Landau resonance mechanism for plasma and wind-generated water waves

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A profound analogy between the Landau damping of plasma waves and the resonant mechanism of wave generation on a water surface by wind is discussed. A new simple derivation of the former effect is presented which explicitly demonstrates the essential role of the integrals of motion (the vorticity for an inviscid fluid, and the particle distribution function for a collisionless plasma) in these phenomena.

I. INTRODUCTION

Although collisionless damping of plasma waves, predicted by Landau in 1946,[1] is now one of the bases of modern plasma physics, it was not until the early 60's that this effect was fully understood and accepted by the wider plasma physics community. The difficulty was with the physics behind this ''Landau damping.'' How can it occur in a system without dissipation such as a collisionless plasma? The breakthrough came when the physical mechanism, namely, interaction of a wave with the resonant particles, had been elucidated in Ref. 2 and Landau damping had been observed experimentally[3] (see also the recent discussions in Refs. 4 and 5).

Later, it was realized that such resonant damping is not a unique feature of plasmas, but is just one manifestation of the universal phenomenon of wave damping (or amplification) in a medium with a continuous oscillation spectrum. Similar effects are now well known in such diverse fields as liquids with gas bubbles,[6] high energy particle beams,[7] superfluids,[8] and quarks.[9] The analogs of Landau damping have also been discovered in biology, ranging from the flashing of fireflies[10] to pacemaker cells controlling the beating of the heart.[11]

The list can be extended with even more exotic (to a physicist's eye) examples, but the aim of the present paper is to discuss an example with which everyone is very familiar, namely, wind-generated waves on a water surface. It is amazing that this phenomenon, observed for millennia, has received a proper theoretical explanation only quite recently.[12] According to this theory, a wave is generated by resonant energy transfer from the wind layer, which moves with a velocity equal to the phase velocity of the surface wave, a mechanism analogous to Landau damping. Various mathematical aspects of such resonant effects in fluids and plasmas were reviewed in Ref. 13. The present paper, instead, puts emphasis on the physics behind the two processes. Although the theory,[12] as well as its remarkable physical interpretation,[14] were developed at the same time as the resonant wave–particle interaction in plasma physics, a straightforward analogy between these effects was not realized at the time, and is not so well known, even today.

The paper is organized as follows. In Sec. II general properties of the surface waves are summarized, and it is also shown why the Kelvin–Helmholtz instability is irrelevant in the wave generation mechanism. In Sec. III the resonant wave–wind interaction is discussed, based on the approach suggested in Ref. 14. Then, in Sec. IV a simple derivation of Landau damping, inspired by Ref. 14, is presented. It shows explicitly how similar these mechanisms are, and also clarifies the important role of the integrals of motion (the vorticity in an inviscid fluid, the particle distribution function in a collisionless plasma), which originate from the absence of dissipation.

II. SURFACE WAVES: BASIC PROPERTIES AND STABILITY

Let us start with the well-known dispersion equation for a surface wave on water which relates the frequency ω to the wave vector k (the wavelength $\lambda=2\pi/k$):

$$\omega=\left(gk+\frac{\alpha k^3}{\rho}\right)^{1/2}, \tag{1}$$

where g is the gravitational acceleration, α is the surface tension coefficient, and ρ is the density of water. As seen from (1), the restoring force which produces these surface oscillations originates from gravity and surface tension. The former dominates for large wavelengths ($k<k^*=\sqrt{\rho g/\alpha}$), giving $\omega=\sqrt{gk}$—the gravity waves. The small scale oscillations ($k>k^*$) are due entirely to surface tension, and for them $\omega=\sqrt{\alpha k^3/\rho}$—the capillary or surface tension waves. Putting realistic numbers ($g\approx 9.8$ m/s^2, $\alpha\approx 7.3\times 10^{-2}$ N/m, $\rho\approx 10^3$ kg/m^3) into these formulas, one finds that $k^*\approx 3.7\times 10^2$ m^{-1}, therefore small-scale ripples on a water surface with a wavelength $\lambda<\lambda^*=2\pi/k^*\approx 1.7$ cm are capillary waves. Other important characteristics of the wave motion are its phase velocity $v_p=\omega/k$ and group velocity $v_g=\partial\omega/\partial k$. For the gravity-capillary waves (1),

$$v_p=\left(\frac{g}{k}+\frac{\alpha k}{\rho}\right)^{1/2}, \quad v_g=\frac{(g+3\alpha k^2/\rho)}{2(gk+\alpha k^3/\rho)^{1/2}}. \tag{2}$$

