$

By induction we can find $\eta(i5^k t)/\eta(i5^{k+1}t)$ for an integer k. Let $k > 0$ and suppose $\eta(i5^{k-1}t)/\eta(i5^k t)$ is known. Take

$$\alpha = 5^k t,\quad \beta = 1/5^k t,\quad \gamma = 5^{k-1}t.$$

Then

$$\left(\frac{1}{A(2i\gamma)}\right)^5 - 11 - (A(2i\gamma))^5 = \left(\frac{\eta(i5^{k-1}t)}{\eta(i5^k t)}\right)^6$$

shows that $A(2i\gamma)$ is determined uniquely. From (26) therefore $A(2i\alpha)$ is known. But from (7) we see that $\eta(i5^{k-1}t)/\eta(i5^k t)$ is evaluated. We therefore obtain the value of $\eta(i5^k t)/\eta(i5^{k+1}t)$.

A similar method gives the result for k negative. Now if $m < l$ (without loss in generality) and $m > 0$

$$\frac{\eta(i5^m t)}{\eta(i5^l t)} = \prod_{k=0}^{l-1}\left(\eta(i5^k t)/\eta(i5^{k+1}t)\right)\Big/\prod_{k=0}^{m-1}\left(\eta(i5^k t)/\eta(i5^{k+1}t)\right).$$

In order to find the value of $\eta(i5^m t)$ for all m, it is thus necessary to evaluate $\eta(it)$. This is done by the Kronecker limit formula.

For example if $t = 1$, we get the value of $A(2i)$ from (24′). Using (6) and (7) we get the values of $\eta(i/5)/\eta(5i)$ and $\eta(i)/\eta(5i)$. From Kronecker limit formula we have

$$\eta(i) = \Gamma(1/4)/2\pi^{3/4} \tag{34}$$

and hence we get the following: From (24′)

$$A(2i) = \sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{2}} - \frac{\sqrt{5}+1}{2}.$$

Therefore

$$\frac{1}{A(2i)} = \sqrt{\frac{\sqrt{5}(\sqrt{5}+1)}{2}} + \frac{\sqrt{5}+1}{2}.$$

Hence from (6)

$$\frac{\eta(i/5)}{\eta(5i)} = \frac{1}{A(2i)} - 1 - A(2i) = \sqrt{5}. \tag{35}$$

In a similar way

$$\left(\frac{\eta(i)}{\eta(5i)}\right)^6 = \left(\frac{1}{A(2i)}\right)^5 - 11 - \left(A(2i)\right)^5 = 5^3\left(\frac{\sqrt{5}+1}{2}\right)^3 \tag{36}$$

so that

$$\frac{\eta(i)}{\eta(5i)} = \sqrt{5}\sqrt{\frac{\sqrt{5}+1}{2}}. \tag{37}$$

Thus

$$\eta(5i) = \Gamma(1/4)\Big/2\pi^{3/4}\sqrt{\frac{5(\sqrt{5}+1)}{2}}$$

and

$$\eta(i/5) = \Gamma(1/4)\Big/2\pi^{3/4}\sqrt{\frac{\sqrt{5}+1}{2}}.$$

7. Another result which belongs to the order of ideas in the previous sections is the following.

Let α and β be two positive real numbers and

$$x = e^{-2\pi\alpha}, \quad y = e^{-2\pi\beta}. \tag{38}$$

Let

$$f = \frac{x^{1/5}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\cdots,$$

$$\varphi = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\cdots, \tag{39}$$

$$g = \frac{y^{1/25}}{1+}\,\frac{y^{1/5}}{1+}\,\frac{y^{2/5}}{1+}\cdots$$

Choose now α, β so that

$$\alpha\beta = 1. \tag{40}$$

Then $f = A(2i\alpha)$, $\varphi = A(2i\cdot 5\alpha)$, $g = A\big(2i\cdot(\beta/5)\big)$. Because of (40), φ and g satisfy Theorem 1. Therefore

$$\left(\frac{\sqrt{5}+1}{2} + \varphi\right)\left(\frac{\sqrt{5}+1}{2} + g\right) = \sqrt{5}\cdot\frac{\sqrt{5}+1}{2}$$

which is the same as

$$g = \frac{1 - \frac{\sqrt{5}+1}{2}\varphi}{\varphi + (\sqrt{5}+1)/2}. \tag{41}$$

