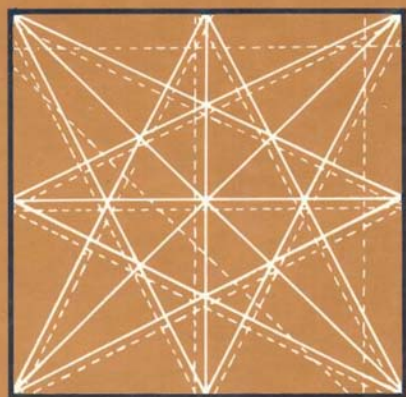


The Physics of Intense Beams and Storage Rings



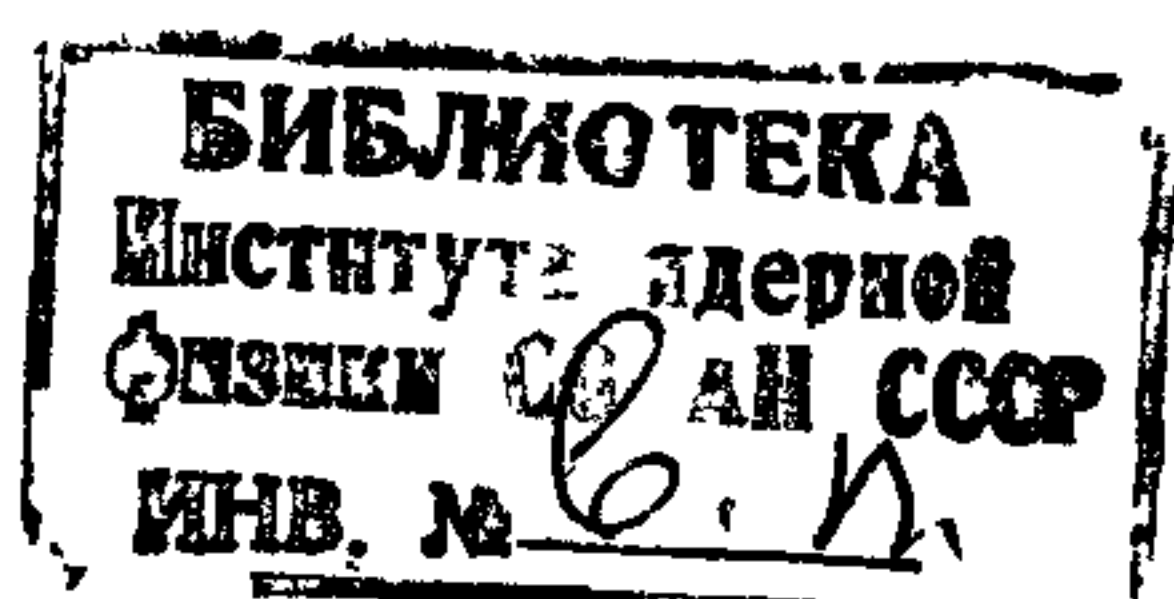
N. S. Dikansky
D. V. Pestrikov

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Nikolai Dikansky
Dmitri Pestrikov

*Institute of Nuclear Physics–Novosibirsk
Novosibirsk, Russia*



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Preface

Our objective in writing this book is to discuss the physical processes limiting or helping the performance of intense beams in particle accelerators, colliders, and storage rings. Initially it was supposed to be almost a direct translation from Russian of our book *Physics of Intense Beams in Storage Rings* published by Novosibirsk Division of Nauka in 1989. However, fast developments in this field of science as well as discussions about the subject with our colleagues convinced us to bring into the translated text numerous additions. An additional motivation for these alterations was our lectures on particle accelerators and on collective phenomena in stored beams which we gave for years in Novosibirsk State University. Some new items added have a heuristic (or maybe methodical) value, illustrating the reliability of subsequent general treatments, while others describe completely new results that are interesting even for experts in the field.

When writing this book we had to choose between two alternatives. From one side we tried to give the most general description of both phenomena and technical methods used for that description. In successful cases, this enables the reader to estimate the mutual coupling of looking at different effects as the issues for the same physical theory. Therefore our emphasis is on the application of statistical mechanics methods to the physics of stored beams. As a result, the book treats the wide scope of problems of beam physics from single-particle dynamics, through the theory of linear coherent oscillations and cooling techniques, toward the description of kinetic effects in intense beams and nonlinear collective phenomena.

On the other hand, general description of the current dependent phenomena in stored beams frequently meets serious mathematical problems that can be difficult even for modern computers. In these circumstances, particular predictions only can be done using simplifying assumptions. When the use of these simplifications conflicted with adopted rigorous description, we usually gave the priority to the simplified one if it enhanced the understanding of the physics.

For many years many results in beam physics exclusive of experimental studies have been obtained using computer simulations. This is particularly true for the beam-beam instability in colliders. In this book we paid more attention to the methods of analytic description that usually yield some complementary issues having both general worth and the base for new computer codes. For this reason we did not discuss details of computer codes or simulations, which are

here used mostly as illustrations. The same is true concerning our discussion of experimental results and techniques.