Thus, both v_p and v_g increase for long ($k\rightarrow 0$) and short ($k\rightarrow\propto$) wavelengths, having their minimum at $k\sim k_*$. For the phase velocity, which is of particular interest in what follows, the minimum at $k=k_*$ is

$$v_p^{(\min)}=(4\alpha g/\rho)^{1/4}\approx 0.23 \text{ m/s}. \tag{3}$$

Since this velocity is far smaller than the sound speed in both water and air, these fluids can be assumed to be incompressible as far as generation of surface waves is concerned.

What difference does the air motion (wind) make, and what is the critical wind velocity required for the generation of surface waves? The simplest model used for tackling this question is the ''sharp boundary'' model, for which the initial equilibrium comprises a tangential discontinuity between the stationary water occupying the half-space $z<0$, and the uniformly moving air above ($z>0$). Wave generation occurs when the amplitude of a small wave-like boundary surface perturbation grows with time, i.e., the initial wave-free equilibrium is unstable.

Suppose the wind speed is equal to V_0 directed along the x axis. It is then reasonable to assume that the most unstable waves are those propagating in the same direction as the wind. Therefore it is sufficient to consider only two-dimensional fluid motion, for which all characteristics depend only on x and the vertical coordinate z. As mentioned above, both air and water can be considered here as incompressible fluids for which the velocity $\mathbf{v}$ is divergence-free: $\nabla\cdot\mathbf{v}=0$. For y-invariant flows this condition is satisfied identically if $\mathbf{v}$ is represented in the form

$$\mathbf{v}(x,z,t)=\nabla\psi(x,z,t)\times\hat{\mathbf{y}}, \quad \text{i.e.,} \quad v_x=-\partial\psi/\partial z, \quad v_z=\partial\psi/\partial x, \tag{4}$$

where ψ is the so-called ''streamfunction'' [the name originates from the fact that streamlines of the flow (4) are identified by the condition $\psi=$const]. Since in the absence of a surface wave air is already moving with the velocity V_0 along the x axis, its streamfunction can be written as

$$\psi'=-V_0 z+\psi_1'(x,z,t), \tag{5}$$

where the second term, ψ_1', is due to the wave perturbation. The corresponding streamfunction in water, which is initially at rest, is then simply $\psi_1(x,z,t)$.

The variation of ψ_1 and ψ_1' with x and t for a wave-like perturbation is of the form

$$\psi_1(x,z,t)=f(z)e^{i(kx-\omega t)}, \quad \psi_1'(x,z,t)=f'(z)e^{i(kx-\omega t)}, \tag{6}$$

where functions f and f' can be derived from the equations of motion for water,

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\rho\mathbf{g}, \tag{7}$$

and the similar one for air. Taking the curl of (7), one concludes that

$$\nabla^2\psi_1=\nabla^2\psi_1'=0,$$

so together with (6) this yields the well-known solutions

$$f(z)=Ae^{kz}, \quad z\leqslant 0; \quad f'(z)=Be^{-kz}, \quad z\geqslant 0, \tag{8}$$

which show that the thickness of the fluid layer involved in the motion is of the order of the perturbation wavelength λ.

The constants A and B in (8) are related to the amplitude ξ_0 of the vertical displacement of the water surface,

$$\xi_s(x,t)=\xi_0 e^{i(kx-\omega t)}, \tag{9}$$

as $A=-(\omega/k)\xi_0$, $\quad B=(-\omega/k+V_0)\xi_0$.

