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70+ Years of the Watson Integrals

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Abstract Watson (Q. J. Math. Oxford 10:266–276, 1939) published in 1939 the evaluation of three integrals submitted to him, which had arisen from a problem in physics (van Pepye in Physica 5:465–482, 1938). Over the years these integrals have continued to occur in other aspects of physics such as random walk problems. This article reviews these integrals and generalisations over the past 70 years.

Keywords Cubic LN curve · Convolution curve · Minkowski sum · Offsets · Linear normal map · G-1 Bezier approximation · Error estimate

1 Introduction

This is a short history of three triple integrals and their generalisations which first appeared some seventy years ago. It provides an example of how physicists and mathematicians like to elaborate particular results and extend them to more general areas. In doing so new techniques are developed which eventually become standard methods. It is also an example of how chance plays a part in bringing together researchers with knowledge in different fields who collaborate to produce results which might not otherwise have been obtained. This will be a rather personal account as many hundreds of papers and articles concerning these

It is with great pleasure that I dedicate this to my esteemed mentor Cyril Domb on his 90th birthday. In 1954 when he took up his appointment as Professor of Theoretical Physics at King's College, London, I became his first post-graduate student there. Originally he asked me to learn about series expansions, but quickly realised this was a step in the wrong direction for me. So instead I studied intermolecular forces in rare gas crystals which at that time was another interest of Cyril. Thus I never became part of that wonderful school of statistical mechanics built by Cyril at Kings. However, I always remained in contact with various members of that group and worked together whenever our interests coincided. Lately a collaboration resulting in some joint work on Watson integrals occurred. It is allowable to reminisce and review when celebrating a 90th anniversary, hence the following.

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integrals have been published, and it is not possible in limited space to refer to every contribution, so at the start I apologise to authors whose work I do not cite.

The three original integrals are

$$W_B = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-\cos x\cos y\cos z}, \tag{1.1}$$

$$W_F = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x\cos y-\cos y\cos z-\cos z\cos x}, \tag{1.2}$$

$$W_S = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x-\cos y-\cos z}. \tag{1.3}$$

They made their first appearance in a paper on magnetic anisotropy by van Pepye [1] in 1938. The integrals emerged when dealing with three well-known cubic structures formed by real crystals, namely the body centered (B), the face-centered (F) and the simple cubic (S) lattices. Van Pepye was a student of the Dutch physicist H.A. Kramers, and clearly these integrals intrigued the latter. He sent them on to R.H. Fowler, the son-in-law of Rutherford, in Cambridge and then... Well let Watson [2] in his own words describe how the problem reached him.

'The problem of evaluating them was proposed by Kramers to R.H. Fowler who communicated them to G.H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the journey to Birmingham without difficulty.'

Whatever the motive was for this last somewhat barbed comment, Watson found closed forms for these integrals and they are now universally known by his name. They will now be referred to as WI. The first, as Watson acknowledged, is fairly well-known. Indeed van Pepye himself had evaluated it and the result can be traced back to Kummer [3, 4]; also its generalisation is simple to find Maradudin et al. [5] so the closed form for this more general version of (1.1) is now given. We have

$$W_B(w_b) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_b-\cos x\cos y\cos z} = \frac{4}{\pi^2}K^2\left(\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{1}{w_b^2}}}\right), \tag{1.4}$$

when $w_b = 1$

$$W_B(1) = \frac{4}{\pi^2}K^2\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\pi^3}\Gamma^4\left(\frac{1}{4}\right) = 1.39320\,39297\ldots, \tag{1.5}$$

where K is the complete elliptic integral of the first kind.

As complete elliptic integrals of the first kind play a primary role in all aspects of WI and their generalisations it is apposite here to give a short account of them. The complete elliptic integral of the first kind is defined by

$$K(k) := \int_0^{\pi/2} \frac{\mathrm{d}t}{(1-k^2\sin^2 t)^{1/2}} = \int_0^1 \frac{\mathrm{d}x}{[(1-x^2)(1-k^2x^2)]^{1/2}}, \tag{1.6}$$

for $0 \le k < 1$. k is known as the modulus. The complementary modulus k' is defined by the relation $k^2 + k'^2 = 1$ and we write $K(k') := K'(k)$. K and iK' play the equivalent roles in the theory of the Jacobian elliptic functions as does $\pi/2$ in the theory of the circular functions, namely they are quarter periods. An important property of $K(k)$ is that it can be expressed as a hypergeometric function, namely

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 : k^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}k^{2n}, \tag{1.7}$$

which will later be seen as crucial in further analysis.

Although (1.1) and its generalisation were easily solved, (1.2) and (1.3) were a different matter. Watson's [2] approach was to use an inspired sequence of changes of variable and integrations to reduce the triple integrals to single integrals involving K. Then using expansions of K and K' he had obtained himself 30 years previously—Watson [6]—he was able to evaluate these last integrals. One should go to the original paper to admire the ingenuity displayed in finding (1.8) and to enjoy the brilliance of his derivation of (1.9). His results are

$$W_F = \frac{\sqrt{3}}{\pi^2}K^2\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4} = 0.44822\,03944\ldots, \tag{1.8}$$

$$W_S = \frac{4(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})}{\pi^2}K^2\big[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\big] = 0.50546\,20197. \tag{1.9}$$

It will be observed that whereas W_B and W_F were expressed in terms of gamma functions, this was not done for W_S. This had to await a later investigation.

The next appearance of WI in the literature (though not by name) followed almost immediately. In an odd paper by McCrea and Whipple [7], they took up the problem of the "Drunkards Walk" in three dimensions. The strangeness of the paper was not in its content, but in the complete oblivion of the authors of any other work in this field. There were only two references given. The first was to a note of McCrea in the Mathematical Gazette [8], in which he had considered the two dimensional square lattice, and as a kind of an afterthought a paper by Courant, Friedrichs and Lewy [9] which had been brought to their attention. The problem may be formulated as follows. An intoxicated person drops his housekeys under a lamppost—the origin. Suppose the lamppost is one of an infinite number of posts equally spaced along a line. The inebriate staggers from lamppost to lamppost with an equal probability of going to the right or left while looking for the keys. What is the probability that he or she will return to the origin? The same question may be asked if the lampposts form a two dimensional square array or a three dimensional cubic array. In fact this problem was first discussed by Pòlya [10]. He showed that in the one and two dimensional cases the probability of return to the origin was one, i.e the keys would eventually be found. However, in three dimensions the probability was less than one, but Pòlya did not find a value for it.