In this book we assume that the reader has basic knowledge of particle accelerators physics, statistical mechanics, and the mathematics within the framework of the usual university course. For this reason the book is mainly addressed to the physicists and engineers working on particle accelerators, storage rings, and colliders. For beginners we have supplied the book with two short introductory chapters. Chapter 1 contains the base of the particle accelerators theory used in subsequent analysis. Chapter 2 treats the basic questions of physical kinetics and its simplest applications to the physics of intense beams. Our experience shows that this is enough to address students of the higher grades and postgraduate students.

Chapters 3 through 6 describe the general theory of the linear coherent oscillations of the beams in the particle accelerators and storage rings and its applications to particular problems. The examples of such problems selected for this book in our time were considered as the issues of the intensity upgrade studies in real machines. Chapter 7 describes the Schottky noise phenomenon, the nondestructive beam diagnostic based on the Schottky noise measurements, the physics of the stochastic cooling, and the application of this cooling technique to the accumulation of particles. Chapter 8 describes the method of electron cooling and related phenomena in intense beams. Chapter 9 treats quasi-nonlinear collective effects, the beam heating by its coherent fluctuations, and the bunch lengthening effect in storage rings.

All chapters contain short introductions describing the main scope of problems to be discussed. Chapters are divided in sections, which have common numeration throughout the chapter. In cases when the section treats manifold phenomena, or problems, it is divided into subsections, numerated by Arabic numbers. Equations have common numeration through the section. Equation numbers specify the number of the chapter, the number of the section, and the number of equation in the section. Usually this simplifies the search for necessary equations in the book. The number of the figure specifies the number of the chapter and the number of the figure in the chapter. Typically we tried to use definitions that are common for the particle accelerator or general physics literature.

The bibliography in the book is far from exhausting, especially concerning papers treating the collective stability of beams, where the number of publications is immensely high. In these cases we tried to cite the papers where the question was discussed for the first time, or where the issue complements the book's description. We hope that our bibliography will open some new entries to our English reader complementary to those found in well-known reviews.

The majority of the results found in this book were original or obtained with our colleagues studying various problems of colliding beam physics mainly in the Institute of Nuclear Physics at Novosibirsk. We would like to thank all our colleagues for numerous useful discussions and comments. In particular we want

to thank professor A. N. Skrinsky, whose permanent interest and support of our work significantly helped us to complete the Russian version of the book. We are grateful to Professors P. R. Zenkevich and A. N. Lebedev for comments and suggestions that have been used in the English version as well.

We are indebted to Dr. F. Cole for his encouragement and support of our work on the English text of this book, and to Dr. T. Knight who had read the first three chapters and made numerous corrections and suggestions. We cannot forget very important technical help from Mrs. L. Bukina, Mrs. E. Lisman, Mr. A. A. Prokopenko, and Mr. A. Kulakov.

CHAPTER 1

The Motion of a Single Particle

Limitations on the beam intensity in storage rings are usually caused by the interaction of beam particles. Typically these occur due to very delicate processes that are strongly affected by specific features of the motion of individual particles. Therefore, it seems reasonable to start with a brief discussion of the basic properties of the motion of a single particle in a storage ring. A more detailed description of this problem can be found elsewhere (for instance, in Refs. 1,2,3, and 4).

1.1. EQUATIONS OF MOTION

One of the main problems of beam dynamics in storage rings is to ensure that the lifetime of a particle inside the vacuum chamber of the ring is as long as possible. Practically, this can be done by special distribution along the ring of magnetic and electric fields, which push deflected particles toward the desired closed orbit. Thus the motion of a particle in the storage ring becomes the combination of the motion along this closed orbit $\mathbf{r}=\mathbf{r}_0(t)$ and small oscillations near it. Due to many factors, the character of particle trajectories can be very complicated. However, in typical cases it is possible to specify some particular factor as the leading one and to consider others as perturbations; thus one can study the motion of a particle in a storage ring by subsequent steps, taking into account with each new step a new feature of the trajectory.

As a first approximation, the motion of a particle with constant energy in a given magnetic field $\mathbf{H}(\mathbf{r})$ is usually treated. For this case the equation of motion reads

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c} \mathbf{v} \times \mathbf{H}(\mathbf{r}), \quad (1.1.1)$$

where $\mathbf{p}=\gamma m\mathbf{v}$ is the momentum of the particle, $\mathbf{v}=d\mathbf{r}/dt$ is its velocity, c is the speed of light, $\gamma=1/\sqrt{1-(v/c)^2}$, and e is the charge of the particle.