To obtain the frequency ω which determines stability (or instability) of the system, one needs to derive first the pressure perturbations in water and air, δp and $\delta p'$, respectively, using the linearized equation of motion (7). The result is

$$\delta p=\frac{\omega^2}{k}\rho\xi_0 e^{kz}e^{i(kx-\omega t)}, \tag{10a}$$

$$\delta p'=-\frac{(\omega-kV_0)^2}{k}\rho'\xi_0 e^{-kz}e^{i(kx-\omega t)}. \tag{10b}$$

Then, matching δp and $\delta p'$ at the boundary surface $z=\xi_s(x,t)$ by the Laplace formula

$$(p'+\delta p')_{z=\xi_s}-(p+\delta p)_{z=\xi_r}=\alpha/R,$$

where R is the radius of curvature of the water surface (p and p' are the initial equilibrium pressure in water and air), the following dispersion equation for $\omega(k)$ is found:

$$\omega=\frac{\epsilon k V_0}{(1+\epsilon)}\pm\left[\frac{\epsilon^2k^2V_0^2}{(1+\epsilon)^2}+\left(gk\,\frac{(1-\epsilon)}{(1+\epsilon)}+\frac{\alpha k^3}{\rho(1+\epsilon)}-\epsilon k^2V_0^2\right)\right]^{1/2}, \tag{11}$$

where $\epsilon=\rho'/\rho$ is the air–water density ratio. Since in normal conditions the latter is very small ($\epsilon\cong 0.0012$), the simple dispersion law (1) follows from (11) for $V_0=0$ (no wind). For the very same reason a high wind velocity is required to make a difference in the dispersion equation (11). Indeed, the surface perturbation becomes unstable when $\mathrm{Im}\,\omega>0$, and this occurs only if

$$V_0>\epsilon^{-1/2}\left(\frac{g}{k}+\frac{\alpha k}{\rho}\right)^{1/2}. \tag{12}$$

Comparing (12) with (2), one easily finds that a perturbation with a given wavelength is unstable if the wind velocity V_0 exceeds the phase velocity of the corresponding surface wave (the wave with the same k) times the large factor $\epsilon^{-1/2}\cong 29$. Therefore, the minimum wind velocity $V_0^{(\min)}$ required for instability is determined by the surface wave with the minimum phase velocity, which is a ripple with $\lambda\cong 1.7$ cm [see Eq. (3)], and $V_0^{(\min)}=\epsilon^{-1/2}v_p^{(\min)}\approx 6.6$ m/s. This is already quite a strong wind, but it needs almost a gale force wind to generate longer waves (for example, $V_0\cong 36$ m/s for $\lambda\cong 1$ m), which is in apparent contradiction with the fact that even a light breeze generates water surface waves.

Thus the above mechanism, commonly known as Kelvin–Helmholtz instability, is irrelevant to this phenomenon. It is also worth noting that this instability of the air–water boundary, even when it does occur, does not generate surface waves because they actually no longer exist: the required airflow is so strong that it overcomes the restoring forces of gravity and surface tension which are responsible for such waves. That is why the mechanism of this robust hydrodynamic instability is called a ''stiffness reduction'': as the wind speed increases, it first reduces the wave frequency and eventually gives ''negative stiffness'', i.e., instability [see the dispersion equation (11)].

III. SURFACE WAVES GENERATION BY RESONANT WIND–WAVE INTERACTION

The actual mechanism of water wave generation by wind cannot be understood in the framework of the above sharp-boundary model, which assumes a uniform airflow above the water. In reality, a finite boundary layer with a sheared air flow is always present, and this is at the heart of the problem. Inside this layer the velocity of airflow (say, still in the x direction), $V(z)$, gradually increases from zero at the water surface to the maximum value V_0, which we will call, as in the previous section, the wind speed. If at some height $z=z_r$ the air velocity $V(z_r)$ is equal to the phase velocity of a surface wave propagating in the same direction, a resonant interaction between the air flow and the wave becomes possible, which results in surface wave instability.