This problem is mirrored in one dimension exactly by tossing a coin heads or tails, in two dimensions by throwing a tetrahedral die and in three dimensions by casting a common six-sided die. Stewart [11] discusses this in some detail. He points out that in tossing a fair coin the starting point is equal numbers of heads and tails. As one continues tossing an imbalance between numbers of heads and tails will occur, but no matter how large this imbalance becomes if you carry on flipping the coin the imbalance will eventually correct itself. Similarly in throwing an unbiased tetrahedral die, the digits 1, 2, 3 and 4 have an equal probability of $1/4$ of appearing. As before one starts with equal numbers of the 4 digits, and again as one continues throwing, deviations from the initial state will appear. But as in the coin case if one continues to throw the four sided die, the state of equal numbers for the four digits will eventually reoccur. However in throwing an ordinary 6-sided die each of the digits 1–6 has a probability of $1/6$ of occurring. Starting from equal numbers of the six digits imbalances will develop, but the probability of returning to equal numbers of the six digits is less than one. He actually credits Ulam as proving this but gives no reference, but I think Pòlya [10] must be credited as being first with this result.

However, it was McCrea and Whipple who first set out to find a value for this probability. Clearly they did not know of Pòlya's paper for no reference to him is made, but they succeeded in showing that the probability of return to the origin was $1-1/3W_S$! (I have converted their result into the notation used here. All authors in this field use their own symbols. So all formulae given here are translations of their results into notation consistent with that used in this place.) Just as McCrea and Whipple were unaware of Pòlya they also were in ignorance of Watson, and it is interesting to follow their attempt to evaluate $3W_S$. They rapidly reduced W_S to a single integral from $\pi/6$ to $\pi/2$ of a complete elliptic integral multiplied by an angular factor and then resorted to numerical integration. Such was the inaccuracy of their computation that they only gave the value to two significant figures, $3W_S=1.53$, which compared to the value 1.51638606 found from (1.9) is woefully out. They thus found the probability of returning to the origin was 0.34—surprisingly small. This seems to be the very first numerical evaluation of this probability.

A much more accurate result was found by Domb [12] in a completely different manner with no reference to W_S whatsoever. He calculated directly the probability of the return to the origin, p_n, after n steps. For n odd you cannot return to your starting point and thus $p_{2n+1}=0$. For small n, p_{2n} can be calculated exactly from a rather complex formula. In this way he directly calculated $p_2 \dots p_{18}$, then applying Euler-McLaurin summation to the asymptotic formula for p_{2n} he obtained the excellent estimate for the equivalent of $3W_S$ of 1.51639 giving 0.34054 for the probability of return to the origin. If we use the more accurate value of Watson, the probability of return to the origin is 0.340537330.

Now the Courant paper referred to by McCrea and Whipple was actually a seminal paper on how limiting forms of *difference* equations could equal the solution of the equivalent differential equation. It was noted that the random walk problem was set up as a difference equation and the solution was none other than the Green's function of the equation $\Delta u=0$. This idea was elaborated on by Duffin [13] who considered an infinite lattice in which every point was connected to its nearest neighbour by a unit resistance. If a current is introduced at some lattice point, then it will split equally among all connections and so on at every lattice point and this again mimics the random walk problem. Duffin then considered an infinite simple cubic lattice in which every lattice point is labelled (l,m,n) where each of (l,m,n) may take on every integer value from $-\infty$ to ∞. A unit current is introduced into the source point $(0,0,0)$. The Green's function $G(l,m,n)$ is defined by the solution to

$$DG(0,0,0)=-1, \qquad DG(l,m,n)=0 \quad \text{otherwise}$$

where D is the difference operator corresponding to the Laplacian differential operator Δ in three dimensions. The solution is found to be

$$G(l,m,n)=\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos lx\cos my\cos nz}{3-\cos x-\cos y-\cos z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1.10}$$

This result was ascribed to Courant [14]. A similar result was then obtained by Davies [15] without reference to Duffin or Courant. Davies noted that the resistance between the source point and a point infinitely far away was just $G(0,0,0)$. This of course is just the Watson integral (1.3). Also the resistance between the source point and any other point (l,m,n)

on the lattice was $R_{l,m,n} = G(0,0,0) - G(l,m,n)$. An excellent modern account of the relationship between random walks on lattices and resistance networks has been given by Doyle and Snell [16].

The association of WI as special values of Green's functions of a particular lattice has led to the term Green's function being applied to simple generalisations of WI. Thus it has became customary to refer to any integral of the form

$$\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w-\phi(\cos x,\cos y,\cos z)}, \tag{1.11}$$

where $\phi(\cos x, \cos y, \cos z)$ is some ternary trigonometric polynomial, as a lattice Green's function. However, it would be better to think about objects having the form (1.11) as functions of a complex variable w. What Watson had done was to evaluate the original integrals at a certain critical value of w relevant to each integral, where the integrand became infinite at the origin. It thus became a challenge to extend his analysis to general w, i.e. to evaluate

$$W_F(w_f) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_f-\cos x\cos y-\cos y\cos z-\cos z\cos x}, \tag{1.12}$$

$$W_S(w_s) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_s-\cos x-\cos y-\cos z}. \tag{1.13}$$

Then (1.12) defines a single-valued analytic function in the w_f plane, provided that a cut is made along the real axis $-1 < w_f \le 3$. A similar property holds for (1.13) provided the cut on the real axis is made for $-3 < w_s \le 3$. Also rather than having a different w for each lattice many authors considered a modified form of WI namely

$$P(u) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1-\frac{u}{d}\phi(\cos x,\cos y,\cos z)}, \tag{1.14}$$

where d is the number of terms in ϕ. Sometimes it will be more convenient to deal with P. It was the evaluation of such quantities that has exercised investigators from the mid 1950's to present times.

The first extension of Watson's result was achieved by Montroll [17]. He considered the integral

$$W_S(2+\alpha,\alpha) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{2+\alpha-\alpha\cos x-\cos y-\cos z} \tag{1.15}$$

and following *exactly* the same path as Watson, Montroll found

$$W_S(2+\alpha,\alpha) = \frac{2^{5/2}\sqrt{k_1k_2}K(k_1)K(k_2)}{\sqrt{\alpha\pi^2}}, \tag{1.16}$$

where

$$k_1 = \frac{1}{2\alpha}\big[(\sqrt{4+2\alpha}-2)(2\sqrt{1+\alpha}-\sqrt{4+2\alpha})\big],$$

$$k_2 = \frac{1}{2\alpha}\big[(\sqrt{4+2\alpha}+2)(2\sqrt{1+\alpha}-\sqrt{4+2\alpha})\big].$$

Although this indicated that further generalisations might be possible, no new methods were introduced and the chances of improving on Watson's ingenuity seemed small. Thus two big advances were made when Iwata [18] produced a closed form solution for $W_F(w_f)$, and Joyce [19, 20] who further generalised W_F, but more importantly found a closed form for $W_S(w_s)$.