Since the instantaneous trajectory of the particle is expected to be close to the given $\mathbf{r}=\mathbf{r}_0(t)$, one can find it using the so-called natural coordinate system, which refers to the orbit $\mathbf{r}_0(t)$. In this system the path s , which is covered by the particle along the trajectory, is used as the independent variable. In these coordinates the deviation of any vector from $\mathbf{r}_0(t)$ can be expanded using three unit vectors, which are defined along the orbit: the tangential to the orbit vector $\boldsymbol{\tau}(s)$, the normal vector $\mathbf{n}(s)$, and binormal vector $\mathbf{b}(s)$. These vectors satisfy the

Serret-Frenet relationships, which for the most important case of flat orbits have the form

$$\tau = \frac{dr_0}{ds}, \quad \frac{n(s)}{R(s)} = -\frac{d\tau}{ds}, \quad \frac{dn}{ds} = \frac{\tau}{R(s)}. \quad (1.1.2)$$

Here, $R(s)$ is the radius of curvature of the trajectory at the point s . From Eqs. (1.1.1) and (1.1.2) one can find that for closed orbits, without self-crossings

$$\oint \frac{ds}{R(s)} = 2\pi, \quad (1.1.3)$$

the guiding magnetic field $\mathbf{H}$ satisfies the equation

$$pc = -\frac{e}{2\pi} \oint ds H_z(s), \quad H_z = \mathbf{bH}. \quad (1.1.4)$$

The integration in Eqs. (1.1.3) and (1.1.4) is executed along the trajectory $\mathbf{r} = \mathbf{r}_0(s)$. The radius of curvature of the orbit $R(s)$ and H_z are related through the equation

$$pc = -eH_z R(s). \quad (1.1.5)$$

The motion of the particle along the closed orbit has the period T_0 , which for the path perimeter Π and particle velocity v is

$$T_0 = \Pi/v. \quad (1.1.6)$$

This equation can also be rewritten in the form

$$T_0 = 2\pi R_0/v = 2\pi/\omega_0, \quad (1.1.7)$$

where $R_0 = \Pi/2\pi$ is the so-called average radius of the orbit, and $\omega_0 = v/R_0$ is the revolution frequency of the particle. Using Eq. (1.1.4) one can write

$$1/R_0 = -\frac{e}{pc} \int_0^\Pi \frac{ds}{\Pi} H_z(s) = -\frac{\overline{eH_z}}{pc} \quad (1.1.8)$$

and therefore

$$\omega_0 = -e\overline{H_z}/(\gamma mc). \quad (1.1.9)$$

Let us now find equations that describe the motion of particles deviating from the closed orbit $\mathbf{r}_0(s)$. For such particles one can write

$$\mathbf{r}(s) = \mathbf{r}_0(s) + \mathbf{n}(s)x + \mathbf{b}(s)z \quad (1.1.10)$$

and consider x, z, s as the curvilinear coordinates of the particle. In the natural coordinate system Eq. (1.1.1) has the form

$$\frac{d}{ds} \left(\frac{ds}{dt} \frac{d\mathbf{r}}{ds} \right) = \frac{e}{\gamma mc} \frac{d\mathbf{r}}{ds} \times \mathbf{H}(\mathbf{r}), \quad (1.1.11)$$

where

$$d\mathbf{r}/ds = \tau(1+x/R) + \mathbf{n}x' + \mathbf{b}z', \quad x' = dx/ds, \quad (1.1.12)$$

$$\frac{ds}{dt} = \frac{v}{[(1+x/R)^2 + x'^2 + z'^2]^{1/2}}. \quad (1.1.13)$$

Assuming no longitudinal magnetic field along the closed orbit and projecting Eq. (1.1.11) onto directions $\mathbf{n}$ and $\mathbf{b}$, one gets

$$\left(\frac{ds}{dt} x' \right)' = (1+x/R) \left(\frac{eH_z}{\gamma mc} + \frac{ds}{dt} \right), \quad (1.1.14)$$

$$\left(\frac{ds}{dt} z' \right)' = -(1+x/R) \frac{eH_x}{\gamma mc}. \quad (1.1.15)$$

One of the most important applications of Eqs. (1.1.14) and (1.1.15) is the study of the stability of the orbit. For particles with small deviations from the closed orbit, this problem can be solved using Eqs. (1.1.14) and (1.1.15) linearized with respect to x and z . Taking into account the symmetry of the magnetic field, which applies for flat orbits,

$$H_x(x, z) = H_z(x, -z), \quad H_x(x, z) = -H_x(x, -z), \quad (1.1.16)$$

from Eqs. (1.1.14) and (1.1.15) one can get

$$x'' + g_x(s)x = 0, \quad g_x(s) = \frac{1}{R^2} - \frac{e}{pc} \left(\frac{\partial H_z}{\partial x} \right)_{\mathbf{r}=\mathbf{r}_0(s)}; \quad (1.1.17)$$

$$z'' + g_z(s)z = 0, \quad g_z(s) = \frac{e}{pc} \left(\frac{\partial H_z}{\partial x} \right)_{\mathbf{r}=\mathbf{r}_0(s)}. \quad (1.1.18)$$