Since the air velocity required for resonant interaction is much lower than that in (12), such a wind has no effect on the gravity-capillary restoring force, so the simple dispersion law (1) holds for surface waves. In other words, this wave generation mechanism represents a weak wind–wave coupling (contrary to the Kelvin–Helmholtz instability, which requires a strong coupling), such that the amplitude of the generated wave grows in time due to a slow energy and momentum transfer from the airflow. Therefore let us turn now to the details of this transfer, in which an essential role is played by the vorticity $\Omega=\nabla\times\mathbf{v}'$ of the airflow. By the way, this is another important difference between the above sharp-boundary instability, where the flow of air and water is potential, and the resonant wind–wave interaction, where vorticity in the airflow comes from the initial sheared velocity profile $V(z)$ inside the boundary layer. In an inviscid fluid there is a conservation law of the velocity circulation known as the Kelvin theorem (see, e.g., Ref. 15). Here, where airflow is also incompressible and two dimensional, a stricter conservation of the vorticity itself holds. Indeed, the vector identity,

$$\mathbf{v}\times(\nabla\times\mathbf{v})=\nabla(v^2/2)-(\mathbf{v}\nabla)\mathbf{v},$$

allows us to rewrite the equation of motion (7) as

$$\frac{\partial v'}{\partial t}=(\mathbf{v}'\times\Omega)-\nabla\left(\frac{p'}{\rho'}+\frac{v'^2}{2}+gz\right). \tag{13}$$

Taking the curl of (13) yields

$$\frac{\partial\Omega}{\partial t}=\nabla\times(\mathbf{v}'\times\Omega)=\mathbf{v}'(\nabla\Omega)+(\Omega\nabla)\mathbf{v}'-\Omega(\nabla\mathbf{v}')-(\mathbf{v}'\nabla)\Omega.$$

Since $\nabla\mathbf{v}'=\nabla\Omega=0$, and because for two-dimensional, y-invariant flow the vorticity has only one nonzero component $\Omega_y=\Omega(x,z)$, only the last term on the right-hand side of the above equation does not vanish, giving

$$\frac{\partial\Omega}{\partial t}+(\mathbf{v}'\nabla)\Omega\equiv\frac{d\Omega}{dt}=0, \tag{14}$$

which shows that vorticity $\Omega(x,z)$ is conserved for each ''fluid element'' in the course of its motion.

The airflow velocity is a superposition of the initial sheared component and the perturbation caused by the surface wave

$$\mathbf{v}'(x,z,t)=V(z)\hat{\mathbf{x}}+\mathbf{u}(z)\sin(kx-\omega t). \tag{15}$$

Let us calculate now the rate of change of the x component of the air momentum per unit horizontal area by integrating the x component of (13) over z and averaging it over a wavelength along x. Then

$$\frac{dP_x'}{dt}=\int_0^\infty dz\rho'\left\langle\frac{\partial v_x'}{\partial t}\right\rangle=\int_0^\infty dz\rho'\langle(\mathbf{v}'\times\Omega)_x\rangle=-\rho'\int_0^\infty dz\langle v_z'\Omega\rangle, \tag{16}$$

where $\langle\ \rangle$ signifies the above average. [The last term in (13), being the gradient of a function periodic in x, makes a zero average contribution.] It is seen from (16) that to get a finite averaged result one needs to take into account the variations in vorticity because $\langle \nu_z' \rangle = 0$ according to (15). Since Ω is conserved not locally but for a moving fluid element, it is convenient to consider the trajectory of such an air element, say one which initially (at $t=0$) was located at $x=x_0$ and $z=z_0$. Due to the starting sheared flow with the velocity $V(z_0)$, its x-coordinate is then

$$x(t)=x_0+V(z_0)t,$$

so the vertical velocity acquired by this element due to the presence of the surface wave, according to (15), is

$$\begin{aligned}\dot{z}&=u_z(z_0)\sin[k(x_0+V(z_0)t)-\omega t]\\&=u_z(z_0)\sin[kx_0-(\omega-kV(z_0))t].\end{aligned} \tag{17}$$

Therefore, it also undergoes a vertical displacement Δz, so its height varies with time as

$$z=z_0+\Delta z=z_0+\frac{u_z(z_0)}{[\omega-kV(z_0)]}\times\{\cos[kx_0-(\omega-kV(z_0)t)]-\cos kx_0\}. \tag{18}$$

This results in variations of the local vorticity as well, because air elements with different initial location z_0 reach a fixed height z and bring with them their own conserved vorticity $\Omega_0(z_0)=V'(z_0)$, as determined by the initial sheared flow $V(z)$.