2 Solutions for $W_F(w_f)$, $W_F(\alpha_f, w_f)$ and $W_S(w_s)$

Iwata's method of finding a closed form for (1.12) was as follows. He copied Watson in reducing the triple integral to a single integral but in an entirely different fashion. He made use of two successive well-known integrations, and then solved this last integral in such a way that this now has become a standard integration. The first integration was accomplished using the established result

$$\int_0^\pi \frac{\mathrm{d}z}{a - b\cos z} = \frac{\pi}{\sqrt{a^2 - b^2}}, \tag{2.1}$$

which after integrating (1.12) over z gives

$$W_F(w_f) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{(w_f - \cos x\cos y + \cos x + \cos y)(w_f - \cos x\cos y - \cos x - \cos y)}}.$$

Then substituting $\cos y = u$, and after some re-arrangement one obtains

$$W_F(w_f) = \int_0^\pi \frac{\mathrm{d}x}{\sqrt{\cos^2 x - 1}}\int_{-1}^1 \frac{\mathrm{d}u}{\sqrt{(\frac{w_f + \cos x}{\cos x - 1} - u)(\frac{w_f - \cos x}{\cos x + 1} - u)(1 - u^2)}}. \tag{2.2}$$

The standard result

$$\int_{-1}^1 \frac{\mathrm{d}u}{\sqrt{(a-u)(b-u)\big(1-u^2\big)}} = \frac{2}{\sqrt{(a-1)(b+1)}}K(k), \tag{2.3}$$

where

$$k^2 = \frac{2(a-b)}{(a-1)(b+1)}$$

is now employed on (2.2) and after some algebra $W_F(w_f)$ is obtained as a single integral namely

$$W_F(w_f) = \frac{2}{\pi^2(w_f+1)}\int_0^\pi K\left[\frac{2\sqrt{w_f + \cos^2 x}}{(w_f+1)}\right]\mathrm{d}x. \tag{2.4}$$

There are a number of other ways in which this integral may be expressed, thus

$$W_F(w_f) = \frac{2}{\pi^2(w_f+1)}\int_0^\pi K(\sqrt{A + B\cos x})\,\mathrm{d}x,$$
$$A = \frac{4w_f + 2}{(w_f+1)^2}, \qquad B = \frac{2}{(w_f+1)^2}, \tag{2.5}$$

$$W_F(w_f) = \frac{2}{\pi^2(w_f+1)} \int_0^\pi K\left(\sqrt{a^2\cos^2 x + b^2 \sin^2 x}\right) \mathrm{d}x,$$
$$a = \frac{2\sqrt{w_f+1}}{w_f+1}, \qquad b = \frac{2\sqrt{w_f}}{w_f+1}. \tag{2.6}$$

Iwata chose to work with (2.6) using the representation of K as a hypergeometric function, thus

$$\int_0^\pi K\left(\sqrt{a^2\cos^2 x + b^2\sin^2 x}\right) \mathrm{d}x$$
$$= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \int_0^\pi \left(a^2\cos^2 x + b^2 \sin^2 x\right)^n \mathrm{d}x. \tag{2.7}$$

The integral on the RHS of (2.7) can be evaluated as an infinite sum, and the double sum so produced was identified as $\mathcal{F}_4$, an Appell hypergeometric function of two variables—Whittaker and Watson [21, p. 300]. In this case a theorem derived by Bailey [22] showed that this $\mathcal{F}_4$ was in fact the product of two complete elliptic integrals. It is more straightforward to express the result in terms of (2.5) so

$$\int_0^\pi K(\sqrt{A+B\cos x})\,\mathrm{d}x = 2K(k_+)K(k_-),$$

where

$$2k_\pm^2 = 1 \pm \sqrt{A^2-B^2} - \sqrt{(1-A)^2-B^2}. \tag{2.8}$$

This result may now be accepted as a standard form. Hence for $W_F(w_f)$ we have

$$W_F(w_f) = \frac{4}{\pi^2(w_f+1)} K(k_{f+})K(k_{f-}),$$
$$2k_{f\pm}^2 = 1 \pm \frac{4\sqrt{w_f}}{(w_f+1)^{3/2}} - \frac{(w_f-1)\sqrt{w_f-3}}{(w_f+1)^{3/2}}. \tag{2.9}$$

A further advance was made by Joyce [19] who published a generalisation of Iwata. Thus he solved

$$W_F(\alpha_f, w_f) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_f - \alpha_f \cos x \cos y - \cos y \cos z - \cos z \cos x}, \tag{2.10}$$

the solution being

$$W_F(\alpha_f, w_f) = \frac{4}{\pi^2(w_f+\alpha_f)} K(k_+)K(k_-),$$

where

$$2k_\pm^2(\alpha_f, w_f) = 1 \pm \frac{4\alpha_f w_f}{(w_f+\alpha_f)^2}\left(1 + \frac{1}{w_f\alpha_f}\right)^{1/2}$$
$$- \frac{(w_f-\alpha_f)}{(w_f+\alpha_f)^2}\left[w_f + (2-\alpha_f)\right]^{1/2}\left[w_f - (2+\alpha_f)\right]^{1/2}. \tag{2.11}$$

His analysis involved showing that $W_F(\alpha_f, w_f)$ could be expressed as

$$\int_0^\infty J_0(at)J_0(bt)J_0(t)\,\mathrm{d}t, \tag{2.12}$$

where a and b are functions of w_f and α_f, and J_0 is a Bessel function. A theorem of Bailey [23] then enabled (2.12) to be evaluated directly. Joyce indicated that he would describe the process in detail later but he never did! Instead 32 years later under the prompting of colleagues the result was republished, but shown to be derived by Iwata's method. We are assured that the promised other proof will be forthcoming.

Rather interestingly in his evaluation of W_S Watson [2] had employed the same strategy as Iwata in evaluating his last integral using Appell and Bailey, but his result was confined to a special case. Also Iwata ends his paper with the remark that W_S might have a similar result to (2.6). Now one can be assured that Iwata must have tried his technique on $W_S(w_s)$, since the latter looks a considerably simpler form than $W_F(w_f)$, but if indeed one performs the first two integrations one obtains

$$\begin{aligned} W_S(w_s) &= \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_s - \cos x - \cos y - \cos z} \\ &= \frac{1}{\pi^2}\int_0^\pi \frac{2}{w_s - \cos x} K\left(\frac{2}{w_s - \cos x}\right). \end{aligned} \tag{2.13}$$

This form was actually obtained by Tikson [24]. However, this does not succumb to Iwata's approach and it seems that a different technique was required to obtain a closed form for $W_S(w_s)$. This was accomplished by Joyce [20] in a classic paper, and a brief description of his method is now given.