Typically, the value of the gradient of the magnetic field in these equations is specified by the so-called field index

$$n(s) = - \left(\frac{R}{H_z} \frac{\partial H_z}{\partial x} \right)_{\mathbf{r}=\mathbf{r}_0(s)}. \quad (1.1.19)$$

This yields for the rigidity coefficients in Eqs. (1.1.17) and (1.1.18)

$$g_x(s) = [1 - n(s)]/R^2, \quad g_z(s) = n(s)/R^2. \quad (1.1.20)$$

General properties of the solutions of Eqs. (1.1.17) and (1.1.18) will be discussed in §1.2. Here we note only that in the simplest case of circular orbits

[$R=\text{const}$, $n(s)=n_0$] the motion of deflected particles is described by the formulas

$$x = a_x \cos(sv_x/R + \varphi_x), \quad z = a_z \cos(sv_z/R + \varphi_z),$$

$$v_x = \sqrt{1-n_0}, \quad v_z = \sqrt{n_0}. \quad (1.1.21)$$

It oscillates near the closed orbit, provided

$$0 \leq n_0 \leq 1. \quad (1.1.22)$$

1.2. BETATRON OSCILLATIONS

As a rule, for the arbitrary distribution of focusing and defocusing elements along the ring, solutions of Eq. (1.1.17) and Eq. (1.1.18) cannot be found analytically. However, due to the periodicity of the coefficients g_x and g_z , general properties of these solutions can be studied in many details.⁵

Since the motion of particles along the ring is cyclic, the functions $g_x(s)$ and $g_z(s)$ have at least the period $L_0 = \Pi$. If the lattice of the ring consists of N_B identical cells, then the period of $g_x(s)$ and $g_z(s)$ is $L_0 = \Pi/N_B$

$$g_x(s+L_0) = g_x(s), \quad g_z(s+L_0) = g_z(s). \quad (1.2.1)$$

As the following analysis will be similar for both horizontal x and vertical z oscillations, below we shall limit the discussion to x -oscillations. Using for convenience the dimensionless independent variable $\vartheta = s/R$ - the so-called generalized azimuth of the particle, we shall rewrite Eq. (1.1.17) in the form

$$x'' + g_x(\vartheta)x = 0, \quad (1.2.2)$$

where now $x' = dx/d\vartheta$,

$$g_x(\vartheta) = \left(\frac{R_0}{R(\vartheta)} \right)^2 [1 - n(\vartheta)], \quad (1.2.3)$$

and $L_0 = R_0\vartheta_0$; $\vartheta_0 = 2\pi/N_B$ is the azimuthal length of the cell.

Let $x_1(\vartheta)$ and $x_2(\vartheta)$ be two linearly independent solutions of Eq. (1.2.2). They can be used for the construction of the Wronskian

$$w = x_1 x_2' - x_2 x_1', \quad (1.2.4)$$

which is the integral of the motion for Eq. (1.2.2). Any other solution of Eq. (1.2.2) can be written as a linear combination of $x_1(\vartheta)$ and $x_2(\vartheta)$

$$x(\vartheta) = \alpha x_1(\vartheta) + \beta x_2(\vartheta). \quad (1.2.5)$$

The constants α and β can be calculated via initial conditions, say, at $\vartheta = \vartheta_1$:

$$\alpha = \frac{1}{w} [x(\vartheta_1)x'_2(\vartheta_1) - x_2(\vartheta_1)x'(\vartheta_1)], \quad \beta = \frac{1}{w} [x_1(\vartheta_1)x'(\vartheta_1) - x(\vartheta_1)x'_1(\vartheta_1)] \quad (1.2.6)$$

or, using the matrix symbols,

$$\alpha = \frac{1}{w} [x'_2(\vartheta_1), -x_2(\vartheta_1)] \begin{pmatrix} x(\vartheta_1) \\ x'(\vartheta_1) \end{pmatrix} = \frac{1}{w} [x'_2(\vartheta_1), -x_2(\vartheta_1)] \hat{X}(\vartheta_1), \quad (1.2.7)$$

$$\beta = \frac{1}{w} [-x'_1(\vartheta_1), x_1(\vartheta_1)] \hat{X}(\vartheta_1), \quad (1.2.8)$$

where

$$\hat{X}(\vartheta) = \begin{pmatrix} x(\vartheta) \\ x'(\vartheta) \end{pmatrix}. \quad (1.2.9)$$