Then it follows from (18) that

$$\begin{aligned}\Omega(z)&=\Omega_0(z_0)\\&=V'(z-\Delta z)\\&\approx V'(z)-V''(z)\Delta z\\&=V'(z)-V''(z)\frac{u_z(z)}{[\omega-kV(z)]}\\&\quad\times\{\cos[kx_0-(\omega-kV(z))t]-\cos kx_0\},\end{aligned} \tag{19}$$

so the integrand in (16) can be written as

$$\left\langle \nu_z'\Omega\right\rangle=-\left\langle u_z(z)\sin[kx_0-(\omega-kV(z))t]\times V''(z)\frac{u_z(z)}{[\omega-kV(z)]}\times\{\cos[kx_0-(\omega-kV(z))t]-\cos kx_0\}\right\rangle.$$

After averaging the above expression over x_0 (using $\langle \sin^2 kx_0\rangle=\langle\cos^2 kx_0\rangle=\frac{1}{2}$, $\langle \sin kx_0 \cos kx_0\rangle=0$) and inserting the result into (16), one comes to the following rate of change for the x component of the air momentum:

$$\frac{dP_x'}{dt}=\frac{\rho'}{2}\int_0^\infty u_z^2(z)V''(z)dz\times\frac{\sin[\omega-kV(z)]t}{[\omega-kV(z)]}. \tag{20}$$

Though the integral in (20) is formally carried over the whole half-space $z>0$ occupied by air, the main contribution comes from the very thin layer around the level $z=z_r$, where the resonance condition,

$$\omega-kV(z_r)=0, \tag{21}$$

is satisfied. This is because the factor

$$f(z)=\frac{\sin[\omega-kV(z)]t}{[\omega-kV(z)]}$$

in the integrand of (20) is sharply peaked at $z=z_r$, with the amplitude increasing and the width decreasing as time progresses. Therefore after several wave periods it can be replaced by a delta function, giving

$$f(z)\approx\pi\delta[\omega-kV(z)]=\frac{\pi}{k|V'(z_r)|}\delta(z-z_r),$$

which makes the momentum transfer rate (20) equal to

$$\frac{dP_x'}{dt}=\frac{\pi\rho'}{2k}\frac{u_z^2(z_r)}{|V'(z_r)|}V''(z_r). \tag{22}$$

It follows then from (22) that if $V''(z_r)<0$, the momentum of the airflow is transferred to the surface wave, thus providing a mechanism for instability. If, however, $V''(z_r)>0$, the momentum transfer is in the opposite direction, and the wave damps. Therefore, it may seem at first glance that whether or not wind generates water waves is decided more or less by chance, depending on particular details of the velocity profile $V(z)$. Nature is, however, strongly in favor of wave generation. The reason is that the viscosity of air is so small ($\eta'\approx 1.8\times 10^{-5}$ N s/m^2) that even a weak wind is associated with an airflow with large Reynolds number (for example, airflow with the velocity $v'\sim 1$ m/s and scale length $L\sim 1$ m has Reynolds number $R=Lv'\rho'/\eta'\approx 10^5$). Therefore, this airflow is normally turbulent, and the above function $V(z)$ has to be understood as the mean velocity profile averaged over numerous turbulent eddies. But such a function possesses the universal behavior known as the ''logarithmic profile'' (see, e.g., Ref. 16), for which $V(z)\propto\log z$; hence $V''(z_r)<0$. That is why wind above a water surface makes waves, provided that the resonant condition (21) holds at some height. This occurs if the wind speed V_0 exceeds the phase velocity of a surface wave; therefore, in principle, a wind speed of $V_0=0.23$ m/s suffices to ripple a water surface, while $V_0=1.25$ m/s is enough to generate surface gravity waves with $\lambda=1$ m.