In a similar way because of (40), f and g satisfy Theorem 3 which gives

$$f^5 = \frac{1 - \left(\frac{\sqrt{5}+1}{2}\right)^5\cdot g^5}{g^5 + \left(\frac{\sqrt{5}+1}{2}\right)^5}. \tag{42}$$

If we substitute for g in (42) using (41), we get

$$f^5 = \frac{\left(\varphi + \frac{\sqrt{5}+1}{2}\right)^5 - \left(\frac{\sqrt{5}+1}{2}\right)^5\left(1 - \frac{\sqrt{5}+1}{2}\varphi\right)^5}{\left(1 - \frac{\sqrt{5}+1}{2}\varphi\right)^5 + \left(\frac{\sqrt{5}+1}{2}\right)^5\left(\varphi + \frac{\sqrt{5}+1}{2}\right)^5}$$

which gives

$$f^5 = \varphi\,\frac{1 - 2\varphi + 4\varphi^2 - 3\varphi^3 + \varphi^4}{1 + 3\varphi + 4\varphi^2 + 2\varphi^3 + \varphi^4}. \tag{43}$$

This result was stated by Ramanujan in his first letter to Hardy. It was proved first by G. N. Watson [16].

In exactly a similar way using Theorems 2 and 4 we have the relation

$$(44)\qquad \bar{f}^5 = \bar{\varphi}\,\frac{1+2\bar{\varphi}+4\bar{\varphi}^2+3\bar{\varphi}^3+\bar{\varphi}^4}{1-3\bar{\varphi}+4\bar{\varphi}^2-2\bar{\varphi}^3+\bar{\varphi}^4},$$

where

$$\bar{f} = \frac{x^{1/5}}{1-}\,\frac{x}{1+}\,\frac{x^2}{1-}\cdots,\qquad \bar{\varphi} = \frac{x}{1-}\,\frac{x^5}{1+}\,\frac{x^{10}}{1-}\cdots$$

8. Let $x = e^{2\pi i\tau}$, $\tau = \xi + i\eta$, $\eta > 0$. Then

$$R = \frac{x^{1/5}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\cdots = x^{1/5}\,\frac{h(x)}{g(x)}$$

where

$$h = h(x) = \sum_{-\infty}^{\infty}(-1)^k x^{(5k^2+k)/2},\qquad g = g(x) = \sum_{-\infty}^{\infty}(-1)^k x^{(5k^2+3k)/2}.$$

If $\tau = ia$, $a > 0$, then (15) can be written as

$$(45)\qquad \left(\frac{1}{\sqrt{R}}+\frac{\sqrt{5}-1}{2}\sqrt{R}\right)\left(\frac{1}{\sqrt{R}}-\frac{\sqrt{5}+1}{2}\sqrt{R}\right) = \frac{\eta(ia/5)}{\eta(i5a)},$$

taking positive square roots. We can now write

$$(46)\qquad \frac{1}{\sqrt{R}}+\frac{\sqrt{5}-1}{2}\sqrt{R} = \left(g+\frac{\sqrt{5}-1}{2}\,x^{1/5}h\right)\Big/(x^{1/5}gh)^{1/2},$$

$$(47)\qquad \frac{1}{\sqrt{R}}-\frac{\sqrt{5}+1}{2}\sqrt{R} = \frac{g-\dfrac{\sqrt{5}+1}{2}\,x^{1/5}h}{(x^{1/5}gh)^{1/2}}.$$