The procedure used by Joyce was entirely different to that of Iwata. He began by considering the P form of the simple cubic

$$P_S(u) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1 - \frac{1}{3}u(\cos x + \cos y + \cos z)} = \left(\frac{3}{u}\right) W_S\left(\frac{3}{u}\right). \tag{2.14}$$

By expanding the integrand in (2.14) in powers of u a power series for $P(u)$ was found. Thus

$$P_S(u) = \sum_{n=0}^\infty p_n u^{2n} \quad (|u| < 1), \tag{2.15}$$

where

$$p_n = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \left[\frac{1}{3}(\cos x + \cos y + \cos z)\right]^{2n} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{2.16}$$

In the theory of random walks the coefficient p_n gives the probability that a random walker will return to his starting point (not necessarily for the first time) after a walk of $2n$ steps on a SC lattice. For $n \to \infty$ the behaviour of p_n was described by an asymptotic formula given by Domb [12], which showed that the range of validity of (2.15) could be extended to $|u| = 1$. So that $P_S(1)$ which is equal to $3W_S$ may be written as

$$P_S(1) = \sum_{n=0}^\infty p_n. \tag{2.17}$$

Joyce then proceeded to find a closed form for p_n and establish a three term recurrence relation amongst the p_n. This enabled him to construct the following third order differential equation for P_S

$$4x^2(x-1)(x-9)\frac{d^3P_S}{dx^3}+12x\big(2x^2-15x+9\big)\frac{d^2P_S}{dx^2}$$
$$+3\big(9x^2-44x+12\big)\frac{dP_S}{d}x+3(x-2)P_S=0, \tag{2.18}$$

where $x=u^2$. A remarkable result of Appell [25] allowed this third order equation to be reduced to a second order equation and the solutions of this were identified as Heun functions. Then applying various standard transformation to these, they were turned into hypergeometric functions and finally produced the important result

$$\begin{aligned} W_S(w_s) &= \frac{2}{\pi^2 w_s}\frac{\sqrt{4-3t}}{1-t}K(k_+)K(k_-) \\ 2k_\pm^2 &= 1 \pm \frac{1}{2}v\sqrt{4-v}-\frac{1}{2}(2-v)\sqrt{1-v} \\ 2w_s^2 t &= w_s^2+3-\sqrt{(w_s^2-9)(w_s^2-1)}, \quad v=t/(t-1). \end{aligned} \tag{2.19}$$

I have set out here in a few words an investigation which occupies almost 40 printed pages, and once again suggest that the original paper be viewed in order to appreciate the effort put into obtaining (2.19).

3 The Watson Integrals Between 1970 and 2000

In the years that followed Iwata's solution for $W_F(w_f)$ several authors—e.g. Glasser [26], Hioe [27], Rashid [28], Montaldi [29] investigated increasingly complex forms of (1.11). All succumbed to Iwata's method. As an example we give Glasser's [26] result of (1.11) for

$$\phi = \cos x\cos y\cos z+\cos x\cos y+\cos y\cos z+\cos z\cos x$$
$$+\cos x+\cos y+\cos z \tag{3.1}$$

with solution

$$\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi\frac{dx\,dy\,dz}{w-\phi}=\frac{4}{\pi^2(w+1)}K^2(k), \quad \text{where } k^2=\frac{1}{2}\left[1-\left(\frac{w-7}{w+1}\right)^{1/2}\right]. \tag{3.2}$$

What real lattice this represents is open to question, but it is a tribute to Iwata's approach that such a complex integral was evaluated by his method.

At around this time the present author stumbled accidentally into WI and helped in making an unconvincing first contribution. To explain how this occurred more information on complete elliptic integrals needs to be given. It had been shown by Abel that when

$$\frac{K'(k)}{K(k)}=\frac{a+b\sqrt{N}}{c+d\sqrt{N}}, \tag{3.3}$$

with a, b, c, d, N all integers, then k is a root of an algebraic equation with integral coefficients which could be solved in radicals. In the particular case where $K'/K=\sqrt{N}$ the $k(N)$ are called singular values and for convenience $K[k(N)]$ will be denoted by $K[N]$. Now it has long been known for $N=1,3$ and 4 that $K[N]$ can be expressed in terms of gamma functions of rational arguments, the values for $K[1]$ and $K[3]$ having been found by Legendre [21, p. 524]. Now the result given in (1.5) for W_B contains the square of $K(1/\sqrt{2})$ which is indeed equal to $K[1]$, and the result (1.6) for W_F contains the square of $K(\{\sqrt{3}-1\}/2\sqrt{2})$ which is $K[3]$. Of course Watson knew this, hence his translation of the results in terms of K into gamma functions. However, the K involved in W_S is actually $K[6]$ and its expression in gamma functions was unknown. Glasser and Wood [30] had actually given $K[2]$ for which $k=\sqrt{2}-1$ in terms of gamma functions. This might have been extracted from a result of Ramanujan [31] who actually found the complete elliptic integral of the *second* kind for $k=\sqrt{2}-1$, but no one seems to have noticed. Now in investigating properties of the following double sums

$$\sum_{(m,n\neq 0,0)}^{\infty}(am^2+bmn+cn^2)^{-s},$$

a comparatively straightforward procedure was found for finding $K[N]$ in terms of Γ functions for many N—Zucker [32]. Indeed $K[N]$ for $N=1,\dots,16$ (excluding $N=14$ which did not succumb to this technique) were thus evaluated. On speaking to Joyce about this he sent me a paper by Selberg and Chowla [33] which proved that all $K[N]$ could—with sufficient labour—be evaluated in terms of gamma functions, but they had only given explicit values for $N=5$ and $N=7$. Thus the values found were new and included $N=6$. However, at that time I had never heard of WI. Fortunately a colleague knew a great deal about them and together we found—Glasser and Zucker [34]—

$$W_S=\frac{4\sqrt{6}}{\pi^2}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right), \tag{3.4}$$

which unfortunately was wrong—a factor of 384π had been omitted! Now I am sure every one at some time has experienced a wrong sign in a calculation or has lost a factor of 2 somewhere, but 384π requires an explanation. The reason is depressingly simple. The result found for the square of $K[6]$ was

$$K^2[6]=\frac{(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})(2+\sqrt{3})\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{384\pi} \tag{3.5}$$

and when this was substituted into (1.9) the fact that

$$(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})(2+\sqrt{3})$$

dramatically collapsed to $\sqrt{6}$ made us forget the 384π in the denominator of (3.5) in the desire to publish quickly. So the correct result is

$$W_S=\frac{\sqrt{6}}{96\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right), \tag{3.6}$$

as is pointed out by everyone who quotes the original. Later, Borwein and Zucker [35], making use of the surprising relation

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3}(2+\sqrt{3})^{1/2}$$

allowed the reduction of (3.4) to the even more compact form

$$W_S = \frac{(\sqrt{3}-1)}{96\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2. \tag{3.7}$$