For Eq. (1.2.5) this yields

$$\hat{X}(\vartheta) = \hat{M}(\vartheta, \vartheta_1) \hat{X}(\vartheta_1), \quad (1.2.10)$$

where the matrix

$$\hat{M}(\vartheta, \vartheta_1) = \{ \hat{X}_1(\vartheta)(x'_2, -x_2)_{\vartheta_1} - \hat{X}_2(\vartheta)(x'_1, -x_1)_{\vartheta_1} \} / w \quad (1.2.11)$$

transforms the solution from the point ϑ_1 to the point ϑ . By direct calculation it is easy to verify that this so-called transport matrix $\hat{M}$ has the properties

$$\det[\hat{M}(\vartheta, \vartheta_1)] = 1, \quad (1.2.12)$$

$$\hat{M}(\vartheta_3, \vartheta_1) = \hat{M}(\vartheta_3, \vartheta_2) \hat{M}(\vartheta_2, \vartheta_1). \quad (1.2.13)$$

Substituting into the second equation $\vartheta_3 = \vartheta_1$ and taking into account that $\hat{M}(\vartheta_1, \vartheta_1)$ gives the identity transformation, one can write

$$\hat{M}(\vartheta_1, \vartheta_2) \hat{M}(\vartheta_2, \vartheta_1) = \hat{I}, \quad (1.2.14)$$

where $\hat{I}$ is the unit matrix

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.2.15)$$

Therefore one has

$$\hat{M}(\vartheta_1, \vartheta_2) = \hat{M}^{-1}(\vartheta_2, \vartheta_1). \quad (1.2.16)$$

Due to the periodicity of $g(\vartheta)$ the solution $x(\vartheta + \vartheta_0)$ can be written in the form of Eq. (1.2.5) using $x_1(\vartheta)$ and $x_2(\vartheta)$. In particular, this property is valid for solutions $x_{1,2}$ itself:

$$x_{1,2}(\vartheta + \vartheta_0) = \alpha_{1,2}x_1(\vartheta) + \beta_{1,2}x_2(\vartheta). \quad (1.2.17)$$

Once we are rather free in the choice of the normal solutions $x_{1,2}$, they can be constructed with the requirement that after the transformation over the period ϑ_0 functions $x_{1,2}$ should be multiplied by the constant:

$$x_1(\vartheta + \vartheta_0) = \lambda_1 x_1(\vartheta), \quad x_2(\vartheta + \vartheta_0) = \lambda_2 x_2(\vartheta). \quad (1.2.18)$$

On the other hand, $\hat{X}_{1,2}(\vartheta + \vartheta_0)$ can be calculated using the transformation in Eq. (1.2.10)

$$\hat{X}_{1,2}(\vartheta + \vartheta_0) = \hat{M}(\vartheta + \vartheta_0) \hat{X}_{1,2}(\vartheta). \quad (1.2.19)$$

This shows that Eq. (1.2.18) is satisfied by the solutions, which are the eigenvectors of the transport matrix over the period of the lattice.

$$\hat{X}(\vartheta + \vartheta_0) = \lambda \hat{X}(\vartheta) = \hat{M}(\vartheta + \vartheta_0) \hat{X}(\vartheta). \quad (1.2.20)$$

The eigenvalues λ can be calculated from the resolvent condition of this system:

$$\det(\hat{M} - \lambda) = 0. \quad (1.2.21)$$

For the 2×2 matrices $\hat{M}$ discussed here Eq. (1.2.21) yields

$$\lambda^2 - \lambda \text{Sp}(\hat{M}) + 1 = 0, \quad (1.2.22)$$

$$\lambda_{1,2} = \frac{\text{Sp}(\hat{M})}{2} \pm i \sqrt{1 - \left(\frac{\text{Sp}(\hat{M})}{2} \right)^2}. \quad (1.2.23)$$

Due to $\lambda_1 \lambda_2 = 1$, it is clear that, provided that $|\text{Sp}(\hat{M})| > 2$, both eigenvalues λ are the real numbers and one of them will exceed 1. Taking into account that

$$\hat{X}(\vartheta + n\vartheta_0) = \hat{M}(\vartheta + n\vartheta_0) \hat{X}(\vartheta) = \lambda \hat{M}(\vartheta + \vartheta_0) \hat{X}[\vartheta + (n-1)\vartheta_0] = \lambda^n \hat{X}(\vartheta), \quad (1.2.24)$$

one can conclude that in this region of the parameters one of the eigensolutions will grow with the time, which corresponds to unstable oscillations.

On the contrary, if

$$|\text{Sp}(\hat{M})| < 2, \quad (1.2.25)$$

then $\lambda_2 = \lambda_1^*$ and so $|\lambda_{1,2}| = 1$. Therefore, in this region of the parameters, the eigenvalues $\lambda_{1,2}$ have the form $\lambda = \exp(\pm i\mu)$, while the transformation of eigenvectors over the period corresponds to their rotation

$$\hat{X}_{1,2}(\vartheta + \vartheta_0) = \exp(\pm i\mu) \hat{X}_{1,2}(\vartheta).$$