Since g and h are theta functions, they have expressions as infinite products; namely

$$(48)\qquad x^{1/5}gh = x^{1/5}\prod_{1}^{\infty}(1-x^n)(1-x^{5n}).$$

The numerators on the right side of (46) and (47) can also be expressed as infinite products; for if

$$\tfrac{1}{2}\theta_2(z, x) = x^{1/4}\cos z + x^{9/4}\cos 3z + x^{25/4}\cos 5z + \ldots$$

then

$$(49)\qquad \begin{aligned} \frac{1}{2}\,\theta_2\left(\frac{\pi}{10}, x^{1/10}\right) &= x^{1/40}\cos\frac{\pi}{10}\left(g+\frac{\sqrt{5}-1}{2}\,x^{1/5}h\right),\\ \frac{1}{2}\,\theta_2\left(\frac{3\pi}{10}, x^{1/10}\right) &= x^{1/40}\cos\frac{3\pi}{10}\left(g-\frac{\sqrt{5}+1}{2}\,x^{1/5}h\right).\end{aligned}$$

Using the infinite product for the theta function on the left of (49) ([18], p. 85) we have

$$(50)\qquad \begin{aligned} g+\frac{\sqrt{5}-1}{2}\,x^{1/5}h &= \prod_{1}^{\infty}\left\{(1-x^{n/5})\left(1+\frac{\sqrt{5}+1}{2}\,x^{n/5}+x^{2n/5}\right)\right\},\\ g-\frac{\sqrt{5}+1}{2}\,x^{1/5}h &= \prod_{1}^{\infty}\left\{(1-x^{n/5})\left(1-\frac{\sqrt{5}-1}{2}\,x^{n/5}+x^{2n/5}\right)\right\}.\end{aligned}$$

Let us now observe from (45) that

$$\frac{1}{\sqrt{R}}+\frac{\sqrt{5}-1}{2}\sqrt{R} = \frac{\eta(ia/5)}{\eta(i5a)}\cdot\left(\frac{1}{\sqrt{R}}-\frac{\sqrt{5}+1}{2}\sqrt{R}\right)^{-1}.$$

Using the infinite product for the η-function and (47), (48) and (50) we obtain Ramanujan's result stated in the "Lost" notebook:

$$(51)\qquad \frac{1}{\sqrt{R}}+\frac{\sqrt{5}-1}{2}\sqrt{R} = x^{-1/10}\left(\frac{\prod(1-x^n)}{\prod(1-x^{5n})}\right)^{1/2}\prod\left(1-\frac{\sqrt{5}-1}{2}\,x^{n/5}+x^{2n/5}\right)^{-1}.$$

Similarly

$$(52)\qquad \frac{1}{\sqrt{R}}-\frac{\sqrt{5}+1}{2}\sqrt{R} = x^{-1/10}\left(\frac{\prod(1-x^n)}{\prod(1-x^{5n})}\right)^{1/2}\prod\left(1+\frac{\sqrt{5}+1}{2}\,x^{n/5}+x^{2n/5}\right)^{-1}.$$

Let us write the two identities in (50) slightly differently:

$$g+\frac{\sqrt{5}-1}{2}\,x^{1/5}h = \prod_{1}^{\infty}(1-x^{n/5})(1+e^{i\theta}x^{n/5})(1+e^{-i\theta}x^{n/5}),$$

$$g-\frac{\sqrt{5}+1}{2}\,x^{1/5}h = \prod_{1}^{\infty}(1-x^{n/5})(1-e^{i\varphi}x^{n/5})(1-e^{-i\varphi}x^{n/5}),$$

where

$$e^{i\theta}+e^{-i\theta} = (\sqrt{5}+1)/2,\qquad e^{i\varphi}+e^{-i\varphi} = (\sqrt{5}-1)/2.$$

Replacing $x^{1/5}$ by $\varrho x^{1/5}$ where ϱ runs through all fifth roots of unity and multiplying for all ϱ – a familiar technique of Ramanujan's – we have since g and h are power series in x,

$$(53)\qquad \begin{aligned} g^5+\left(\frac{\sqrt{5}-1}{2}\right)^5 xh^5 &= \frac{\prod(1-x^n)^8}{\prod(1-x^{5n})^3}\prod\left(1+\frac{\sqrt{5}+1}{2}\,x^n+x^{2n}\right)^5,\\ g^5-\left(\frac{\sqrt{5}+1}{2}\right)^5 xh^5 &= \frac{\prod(1-x^n)^8}{\prod(1-x^{5n})^3}\prod\left(1-\frac{\sqrt{5}-1}{2}\,x^n+x^{2n}\right)^5.\end{aligned}$$