A more substantial improvement to the theory was made by Joyce [36, 37]—which considerably simplified Iwata's result (2.9) for $W_F(w_f)$ and his own expression (2.19) for $W_S(w_s)$. In both cases the original results appeared as the products of two complete elliptic integrals with *different* moduli. In the above referenced papers it was shown that the results could be expressed as the *square* of a single K. In [37] by using the elliptic modular transformation of order 3, the solutions were found in a particularly simple fashion. They are given below in parametric form

$$W_F(w_f) = \frac{4\xi(1-3\xi)(1+\xi)}{(1-\xi)^3(1+3\xi)}\left[\frac{2}{\pi}K(k)\right]^2, \tag{3.8}$$

where

$$k^2 = \frac{16\xi^3}{(1-\xi)^3(1+3\xi)}, \qquad \xi = \frac{\sqrt{w_f+1}-\sqrt{w_f}}{\sqrt{w_f-3}+\sqrt{w_f}},$$

$$W_S(w_s) = \frac{(1-9\xi^4)}{w_s(1-\xi)^3(1+3\xi)}\left[\frac{2}{\pi}K(k)\right]^2, \tag{3.9}$$

where

$$k^2 = \frac{16\xi^3}{(1-\xi)^3(1+3\xi)}, \qquad \xi = \sqrt{\frac{w_s-\sqrt{w_s^2-1}}{w_s+\sqrt{w_s^2-9}}}.$$

Now define the sets of points in the w_f and w_s cut planes by C_f and C_s respectively. There has not been space to expand upon the regions of C_f and C_s over which Iwata's and Joyce's original results were valid. Suffice it to say that their range was severely limited in those sectors. The new results were valid for *all* of C_f and C_s.

The complicated structures of (3.8) and (3.9) may be simplified by applying various ${}_2F_1$ transformation formulae to the complete elliptic integrals in these formulae. For example Delves and Joyce [38] have shown that

$$W_S(w_s) = \frac{2w_s-\sqrt{w_s^2-9}}{w_s^2+3}\left[{}_2F_1\left(\frac{1}{8},\frac{3}{8};1:t(w_s)\right)\right]^2, \tag{3.10}$$

where

$$t(w_s) = \frac{16}{(w_s^2+3)^4}\left[w_s(w_s^2-5)-(w_s^2-1)\sqrt{w_s^2-9}\right]^2.$$

This result may also be used to calculate $W_S(w_s)$ at any point in C_s. For $w_s = 3$ the original Watson integral becomes

$$W_S = \left(\frac{1}{2}\right)\left[{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1 : \frac{1}{9}\right)\right]^2. \tag{3.11}$$

Then making use of a famous result of Clausen [39] this may be written as

$$W_S = \left(\frac{1}{2}\right)\left[{}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{8}; 1, 1 : \frac{1}{9}\right)\right]. \tag{3.12}$$

From this and (3.7) we find the striking result that

$$\sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1}{(48)^{2n-1}} = \frac{\sqrt{3}-1}{\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2. \tag{3.13}$$

4 The Singly Anisotropic Simple Cubic Lattice

The equation

$$W_S(\alpha_s, w_s) = \frac{1}{\pi^3}\int_0^\pi \int_0^\pi \int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_s - \alpha_s \cos x - \cos y - \cos z}, \tag{4.1}$$

is a further generalisation of the original Watson integral W_S. In terms of random walks on a cubic lattice, the spacing between lattice points in one of the dimensions is different from the other two. It might have been better to refer to (4.1) as the Watson integral for a tetragonal lattice. The investigation of Delves and Joyce [38] in finding a closed form for (4.1) is an exceptional work of analysis which again can only be described briefly here. The integrand in (4.1) was expanded in powers of $1/w_s$ and the resulting series integrated term by term. Thus

$$w_s W_S(\alpha_s, w_s) = y = \sum_{n=0}^{\infty} \mu_{2n}(\alpha_s) z^n, \tag{4.2}$$

where $z = 1/w_s^2$ and

$$\mu_{2n} = \frac{1}{\pi^3}\int_0^\pi \int_0^\pi \int_0^\pi (\alpha_s \cos x + \cos y + \cos z)^{2n} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{4.3}$$

An explicit expression was found for $\mu_{2n}(\alpha_s)$ and a complex recurrence relation established amongst $\mu_{2n+2}(\alpha_s)$, $\mu_{2n}(\alpha_s)$, $\mu_{2n-2}(\alpha_s)$, $\mu_{2n-4}(\alpha_s)$ and $\mu_{2n-6}(\alpha_s)$. From this it was shown that y was the solution of a *sixth* order differential equation $L_6(y) = 0$ where L_6 was an extremely long and intricate operator. The authors then showed it is possible to express L_6 as the product of a fourth order, L_4, and a second order differential operator and to reveal that y is the solution of L_4 only. A further step then allowed the solutions of $L_4(y) = 0$ to be expressed as a product of the solutions of two second order differential equations. Then Schwarzian transformation theory enabled both these second order differential equations to be reduced to the standard Gauss hypergeometric differential equation. Finally the hypergeometric functions which are solutions of these equations could be transformed by well-known quadratic transformations into complete elliptic integrals as in (1.9). The final solution for y is given as

$$w_s W_S(\alpha_s, w_s) = \frac{2}{\sqrt{1-(2-\alpha_s)^2 z + \sqrt{1-(2+\alpha_s)^2 z}}} \left[\frac{2}{\pi} K(k_+)\right]\left[\frac{2}{\pi} K(k_-)\right], \quad (4.4)$$

where

$$\begin{aligned} 2k_\pm^2 \equiv 2k_\pm^2(\alpha_s, z) = 1 - \left[\sqrt{1-(2-\alpha_s)^2 z} + \sqrt{1-(2+\alpha_s)^2 z}\right]^{-3} \\ \times \left[\sqrt{1+(2-\alpha_s)\sqrt{z}}\sqrt{1-(2+\alpha_s)\sqrt{z}} + \sqrt{1-(2-\alpha_s)\sqrt{z}}\sqrt{1+(2+\alpha_s)\sqrt{z}}\right] \\ \times \Big\{\pm 16z + \sqrt{1-\alpha_s^2 z}\Big[\sqrt{1+(2-\alpha_s)\sqrt{z}}\sqrt{1-(2+\alpha_s)\sqrt{z}} \\ + \sqrt{1-(2-\alpha_s)\sqrt{z}}\sqrt{1+(2+\alpha_s)\sqrt{z}}\Big]\Big\}. \quad (4.5) \end{aligned}$$

This is a few word summary of some 60 pages of analysis, and for the third time here I recommend readers to view the original paper to appreciate the tenacity that went into producing (4.4) and (4.5), to which the comment of Bornemann et al. [40] may be added. "What a triumph of dedicated men; for such problems current computer algebra systems are of little help."