These solutions describe the stable betatron oscillations of the beam particles. Using the eigenvectors $\hat{X}_{1,2}$ the deviation of a particle from the closed orbit is a superposition of the functions

$$x_{1,2}(\vartheta) = f(\vartheta) e^{\pm i\nu\vartheta \pm \chi(\vartheta)}, \quad (1.2.26)$$

where $f(\vartheta)$ and $\chi(\vartheta)$ are the so-called amplitude and phase of the Floquet function. By definition, these are periodical functions of the azimuth

$$f(\vartheta) = f(\vartheta + \vartheta_0), \quad \chi(\vartheta) = \chi(\vartheta + \vartheta_0). \quad (1.2.27)$$

The value ν in Eq. (1.2.26) is usually called the dimensionless frequency or tune of the betatron oscillations. The ratio

$$\mu = \nu\vartheta_0 = 2\pi\nu/N_B$$

determines the phase advance of the betatron oscillations over the period of the lattice. Since for a given μ one has $\nu = \mu N_B / 2\pi$, it is clear that the frequency ν is generally higher as the number of cells in the lattice increases.

It is convenient to normalize the functions from Eq. (1.2.26) with the condition $w = -2i$. Then the direct calculation of w yields the equation

$$\chi'(\vartheta) + \nu = 1/f^2(\vartheta). \quad (1.2.28)$$

Taking into account the periodicity of χ , after averaging this equation, we get

$$\nu = \int_0^{\vartheta_0} \frac{d\vartheta}{\vartheta_0} \frac{1}{f^2(\vartheta)}. \quad (1.2.29)$$

Finally, the substitution of Eq. (1.2.26) into Eq. (1.2.2) yields

$$f''(\vartheta) + g(\vartheta)f(\vartheta) = 1/f^3(\vartheta). \quad (1.2.30)$$

To clarify the physical sense of the values that have been introduced, let us note that using Eq. (1.2.26) the deviation of the particle from the closed orbit can be written in the form

$$x(\vartheta) = a f(\vartheta) \cos[\nu\vartheta + \chi(\vartheta) + \psi_0], \quad (1.2.31)$$

where a and ψ_0 are respectively the amplitude and the initial phase of the oscillation. Therefore, at the given azimuth ϑ the deviation of the particle executes a harmonic oscillation with so-called frequency ($\vartheta = \omega_s t$) of the betatron oscillations:

$$\omega_b = \omega_s \nu.$$

The functions $f(\vartheta)$ and $\chi(\vartheta)$ describe the modulation of the amplitude and the phase of such oscillations along the orbit. Once the deviation cannot exceed $af(\vartheta)$, the function $f(\vartheta)$ determines the envelope of the beam in the storage ring. As can be seen from Eq. (1.2.30), $f(\vartheta)$ can reach its minimum values at defocusing elements of the cell. At the focusing elements $g \geq 0$, f can reach its maximum values, provided $g \geq 1/f^4(\vartheta)$.

To complete this section, let us express the elements of the transport matrix $\hat{M}$ in terms of the solutions in the form of Eq. (1.2.26). Substituting Eq. (1.2.26) into Eq. (1.2.11) and denoting for short $\psi = \nu \vartheta + \chi(\vartheta)$, we shall find for $\hat{M}(\vartheta, \vartheta_1)$

$$\hat{M}(\vartheta, \vartheta_1) = -\text{Im} \left\{ e^{i\Delta\psi} \begin{pmatrix} f \\ f' - i/f \end{pmatrix}_{\vartheta} \begin{pmatrix} f' - \frac{i}{f}, & -f \end{pmatrix}_{\vartheta_1} \right\}, \quad (1.2.32)$$

where $\Delta\psi = \psi(\vartheta) - \psi(\vartheta_1)$. The calculation of the matrix elements in Eq. (1.2.32) becomes especially simple for the transformation over the period $\vartheta = \vartheta_1 + \vartheta_0$, when $f(\vartheta) = f(\vartheta_1)$, $f'(\vartheta) = f'(\vartheta_1)$, $\Delta\psi = \mu$, and

$$\hat{M}(\vartheta_0) = \begin{pmatrix} \cos\mu - \frac{(f^2)'}{2} \sin\mu & f^2 \sin\mu \\ - (f'^2 + 1/f^2) \sin\mu & \cos\mu + \frac{(f^2)'}{2} \sin\mu \end{pmatrix}. \quad (1.2.33)$$

On the other hand, for particular lattices the elements of $\hat{M}$ can be calculated using vectors with special initial conditions

$$\hat{x}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{x}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.2.34)$$

Then, the comparison with Eq. (1.2.33) yields

$$\begin{aligned} x_1(\vartheta_0) &= \cos\mu - \frac{(f^2)'}{2} \sin\mu, & x'_1(\vartheta_0) &= - \left((f')^2 + \frac{1}{f^2} \right) \sin\mu, \\ x_2(\vartheta_0) &= f^2 \sin\mu, & x'_2(\vartheta_0) &= \cos\mu + \frac{(f^2)'}{2} \sin\mu, \end{aligned} \quad (1.2.35)$$