It is worth noting at this point that resonant condition (21) applies only for waves propagating parallel to the wind (in the x direction above). However, waves at a nonzero angle to the wind can be resonantly generated as well. For this more general case the condition (21) has to be replaced by

$$\omega-\mathbf{k}\cdot\mathbf{V}(z_r)=0, \tag{23}$$

where both $\mathbf{k}$ and $\mathbf{V}$ are now two-dimensional vectors in the horizontal plane. It is no surprise that (23) resembles the general Landau resonance in a plasma.[1]

The important conclusion which is apparent from (21) and (23) is that the critical wind speed required for the resonant generation of surface waves is determined only by the wave's phase velocity, and thus does not depend at all on the air density ρ'.

Where ρ' does matter is in the rate of this generation, because, according to (22), the rate of momentum transfer is proportional to ρ', and so is the wave growth rate γ. To derive this one has to compare (22) with the momentum carried by the surface wave per unit horizontal area. Alternatively, one can calculate the energy transfer rate and compare with the energy of the surface wave. The fact that the result will be the same (as, of course, it should be) becomes apparent from the following simple consideration. Since the momentum transferred from air to wave comes from the narrow resonant layer which moves with the velocity $V(z_r)$, the energy transfer rate dW'/dt is related to the momentum transfer as

$$\frac{dW'}{dt}=V(z_r)\,\frac{dP'}{dt}=\frac{\omega}{k}\,\frac{dP'}{dt}.$$

But the energy W and momentum P of the surface wave with frequency ω and wave vector k have the same relation $W=P\omega/k$ (see, e.g., Ref. 17).

The momentum of a surface wave, to which air has a negligible contribution, can be obtained using the water stream function ψ_1 derived in Sec. II:

$$\psi_1(x,z,t)=-\frac{\omega}{k}\,\xi_0 e^{kz}\cos(kx-\omega t). \tag{24}$$

Though it may seem from (24) that there is no net momentum because $\langle \mathbf{v}\rangle=0$ for any fixed level z in water, the nonzero result comes from the correlation between the horizontal velocity $v_x=-\partial\psi/\partial z$ and water surface elevation ξ_s [see Eq. (9)]: for a wave which propagates to the right along the x axis, v_x is positive for crests where $\xi_s>0$, but is negative for troughs where $\xi_s<0$. Therefore at any time a slightly larger volume of water is moving to the right than to the left, thus providing the net momentum. Putting it formally, it follows from (24) that

$$\begin{aligned} P_x&=\left\langle \int_{-\infty}^{\xi_s}\rho v_x\right\rangle \\ &=-\rho\left\langle \int_{-\infty}^{\xi_s}\frac{\partial\psi_1}{\partial z}\,dz\right\rangle \\ &=-\rho\langle\psi_1(\xi_s)\rangle \\ &\approx\rho\,\frac{\omega}{k}\,\xi_0\langle(1+k\xi_s)\cos(kx-\omega t)\rangle \\ &=\rho\,\frac{\omega}{k}\,\xi_0\langle[1+k\xi_0\cos(kx-\omega t)]\cos(kx-\omega t)\rangle \\ &=\rho\omega\xi_0^2/2. \end{aligned} \tag{25}$$

Then the growth rate of the surface wave is

$$\gamma=-\frac{dP_x'/dt}{2P_x}=-\omega\pi\,\frac{\rho'}{\rho}\,\frac{u_z^2(z_r)}{\omega^2\xi_0^2}\,\frac{V''(z_r)}{kIV'(z_r)I}. \tag{26}$$

Let us make now a simple estimate of γ. The derivatives of the sheared flow velocity which enter (26) can be estimated as $V'(z_r)\sim V_0/z_r$, $V''(z_r)\sim V_0/z_r^2$, while $\omega\xi_0$ is the vertical velocity amplitude at the water surface, i.e., $u_z(0)$. Thus (26) yields

$$\frac{\gamma}{\omega}\sim\frac{\pi}{kz_r}\,\frac{\rho'}{\rho}\,\frac{u_z^2(z_r)}{u_z^2(0)}. \tag{27}$$

Since, according to (8), perturbations of the air flow caused by the surface wave decay with height like $\exp(-kz)$, resonant wave generation by wind is effective only for $kz_r\leqslant 1$ [otherwise the ratio $u_z^2(z_r)/u_z^2(0)$ in (27) is exponentially small]. Therefore, for $kz_r\sim 1$ it follows from (27) that the growth rate ratio

$$\frac{\gamma}{\omega}\sim\frac{\rho'}{\rho}\sim 10^{-3},$$

i.e. the wave growth time $\tau\sim\gamma^{-1}$ is equal to about a hundred wave periods $T=2\pi/\omega$.