Is there a more direct approach to (4.5)? It turns out that there is, but it is doubtful whether it would have been found without the inspiration of the known result. Going back to Tikson's result (2.12), and putting in the anisotropy does not complicate the last integral by very much. In fact doing the first two integrals following Iwata one obtains

$$W_S(\alpha_s, w_s) = \frac{1}{\pi^2}\int_0^\pi \frac{2}{w_s - \alpha_s \cos x} K\left(\frac{2}{w_s - \alpha_s \cos x}\right) \mathrm{d}x. \quad (4.6)$$

But now the following substitution is now made,

$$\cos x = \frac{w_s \cos\psi + \alpha_s}{w_s + \alpha_s \cos\psi}, \quad (4.7)$$

and a much simplified version of (4.6) appears, namely

$$W_S(\alpha_s, w_s) = \frac{1}{\sqrt{w_s^2 - \alpha_s^2}} \frac{2}{\pi^2} \int_0^\pi K(C + D\cos\psi)\,\mathrm{d}\psi. \quad (4.8)$$

where

$$C = \frac{2w_s}{w_s^2 - \alpha_s^2} \quad \text{and} \quad D = \frac{2\alpha_s}{w_s^2 - \alpha_s^2}. \quad (4.9)$$

Now in going through the first two steps of the Iwata process for the anisotropic face-centred cubic lattice we arrived at

$$W_F(\alpha_f, w_f) = \frac{1}{(w_f + \alpha_f)} \frac{2}{\pi^2} \int_0^\pi K(\sqrt{A + B\cos x})\,\mathrm{d}x, \quad (4.10)$$

where

$$A = \frac{2(2\alpha_f w_f + 1)}{(w_f + \alpha_f)^2} \quad \text{and} \quad B = \frac{2}{(w_f + \alpha_f)^2}, \tag{4.11}$$

and the similarity between (4.8) and (4.10) is evident. We have of course the solution of (4.10) with (2.11), and it seemed natural to look for a solution to (4.8) along the same lines as Iwata had done for (4.10). This was accomplished as follows. For convenience we write

$$\begin{aligned} I(A,B) &= \frac{2}{\pi^2}\int_0^\pi K(\sqrt{A+B\cos x})\,\mathrm{d}x, \\ J(C,D) &= \frac{2}{\pi^2}\int_0^\pi K(C+D\cos\psi)\,\mathrm{d}\psi. \end{aligned} \tag{4.12}$$

As Iwata did, the integrand of $I(A,B)$ is expanded and integrated term by term to yield a double series. However, instead of identifying this double series as an Appell hypergeometric function, $I(A,B)$ is represented by a Kampé de Feriet series. Doing the same thing to $J(C,D)$ yields a different Kampé de Feriet series, but by applying appropriate transformation formulae to the $I(A,B)$ series it can be put into the form found for $J(C,D)$. The following remarkable connection formula was found.

$$J(C,D) = (1-A+B)^{1/2} I(A,B), \tag{4.13}$$

where A and B are *appropriate* solutions of the simultaneous equations

$$C^2 = -\frac{(A-B)(1-A+B)}{(1-A+B)^2}, \qquad D^2 = \frac{2B}{(1-A+B)^2}. \tag{4.14}$$

By finding relevant solutions for A and B and substituting into (4.13), after considerable algebraic simplification a standard form for $J(C,D)$ may now be established. It is

$$J(C,D) = \frac{2}{\sqrt{(1-D)^2-C^2}+\sqrt{(1+D)^2-C^2}}\left(\frac{2}{\pi}\right)K(k_+)K(k_-), \tag{4.15}$$

where

$$\begin{aligned} 2k_\pm = 1 - \frac{1}{C}&[\sqrt{(1-D)^2-C^2}+\sqrt{(1+D)^2-C^2}]^{-3} \\ &\times 2\big[(C+D)\sqrt{1-(C-D)^2}+(C-D)\sqrt{1-(C+D)^2}\big] \\ &\times \big\{\pm 4C\sqrt{C^2-D^2}+\big[\sqrt{(1+C)^2-D^2}+\sqrt{(1-C)^2-D^2}\big]\big\}. \end{aligned} \tag{4.16}$$

This can now take its place as a standard integration alongside the result for $I(A,B)$. If the values of C and D given in (4.9) are now substituted in the above, the result obtained for $W_S(\alpha_s, w_s)$ is precisely the same as that given by Delves and Joyce [38]. A further consequence of the relationship between $I(A,B)$ and $J(C,D)$ is that a connection between $W_F(\alpha_f, w_f)$ and $W_S(\alpha_s, w_s)$ may be found. Thus

$$W_F(\alpha_f, w_f) = -\frac{2\alpha_s}{[\sqrt{w_s^2-(2+\alpha_s)^2}+\sqrt{w_s^2-(2-\alpha_s)^2}]} W_S(\alpha_s, w_s), \tag{4.17}$$

with

$$w_f = -\frac{1}{4\alpha_s}\Big\{(w_s^2 - 4 - \alpha_s^2) + \sqrt{w_s^2 - \alpha_s^2}\Big[\sqrt{w_s^2 - (2+\alpha_s)^2} + \sqrt{w_s^2 - (2-\alpha_s)^2}\Big] + \sqrt{w_s^2 - (2+\alpha_s)^2}\sqrt{w_s^2 - (2-\alpha_s)^2}\Big\}, \tag{4.18}$$

and

$$\alpha_f = -\frac{1}{4\alpha_s}\Big\{(w_s^2 - 4 - \alpha_s^2) - \sqrt{w_s^2 - \alpha_s^2}\Big[\sqrt{w_s^2 - (2+\alpha_s)^2} + \sqrt{w_s^2 - (2-\alpha_s)^2}\Big] + \sqrt{w_s^2 - (2+\alpha_s)^2}\sqrt{w_s^2 - (2-\alpha_s)^2}\Big\}. \tag{4.19}$$

The following *inverse* connection formula can also be found.

$$w_s W_S(\alpha_s, w_s) = \left[\frac{\alpha_f(w_f + 2 - \alpha_f)(w_f - 2 - \alpha_f)}{w_f(\alpha_f w_f + 1)}\right]^{1/2} w_f W_F(\alpha_f, w_f), \tag{4.20}$$

where

$$w_s^2 = -\frac{\alpha_f w_f(w_f + 2 - \alpha_f)(w_f - 2 - \alpha_f)}{(\alpha_f w_f + 1)^2}, \qquad \alpha_s = \frac{w_f + \alpha_f}{\alpha_f w_f + 1}. \tag{4.21}$$

Thus we require *either* the result for $W_F(\alpha_f, w_f)$ *or* the result for $W_S(\alpha_s, w_s)$ for the other to be determined. All is explained in detail in a paper of Joyce,Delves and Zucker [41]. The 'final problem' yet to be solved is

$$W_S(\alpha_s, \beta_s, w_s) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{w_s - \alpha_s\cos x - \beta_s\cos y - \cos z}. \tag{4.22}$$

The latter might be referred to as the Watson integral for the doubly anisotropic cubic lattice or the Watson integral for an orthorhombic lattice. The difficulties encountered so far in attempted solutions of this three parameter problem seem insuperable. Even attempts at reducing it to a Montroll type two parameter exercise namely

$$W_S(1 + \alpha_s + \beta_s) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{1 + \alpha_s + \beta_s - \alpha_s\cos x - \beta_s\cos y - \cos z} \tag{4.23}$$

have not yielded any success.