It is clear from Eq. (1.2.35) that the Floquet function generally grows when ν approaches the border of the stability

$$f^2(\vartheta_0) = x_2(\vartheta_0) / \sin\mu, \quad \nu \rightarrow N_B/2. \quad (1.2.36)$$

To calculate the Floquet functions along the ring, one can use Eq. (1.2.33) and the transformation

$$\hat{M}(\vartheta_0 + \Delta\vartheta) = \hat{M}(-\Delta\vartheta) \hat{M}(\vartheta_0) \hat{M}(\Delta\vartheta). \quad (1.2.37)$$

Finally, returning back to the independent variable $s = R_0\vartheta$ we can rewrite Eq. (1.2.33) in the form

$$\hat{M}(s + L_0) = \begin{pmatrix} \cos\mu - \frac{d\beta}{ds} \frac{\sin\mu}{2} & \beta \sin\mu \\ -\frac{\left(\frac{d\beta}{ds}\right)^2 + 4}{4\beta} \sin\mu & \cos\mu + \frac{d\beta}{ds} \frac{\sin\mu}{2} \end{pmatrix} \quad (1.2.38)$$

where $\beta(s) = R_0 f^2$ is the so-called β -function of the ring. With these definitions Eqs. (1.2.28–1.2.31) can be written in the form

$$\frac{d\chi}{ds} + k = \frac{1}{\beta(s)}, \quad (1.2.39)$$

$$\mu = \int_0^{L_0} \frac{ds}{\beta(s)}, \quad (1.2.40)$$

$$\beta'' - \frac{(\beta')^2}{2\beta} + 2g(s)\beta = \frac{2}{\beta(s)}, \quad (1.2.41)$$

$$x(s) = \sqrt{J\beta(s)} \cos[ks + \chi(s) + \psi_0], \quad k = 2\pi\nu/\Pi. \quad (1.2.42)$$

1.3. ADIABATIC INVARIANT OF BETATRON OSCILLATIONS

Both the description of the betatron oscillations and especially the construction of the perturbation theory for particles in a storage ring can be considerably simplified by the use of canonically conjugated action-phase variables $(I_x, \psi_x; I_z, \psi_z)$. This is because for the unperturbed oscillations the action variables $I \equiv (I_x, I_z)$ are integrals of the motion, while phases $\psi \equiv (\psi_x, \psi_z)$ are cyclic variables.

The generating function of this transformation S is determined by the Hamilton-Jacoby equation.⁶ For the motion with constant energy $\mathcal{E}$, this has the form

$$\mathcal{H}(\mathbf{P}, \mathbf{r}) = \mathcal{E}, \quad (1.3.1)$$

where

$$\mathcal{H} = c \sqrt{\left(\mathbf{P} - \frac{e}{c} \mathbf{A}_0\right)^2 + m^2 c^2} \quad (1.3.2)$$

is the Hamiltonian of a particle in the magnetic field, described by the vector potential $\mathbf{A}_0(\mathbf{r})$; $\mathbf{P} = \mathbf{p} + (e/c)\mathbf{A}_0$ is the generalized momentum of the particle.

The transformation from $(\mathbf{r}, \mathbf{p})$ to variables referring to the closed orbit (x, z, ϑ) is generated by the function²

$$S_0 = \mathbf{P}[\mathbf{r}_0(\vartheta) + \mathbf{n}(\vartheta)x + \mathbf{b}(\vartheta)z].$$

It determines the momenta, which are conjugated to (x, z, ϑ)

$$\begin{aligned} p_x &= \frac{\partial S_0}{\partial x} = \mathbf{nP}, & p_z &= \frac{\partial S_0}{\partial z} = \mathbf{bP}, \\ p_2 &= \frac{\partial S_0}{\partial \vartheta} = R_0 \left(1 + \frac{x}{R(\vartheta)} \right) \tau \mathbf{P}. \end{aligned} \quad (1.3.3)$$

In these variables the Hamiltonian (1.3.2) can be rewritten in the form

$$\mathcal{H} = c \sqrt{\underbrace{p_2^2}_{R_0^2} \left(1 - \frac{x}{R(\vartheta)} \right)^2 + \underbrace{p_x^2 + p_z^2}_{p_x^2 \rightarrow p_x^2 - eA_x \text{ etc.}} + m^2 c^2}. \quad (1.3.4)$$

Expanding Eq. (1.3.4) in powers of the deviations from the closed orbit, we get

$$\begin{aligned} \mathcal{H} &\simeq \mathcal{E}_0 + \frac{c^2}{\mathcal{E}_0} \left\{ p_x^2 + p_z^2 + p_0^2 \frac{x}{R(\vartheta)} \left(\frac{x}{R(\vartheta)} - 2 \right) \right\} \\ &\quad - \frac{eR_0}{c} \frac{d\vartheta}{dt} \left(1 + \frac{x}{R(\vartheta)} \right) \delta A_\vartheta(\mathbf{r}), \\ \delta A_\vartheta &= \tau(\vartheta) \delta \mathbf{A}(\mathbf{r}). \end{aligned} \quad (1.3.5)$$