Two remarks are worth making here. First, since $\gamma/\omega\ll 1$, the amplitude of a surface wave increases only slightly over the wave period T. This allows the assumption that the wave amplitude is a constant in the above derivation of the momentum transfer rate. Second, this resonant mechanism describes only the primary transfer of energy and momentum from wind to water. The saturation stage of the resulting surface wave instability is determined by the secondary transfers of energy between surface wave modes, i.e., by a nonlinear wave–wave interaction. A study of this process was pioneered in Ref. 18, and a more general unified description of similar nonlinear effects for different media is reviewed in Ref. 19.

IV. RESONANT WAVE–PARTICLE INTERACTION IN PLASMA

In this section another derivation of Landau damping is presented, which clarifies the analogy between the wave–particle interaction in plasma and the resonant wind–wave interaction discussed above. Let us consider the standard example of the electron Langmuir wave (see, e.g., Ref. 20). This is an electrostatic wave with frequency ω equal to $\omega_{pe}=(4\pi n_0 e^2/m)^{1/2}$—the so-called electron plasma frequency, where n_0 is the number density of electrons, e—the electron charge, and m—its mass. It is assumed that plasma quasineutrality is provided by a uniform background of ions which, however, are not involved in oscillations because of their large mass. If such a wave propagates along the x axis with wave vector k, its electric field has the same direction and can be written as

$$E(x,t)=E_0\sin(kx-\omega t). \tag{28}$$

Our interest is in the momentum transfer from this field to electrons, assuming that the wave (28) has been switched on at $t=0$, and the initial distribution function of electrons, $f(x,v,t=0)$, is $f_0(v)$. Then the momentum transfer rate per unit length along the x axis is

$$\frac{dP_e}{dt}=\langle en(x,t)E(x,t)\rangle=e\int_{-\infty}^{+\infty}dv\langle f(x,v,t)E(x,t)\rangle, \tag{29}$$

where $\langle\,\rangle$ signifies an average over x. This expression is similar to (16) in the sense that to get a nonzero average result the spatial variation in the distribution function f has to be taken into account, as it was for the vorticity in (16). However, the analogy goes much further if one notes that, in a collisionless plasma, the distribution function $f(\mathbf{r},\mathbf{v},t)$ remains unchanged along a particle trajectory, similar to the vorticity, which is preserved for a moving ''fluid element'' in an inviscid fluid. This conservation law for f comes directly from the Liouville's theorem (see, e.g., Ref. 21), which states that temporal evolution of a Hamiltonian system such as plasma without collisions represents an ''incompressible flow'' in the phase space $(\mathbf{r},\mathbf{v})$. Therefore $f(\mathbf{r},\mathbf{v},t)$, which is the ''number density'' in phase space, remains unchanged, and it is useful to follow the trajectory of individual electrons (as we did in Sec. III for individual air elements). Therefore, let us consider one with an initial position $x=x_0$ and velocity $v_x=v_0$. Due to the electric field (28) its velocity will be changed to $v_0+\Delta v$, where the velocity perturbation Δv follows from the equation of motion

$$\frac{d(\Delta v)}{dt}=\frac{e}{m}E_0\sin[kx_0-(\omega-kv_0)t],$$

whose solution with $\Delta v(0)=0$ is

$$\Delta v=\frac{eE_0}{m(\omega-kv_0)}\{\cos[kx_0-(\omega-kv_0)t]-\cos kx_0\}. \tag{30}$$

Then the above-mentioned conservation of the distribution function along a particle trajectory means that

$$\begin{aligned} f(x,v,t)&=f_0(v_0)\\ &=f_0[v-\Delta v(x,t)]\\ &\approx f_0(v)-\frac{\partial f_0}{\partial v}\frac{eE_0}{m(\omega-kv)}\\ &\quad\times\{\cos[kx_0-(\omega-kv)t]-\cos kx_0\}. \end{aligned} \tag{31}$$