We now write (7) in the form

$$\left(\frac{1}{\sqrt{R}}\right)^5+\left(\frac{\sqrt{5}-1}{2}\right)^5(\sqrt{R})^5=\left(\frac{\eta(ia)}{\eta(i5a)}\right)^6\left(\left(\frac{1}{\sqrt{R}}\right)^5-\left(\frac{\sqrt{5}+1}{2}\right)^5(\sqrt{R})^5\right)^{-1}.$$

Using (53) and the product formula for the η-function, we get two of Ramanujan's results in the "Lost" notebook:

(54)

$$\left(\frac{1}{\sqrt{R}}\right)^5-\left(\frac{\sqrt{5}+1}{2}\right)^5(\sqrt{R})^5=x^{-1/2}\left(\frac{\prod(1-x^n)}{\prod(1-x^{5n})}\right)^{1/2}\prod\left(1+\frac{\sqrt{5}+1}{2}x^n+x^{2n}\right)^5,$$

$$\left(\frac{1}{\sqrt{R}}\right)^5+\left(\frac{\sqrt{5}-1}{2}\right)^5(\sqrt{R})^5=x^{-1/2}\left(\frac{\prod(1-x^n)}{\prod(1-x^{5n})}\right)^{1/2}\prod\left(1-\frac{\sqrt{5}-1}{2}x^n+x^{2n}\right)^5.$$

9. Ramanujan seems to have had a great fascination for his continued fraction (1). It turns up at many places in the "Lost" notebook, in the Trinity papers and in the Notebooks (on page 374 of the Notebooks [8]). Ramanujan writes

"If

$$\varphi(a)=1+\frac{ax}{(1-x)(1+bx)}+\frac{a^2x^3}{(1-x)(1-x^2)(1+bx)(1+bx^2)}+\ldots$$

then

$$\frac{\varphi(a)}{\varphi(ax)}=1+\frac{ax}{1+}\,\frac{bx}{1+}\,\frac{ax^2}{1+}\,\frac{bx^2}{1+}\,\frac{ax^3}{1+}\,\frac{bx^3}{1+}\ldots\text{''}.$$

If we write $\varphi(a,b,x)$ instead of $\varphi(a)$ then

$$\varphi(a,b,x)=1+\sum_1^\infty\frac{a^n\cdot x^{n(n+1)/2}}{(1-x)(1-x^2)\ldots(1-x^n)(1+bx)(1+bx^2)\ldots(1+bx^n)}$$

Furthermore

(55) $$\varphi(x^{-1},1,x^2)=1+\frac{x}{1-x^4}+\frac{x^4}{(1-x^4)(1-x^8)}+\ldots$$

and

(56) $$x\varphi(x,1,x^2)=x+\frac{x^4}{1-x^4}+\frac{x^9}{(1-x^4)(1-x^8)}+\ldots$$

so that the ratio

(57) $$\left(\frac{\varphi(x^{-1},1,x^2)}{x\varphi(x,1,x^2)}\right)^{-1}=x+\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\ldots$$

Indeed on page 373 of the Notebooks Ramanujan states

$$\frac{\chi(-x^2)f(-x^5)}{f(-x,-x)^4}=\frac{f(x,x^9)}{f(-x^4,-x^{16})}=1+\frac{x}{1-x^4}+\frac{x^4}{(1-x^4)(1-x^8)}+\ldots,$$

$$\frac{x\chi(-x^2)f(-x^5)}{f(-x^2,-x^3)}=\frac{xf(x^3,x^7)}{f(-x^8,-x^{12})}=x+\frac{x^4}{1-x^4}+\frac{x^9}{(1-x^4)(1-x^8)}+\ldots$$

The ratio of these two, as Ramanujan mentions, is

$$\frac{xf(-x,-x^4)}{f(-x^2,-x^3)}=\frac{x}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\ldots$$

which is the continued fraction of Ramanujan. Note that

$$f(a,b)=1+\sum_1^\infty(ab)^{n(n-1)/2}(a^n+b^n)\quad\text{and}\quad\chi(x)=\prod_1^\infty(1+x^{2n-1})$$

are the functions introduced by Ramanujan ([8], p. 197).