5 The Green's Function of the Simple Cubic Lattice

Recently some significant progress has been made in finding exact product forms for certain forms of

$$G(l, m, n; \alpha_s, w_s) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos lx\cos my\cos nz}{w_s - \alpha_s\cos x - \cos y - \cos z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{5.1}$$

A series of papers by Joyce and Delves [42, 43] and Delves and Joyce [44, 45] have gone into great detail in presenting their results, and these are briefly summarised here.

The new results depend on a beautiful extrapolation of Iwata's result for $I(A,B)$. To see how this extrapolation is best presented we have

$$I(A,B)=\frac{2}{\pi^2}\int_0^{\pi}K(\sqrt{A+B\cos x})\,\mathrm{d}x=\frac{1}{\pi}\int_0^{\pi}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1:A+B\cos x\right)\mathrm{d}x. \quad (5.2)$$

This may be taken to be a special case of

$$I_n(t;A,B)=\frac{1}{\pi}\int_0^{\pi}{}_2F_1(t,1-t;1:A+B\cos x)\cos nx\,\mathrm{d}x, \quad (5.3)$$

where clearly Iwata's $I(A,B)$ is $I_0(\frac{1}{2};A,B)$. Now Delves and Joyce [45] found a closed form for $I_n(t;A,B)$ namely

$$I_n(t;A,B)=\frac{(t)_n(1-t)_n}{(n!)^2}\left[\frac{1}{2B}(\sqrt{1-A-B}-\sqrt{1-A+B})^2\right] \times {}_2F_1(t,1-t;n+1:\theta_+)\,{}_2F_1(t,1-t;n+1:\theta_-) \quad (5.4)$$

where $(t)_n$ is the Pochhammer symbol given by $\Gamma(t+n)/\Gamma(n)$ and

$$2\theta_{\pm}=1\pm\sqrt{A^2-B^2}-\sqrt{(1-A)^2-B^2}. \quad (5.5)$$

This remarkable expression was first obtained by the application of a Lie group addition formula applicable to ${}_2F_1$ first obtained by Miller [46]. It was later confirmed by generalising Iwata's original approach. For some particular choices of (l,m,n) the Fourier transform of the Green's function may be reduced to an elliptic integral. Some of these elliptic integrals correspond to Miller's form and the Green's function can be evaluated in a few pages of hand calculation. For example, $W_S(2n,n,n;\alpha_s,w_s)$ was shown to be

$$W_S(2n,n,n;\alpha_s,w_s)=\left(\frac{1}{w_s^2+4-\alpha_s^2}\right)^{\frac{1}{2}} \times\frac{1}{\pi}\int_0^{\pi}{}_2F_1\left[\frac{1}{4},\frac{3}{4};1:8\frac{(2w_s^2-\alpha_s^2+\alpha_s^2\cos x)}{(w_s^2+4-\alpha_s^2)^2}\right]\cos(nx)\,\mathrm{d}x. \quad (5.6)$$

(It is noteworthy that in obtaining (5.6) the authors met a complex trigonometric integral whose solution as an elliptic integral was found in Jacobi [47]. An alternative procedure originated in Cayley [48, 49] combined with an ${}_2F_1$ transformation of Goursat [50]. When stuck go to the experts.)

Equation (5.6) is clearly of the form (5.4) and thus the following closed form is obtained

$$W_S(2n,n,n;\alpha_s,w_s)=\left(\frac{1}{w_s^2+4-\alpha_s^2}\right)^{\frac{1}{2}}\frac{(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^2} \times\left[\frac{1}{8\alpha_s}\left(\sqrt{w_s^2-(2-\alpha_s)^2}-\sqrt{w_s^2-(2+\alpha_s)^2}\right)^2\right]^{2n} \times{}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1:\eta_+\right){}_2F_1\left(\frac{1}{4},\frac{3}{4};n+1:\eta_-\right), \quad (5.7)$$

where

$$2\eta_{\pm} = 1 + \frac{w_s}{(w_s^2+4-\alpha_s^2)^2}\Big[\pm 16\sqrt{w_s^2-\alpha_s^2} - (w_s^2-4-\alpha_s^2)\sqrt{w_s^2-(2-\alpha)^2}\sqrt{w_s^2-(2+\alpha)^2}\Big]. \tag{5.8}$$

Similarly

$$w_s W_S(n,n,n;1,w_s) = \frac{1}{\pi}\int_0^{\pi} {}_2F_1\left[\frac{1}{3},\frac{2}{3};1:\frac{27}{4w_s^3}(w_s+\cos x)\right]\cos(nx)\,\mathrm{d}x, \tag{5.9}$$

gives the closed form

$$w_s W_S(n,n,n;1,w_s) = \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^2}\left[\frac{1}{3}(w_s-\sqrt{w_s^2-9})\right]^{3n} \times {}_2F_1\left(\frac{1}{3},\frac{2}{3};n+1:\xi_+\right){}_2F_1\left(\frac{1}{3},\frac{2}{3};n+1:\xi_-\right), \tag{5.10}$$

with

$$\xi_{\pm}(w_s) = \frac{1}{8w_s^2}\left[4w_s^2+(9-4w_s^2)\sqrt{1-\frac{9}{w_s^2}} \pm 27\sqrt{1-\frac{1}{w_s^2}}\right]. \tag{5.11}$$

Interestingly if in (5.7) and (5.10) n is made zero and w_s put equal to 3, two new formulae for the original Watson integral W_S may be obtained. Along with (1.9) we have

$$\begin{aligned} W_S &= (18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};1:\left\{(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\right\}^2\right)\right]^2 \\ &= \left(\frac{1}{2}\right)\left[{}_2F_1\left(\frac{1}{8},\frac{3}{8};1:\frac{1}{9}\right)\right]^2 = \left(\frac{1}{2}\right)\left[{}_2F_1\left(\frac{1}{4},\frac{3}{4};1:\frac{(\sqrt{2}-1)^2}{6}\right)\right]^2 \\ &= \left(\frac{\sqrt{2}}{3}\right)\left[{}_2F_1\left(\frac{1}{3},\frac{2}{3};1:\frac{(2-\sqrt{2})^2}{4}\right)\right]^2 = \frac{(\sqrt{3}-1)}{96\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2. \end{aligned} \tag{5.12}$$

I believe it was Littlewood who once remarked “All equations are identities.” Does (5.12) support or give the lie to this statement?