Inserting this expression for f, as well as (28) for E, into (29) and averaging the product over x_0, one finds that the first term in (31) makes no average contribution, while the second results in

$$\langle Ef\rangle=-\frac{eE_0^2}{2m}\frac{\partial f_0}{\partial v}\times\frac{\sin(\omega-kv)t}{(\omega-kv)}, \tag{32}$$

yielding the following momentum transfer rate:

$$\frac{dP_e}{dt}=-\frac{e^2E_0^2}{2m}\int_{-\infty}^{+\infty}\frac{\partial f_0}{\partial v}\times\frac{\sin(\omega-kv)t}{(\omega-kv)}\,dv. \tag{33}$$

After several wave periods, the second factor in the integrand of (33) can be replaced by the delta function as $\pi\,\delta(\omega-kv)$ (see Sec. III), and (33) takes the form

$$\frac{dP_e}{dt}=-\frac{\pi e^2E_0^2}{2mk}\frac{\partial f_0}{\partial v}\left(v=\frac{\omega}{k}\right). \tag{34}$$

The next step in calculating the wave growth (or damping) rate γ is to derive the momentum carried by a Langmuir wave per unit length along the x axis, P_w, so that

$$\gamma=\frac{dP_w/dt}{2P_w}=-\frac{dP_e/dt}{2P_w}, \tag{35}$$

thanks to conservation of the total momentum $P=P_e+P_w$. Since this wave is electrostatic, its field carries no momentum (see, e.g., Ref. 22), and P_w originates entirely from the bulk electron motion. The latter constitutes the main body of electrons (the so-called thermal electrons) whose initial velocity is small compared to the phase velocity of a Langmuir wave, and therefore can be neglected. The wave perturbs the density of these electrons, so that $n=n_0+\tilde{n}$, as well as generating their oscillating velocity $\tilde{v}$; therefore,

$$P_w=\langle nmv\rangle=m\langle\tilde{n}\tilde{v}\rangle. \tag{36}$$

Using the Poisson equation, one obtains from (28) that

$$\tilde{n}=\frac{1}{4\pi e}\frac{\partial E}{\partial x}=\frac{kE_0}{4\pi e}\cos(kx-\omega t),$$

while the equation of motion yields the following expression for $\tilde{v}$:

$$\tilde{v}=\frac{eE_0}{m\omega}\cos(kx-\omega t),$$

so (36) results in

$$P_w=\frac{E_0^2}{8\pi}\frac{k}{\omega_{pe}},$$

and finally it follows from (34) and (35) that

$$\gamma=\frac{2\pi^2e^2\omega_{pe}}{mk^2}\frac{\partial f_0}{\partial v}\left(v=\frac{\omega}{k}\right), \tag{37}$$

reproducing the well-known Landau result.

Unlike the wind–wave interaction, where wind normally generates surface waves, the sign of γ for plasma waves is not so universal. Although at thermodynamic equilibrium, when f_0 is a Maxwellian distribution, $\partial f_0/\partial v<0$ and (37) represents Landau damping, the situation can be easily reversed by injecting into a plasma a fast electron beam. Then, for some range of phase velocities this derivative changes its sign, resulting in the generation of Langmuir waves known in plasma physics as the ''bump-on-tail'' instability (see, e.g., Ref. 23).

In summary it is worth noting that though the nature of the forces which provide resonant wind–wave interaction on a water surface, and wave–particle interaction in plasma, are quite different, the actual mechanisms of these phenomena are remarkably similar. Indeed, as seen from (32), the net momentum transfer from Langmuir wave to electrons originates from the bunching of resonant electrons which is correlated with the electric field: the density of these electrons is slightly higher where the field has one direction, and lower for the opposite case. In the case of surface wave generation by wind, the momentum is transferred by the so-called ''vortex force'' [see (16)], which requires correlation between the vertical motion of air and vorticity variation. In both cases these correlations occur under the resonant condition (21), though for Landau damping in plasma the selection is in the velocity space of the electron distribution function, while for wind–wave interaction it is in the coordinate space of the sheared air flow profile.

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