The two series $\varphi(x^{-1},1,x^2)$ and $\varphi(x,1,x^2)$ had been first encountered by L. J. Rogers in his paper ([10], p. 330), and he gave the expressions

$$\varphi(x^{-1},1,x^2)=\frac{1}{\prod(1+x^{2n})(1-x^{5n-1})(1-x^{5n-4})},$$

$$\varphi(x,1,x^2)=\frac{1}{\prod(1+x^{2n})(1-x^{5n-2})(1-x^{5n-4})}$$

which are the same as those given by Ramanujan. These two statements have recently been proved by K. Venkatachaliengar.

In the "Lost" notebook Ramanujan gave a continued fraction which contains (1) as a particular case. If with Ramanujan we put

$$G(a,\lambda)=1+\frac{x(a+\lambda)}{(1-x)(1+bx)}+\frac{x^3(a+\lambda)(a+\lambda x)}{(1-x)(1-x^2)(1+bx)(1+bx^2)}+\ldots$$

then

(58) $$\frac{G(ax,\lambda x)}{G(a,\lambda)}=\frac{1}{1+}\,\frac{ax+\lambda x}{1+}\,\frac{bx+\lambda x^2}{1+}\,\frac{ax^2+\lambda x^3}{1+}\,\frac{bx^2+\lambda x^4}{1+}\ldots$$

as stated by Ramanujan. If we put x^2 for x and then $a=x^{-1}$, $b=1$ and $\lambda=0$, we get precisely the results given above.

If $\lambda=1$ and we put x^2 for x and ax^{-1} for a, we get a continued fraction which was independently discovered and discussed by Atle Selberg [12]. In fact most of the examples given by Ramanujan in the "Lost" notebook (some of which are also to be found in the Notebooks) are found stated and proved by Selberg.

We shall consider some statements made by Ramanujan in the Trinity papers.

Ramanujan states:

"If $v = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+}\ldots$, then

$$x\left(v+\frac{1}{v}\right)\frac{(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} = 1+\sum_1^\infty\left(\frac{nx^n}{1-x^n}-\frac{25nx^{25n}}{1-x^{25n}}\right).$$

If $u = \frac{x^{1/5}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\ldots$, then

$$x\left(u^5+\frac{1}{u^5}\right)\frac{(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} = 1+6\sum_1^\infty\left(\frac{nx^n}{1-x^n}-\frac{5nx^{5n}}{1-x^{5n}}\right)."$$

These two results are simple consequences of the equations (6) and (7). For, let us put $x = e^{2\pi i\tau}$; then by (6)

$$(59)\qquad \frac{1}{v}-1-v = \frac{\eta(\tau)}{\eta(25\tau)} = \frac{1}{x}\cdot\frac{\prod(1-x^n)}{\prod(1-x^{25n})}.$$

We can take τ purely imaginary so that all quantities are real and positive. If we take derivatives of both sides of (59) with regard to x, we get on the right side

$$(60)\qquad -\frac{1}{x^2}\frac{\prod(1-x^n)}{\prod(1-x^{25n})}\left(1+\sum_1^\infty\left(\frac{nx^n}{1-x^n}-\frac{25nx^{25n}}{1-x^{25n}}\right)\right).$$

On the left side we get

$$(61)\qquad -\frac{1}{v}\left(v+\frac{1}{v}\right)\frac{dv}{dx}.$$

Ramanujan has stated in his Notebooks ([8], p. 234), and W. N. Bailey has given a very simple and elegant proof of,

$$(62)\qquad \frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\frac{\prod(1-x^{5n})^5}{\prod(1-x^{25n})}.$$

Using (59)–(61) we obtain Ramanujan's first formula.