6 Generalisations and Recent Manifestations of Watson Integrals

One obvious generalisation of WI is to higher dimensions. A recent survey by Guttmann [51] is an excellent source of information on these objects. Guttmann notes that for 2-d lattices WI are given in terms of a single K. For 3-d lattices the solutions appear as products of two complete elliptic integrals $K(k_+)K(k_-)$. It appears that this occurs because the underlying third order ordinary differential equation (ODE) obeyed by these lattices has the almost-magical Appell (1880) [25] reduction property allowing their solution to be expressed in terms of an associated second order ODE. For 4-d lattices it appears that the underlying ODE’s are all of the Calabi-Yau type. This is true for all 5-d lattices as well except for the 5-d FCC lattice.

Let us consider the 4-d simple cubic lattice in more detail. We have

$$W_{S,4d}(w_s) = \frac{1}{\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{w_s - \cos x - \cos y - \cos z - \cos t}. \tag{6.1}$$

It's P form is

$$P_{S,4d}(u) = \frac{1}{\pi^4}\int_0^\pi\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t}{1 - \frac{u}{4}u(\cos x + \cos y + \cos z + \cos t)}$$
$$= \left(\frac{4}{u}\right) W_S\left(\frac{4}{u}\right). \tag{6.2}$$

Now it has been shown that the d-dimensional diamond (D) lattice is simply related to the $(d+1)$ dimensional hypercubic lattice via an Abel transform—Guttmann and Prellberg [52], Glasser and Montaldi [53]. The D lattice has not been previously discussed since in 3-d the analysis involved is identical to the FCC case. In terms of their respective P functions we have

$$\frac{1}{4u+4} P_{D,3d}\left(\frac{4}{1+u}\right) = \frac{1}{4u} P_{F,3d}\left(\frac{3}{u}\right). \tag{6.3}$$

As can be seen when $u = 3$ the two are identical apart from a numerical factor. Since everything about the 3-d diamond lattice is known, this suggests an interesting approach to the 4-d simple cubic lattice. Indeed it may be shown that the 4-d simple cubic P form may be expressed as a single integral—Guttmann [51]—namely

$$P_{S,4d}(u) = \frac{8}{\pi^3}\int_0^1 \frac{K(k_+)K(k_-)}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad \text{where}$$
$$k_\pm = \frac{1}{2} \pm \frac{1}{4}u^2t^2\sqrt{4-u^2 * t^2} - \frac{1}{4}(2-u^2t^2)\sqrt{1-u^2t^2}. \tag{6.4}$$

A single integral for $W_{S,4d}(w_s)$ was obtained by the author by doing two integrals following Iwata and a third integral as detailed by Joyce and Zucker [54] leading to

$$W_{S,4d}(w_s) = \frac{8}{\pi^3}\int_0^\pi \frac{K^2(k^2)p\sqrt{(1-p^2)(1-9p^2)}}{(1-p)^3(1+3p)}\,\mathrm{d}t, \tag{6.5}$$

where

$$\gamma = w_s - \cos t, \quad p^2 = \frac{\gamma - \sqrt{\gamma^2 - 1}}{\gamma + \sqrt{\gamma^2 - 9}}, \quad \text{and} \quad k^2 = \frac{16p^3}{(1-p)^3(1+p)}.$$

Both (6.4) and (6.5) will easily provide numerical values for any w_s and u, but one is no closer to a closed form evaluation.

It has already been noted in Sect. 2 that Bessel functions are connected to WI. For example it is simple to express W_S as a single integral as follows. First put W_S

$$W_S = \frac{1}{\pi^3}\int_0^\infty\int_0^\pi\int_0^\pi\int_0^\pi e^{t(-3+\cos x+\cos y+\cos z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t, \tag{6.6}$$

and since

$$\frac{1}{\pi}\int_0^{\pi} e^{t\cos x}\,\mathrm{d}x = I_0(t), \tag{6.7}$$

then

$$W_S = \int_0^{\infty} e^{-3t} I_0^3(t)\,\mathrm{d}t, \tag{6.8}$$

where $I_0(t)$ is a modified Bessel function of the first kind. Now in a recent investigation Bailey et al. [55] have studied integrals of the form

$$c_{n,k} = \int_0^{\infty} t^k K_0^n(t)\,\mathrm{d}t \quad \text{and} \quad t_{n,2k+1} = \int_0^{\infty} t^{2k+1} I_0^2(t) K_0^{n-2}(t)\,\mathrm{d}t, \tag{6.9}$$

amongst many others. Here K_0 is a modified Bessel function of the second kind and should not be confused with a complete elliptic integral of the first kind. These integrals arose from certain Feynmann diagrams, and connections with WI in 3-d have been found though some are still conjectures. Thus it has been established that

$$c_{3,0} = \int_0^{\infty} K_0^3(t)\,\mathrm{d}t = \frac{\pi^3}{2} W_F(3), \tag{6.10}$$

and conjectured that

$$t_{5,1} = \int_0^{\infty} t I_0^2(t) K_0^3(t)\,\mathrm{d}t = \pi^2 W_F(15). \tag{6.11}$$

Also Broadhurst [56] has proved

$$\int_0^{\infty} t I_o^2(t) K_0^2(t) K_0(2t)\,\mathrm{d}t = \frac{\pi^2}{12} W_F(3). \tag{6.12}$$

This may just be the tip of an iceberg and many other similar results possibly remain to be discovered.

Another appearance of WI is in the field of Mahler measures. The connection arises since the integrals may be expressed as the derivative with respect to the appropriate complex variable w of a logarithmic form. For example,

$$W_S(w_s) = \frac{\mathrm{d}}{\mathrm{d}w_s}\frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \log(w_s - \cos x - \cos y - \cos z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{6.13}$$

These logarithmic integrals are involved in the calculation of the total number of spanning trees on a hypercubic lattice (Rosengren [57]), and in the theory of collapsing branch polymers (Madras et al. [58]). A rapid way of evaluating these integrals in all dimensions was devised by Joyce and Zucker [59]. The definition of a Mahler measure, m, of an n-variable polynomial $P(z_1, \ldots, z_n)$ is (the P here is not to be confused with a modified WI)

$$m[P(z_1, \ldots, z_n)] = \int_0^{\infty} \cdots \int_0^{\infty} \log\left|P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right)\right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \tag{6.14}$$

It was noted by Rogers [60] that two of the Mahler measures he was interested in namely

$$g_1(u) = m\left(u + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}\right),$$
$$g_2(u) = m\big[-u + 4 + (x + x^{-1})(y + y^{-1}) + (y + y^{-1})(z + z^{-1}) + (z + z^{-1})(x + x^{-1})\big] \tag{6.15}$$

were closely related to the SC and FCC versions of WI. This clearly came as a surprise to Rogers, as much as it did to the present author, who until now had zero knowledge of Mahler measures. The appearance of WI in these unexpected places shows their ubiquitous nature, and this seems an appropriate juncture to end this review.

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