The second formula can be obtained by differentiating both sides of (7), i.e.,

$$\frac{1}{u^5}-11-u^5 = \frac{1}{x}\cdot\left(\frac{\prod(1-x^n)}{\prod(1-x^{5n})}\right)^6.$$

Ramanujan had stated both in the Notebooks ([8], p. 373) and also in the "Lost" notebook the result

$$(63)\qquad \frac{x^{1/3}}{1+}\frac{x+x^2}{1+}\frac{x^2+x^4}{1+}\frac{x^3+x^6}{1+}\ldots = x^{1/3}\frac{\prod(1-x^{2n-1})}{\prod(1-x^{3(2n-1)})^3}.$$

This is also to be found in the work of Selberg. If we put $x = e^{-\pi\sqrt{10}}$ then one can evaluate

$$(64)\qquad e^{-2\pi\sqrt{10}/3}\frac{\prod(1-e^{-(2n-1)\pi\sqrt{10}})}{\prod(1-e^{-3(2n-1)\pi\sqrt{10}})^3}.$$

Indeed if we use the function g_n of Ramanujan ([8], p. 299), we get

$$\frac{\left\{\left(\frac{\sqrt{6}+3}{4}\right)^{1/2}-\left(\frac{\sqrt{6}-1}{4}\right)^{1/2}\right\}^3}{\sqrt{2}(\sqrt{5}+2)^{1/3}(\sqrt{6}+\sqrt{5})^{1/2}}.$$

We expand the numerator and group the terms suitably. Thus

$$\left((\sqrt{6}+3)^{1/2}-(\sqrt{6}-1)^{1/2}\right)^3 = (\sqrt{6}+3)(\sqrt{6}+3)^{1/2}+3(\sqrt{6}+3)^{1/2}(\sqrt{6}-1)-$$
$$-3(\sqrt{6}-1)^{1/2}(\sqrt{6}+3)-(\sqrt{6}-1)^{1/2}(\sqrt{6}-1).$$

Which gives

$$(\sqrt{6}+3)^{1/2}4\sqrt{6}-(\sqrt{6}-1)^{1/2}(4\sqrt{6}+8).$$

Thus the numerator is

$$\tfrac{1}{8}\left((9+3\sqrt{3})^{1/2}-(7+3\sqrt{6})^{1/2}\right).$$

As for the denominator observe that $(\sqrt{5}+1)^3 = 8(\sqrt{5}+2)$. These prove the result stated in (3).

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Pairs of additive equations IV. Sextic equations

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1. Introduction. H. Davenport and D. J. Lewis [10] remarked that it should be possible in principle to show that the equations, with integer coefficients,

$$F(x) = a_1x_1^k + \ldots + a_Nx_N^k = 0,\quad G(x) = b_1x_1^k + \ldots + b_Nx_N^k = 0 \tag{1}$$

have a non-trivial solution in integers provided that

(i) $N \geqslant 2k^2+1$,
(ii) they have a non-singular real solution,
(iii) for each prime p they have a non-singular p-adic solution,
(iv) if the degree k is even then each form $\lambda F+\mu G$, $(\lambda, \mu) \neq (0, 0)$, contains a reasonable number of variables explicitly.

The condition $N \geqslant 2k^2+1$ is similar to Artin's conjecture for two additive forms. Results of this strength have been established when $k = 2$ ([2]), $k = 3$ ([4], [9] and [13]), $k = 4$ ([4]), $k = 5$ ([6]) and $k \geqslant 18$ ([5] and [11]) since the analytic methods of [5] will also work for even values $k \geqslant 18$. The analytic methods used for quintic equations were based on a method of Davenport [7] for iterating admissible exponents and unfortunately the method just fails to work when $k = 6$. However H. Davenport and P. Erdös ([8], Theorem 2) obtained admissible exponents for 3 sixth powers that improved on the estimates obtainable by Davenport's methods. The basis of the present paper is to establish an analogue for two additive equations of this result of Davenport and Erdös and then to use iterative methods to obtain a sequence of 14 exponents for sixth powers.

Theorem 1. *Let the equations*

$$F(x) = a_1x_1^6 + \ldots + a_Nx_N^6 = 0,\quad G(x) = b_1x_1^6 + \ldots + b_Nx_N^6 = 0 \tag{2}$$