Here, $\mathcal{E}_0$ and p_0 are the energy and momenta of the particle at the closed orbit; $\delta \mathbf{A} = \mathbf{A}_0 - \mathbf{A}_0[\mathbf{r}_0(\vartheta)]$. In many cases of practical interest it is quite sufficient to assume that the characteristics of the magnetic field are constant along the particular element of the cell and vary only from one element to another. In this approximation and without the longitudinal magnetic field one has

$$H_x = \frac{\partial \delta A_\vartheta}{\partial x}; \quad H_z = -\frac{\partial \delta A_\vartheta}{\partial z},$$

while δA_ϑ satisfies the equation

$$\frac{\partial^2 \delta A_\vartheta}{\partial x^2} + \frac{\partial^2 \delta A_\vartheta}{\partial z^2} = 0.$$

In polar coordinates $x = r \cos \varphi$, $z = r \sin \varphi$ and, taking into account the symmetry of $\mathbf{H}$, the solution of this equation can be represented by the expansion over multipoles

$$\delta A_{\vartheta}(r, \varphi) = \sum_{m=1}^{\infty} A_m r^m \frac{\cos m\varphi}{m}. \quad (1.3.6)$$

The calculation of coefficients A_m in terms of the derivatives of H_z on the closed orbit yields

$$\begin{aligned} \delta A_{\vartheta}(x, z) = & H_z x + \frac{1}{2} \frac{\partial H_z}{\partial x} (x^2 - z^2) + \frac{1}{6} \frac{\partial^2 H_z}{\partial x^2} (x^3 - 3xz^2) \\ & + \frac{1}{4!} \frac{\partial^3 H_z}{\partial x^3} (x^4 + z^4 - 6x^2 z^2) + \dots \end{aligned} \quad (1.3.7)$$

The subsequent terms in Eq. (1.3.7) give respectively the dipole, quadrupole, sextupole, and octupole components of the guiding field. If necessary the contributions of higher-order multipoles can be found from Eq. (1.3.6). To calculate the Hamiltonian of the betatron oscillations in the ideal lattice we have to take into account the dipole and quadrupole terms in Eq. (1.3.7). This transforms Eq. (1.3.5) into

$$\begin{aligned} \mathcal{H} \simeq \mathcal{E}_0 + \frac{c^2}{2\mathcal{E}_0} \left[p_1^2 + p_0^2 \frac{x}{R(\vartheta)} \left(\frac{x}{R(\vartheta)} - 2 \right) \right] \\ - \frac{ev}{c} \left(1 + \frac{x}{R(\vartheta)} \right) \left(H_z x + \frac{1}{2} \frac{\partial H_z}{\partial x} (x^2 - z^2) \right). \end{aligned} \quad (1.3.8)$$

Using Eq. (1.1.5) and ϑ as the independent variable, we get for the Hamiltonian of betatron oscillations

$$\begin{aligned} h(\mathbf{p}_1, \mathbf{r}_1, \vartheta) &= \frac{\mathcal{H} - \mathcal{E}_0}{d\vartheta/dt} = \left(1 + \frac{x}{R(\vartheta)} \right) \frac{\mathcal{H} - \mathcal{E}_0}{\omega_0} \\ &= h_x(p_x, x, \vartheta) + h_z(p_z, z, \vartheta) \\ &= \frac{R_0}{2p_0} p_1^2 + \frac{p_0}{2R_0} [g_x(\vartheta)x^2 + g_z(\vartheta)z^2]. \end{aligned} \quad (1.3.9)$$

Since the new Hamiltonian $h(\mathbf{p}_1, \mathbf{r}_1, \vartheta)$ is a function of the independent variable ϑ , the Hamilton-Jacoby equation for the generating function S must be written in the form

$$\partial S / \partial \vartheta + h\{\partial S / \partial \mathbf{r}_1, \mathbf{r}_1, \vartheta\} = 0. \quad (1.3.10)$$

Equation (1.3.10) with the Hamiltonian in Eq. (1.3.9) can be solved using the separation of variables:

$$S(\mathbf{p}_1, \mathbf{r}_1, \vartheta) = S_x(p_x, x, \vartheta) + S_z(p_z, z, \vartheta). \quad (1.3.11)$$

Equations for S_x and S_z differ only by the replacement of the indexes ($x \rightarrow z$), and thus we have to solve only one, say the equation for S_x . It reads

$$\frac{\partial S_x}{\partial \vartheta} + \frac{R_0}{2p_0} \left(\frac{\partial S_x}{\partial x} \right)^2 + \frac{p_0}{2R_0} g_x(\vartheta) x^2 = 0. \quad (1.3.12)$$

By the substitution

$$S_x(x, \psi_x, \vartheta) = \frac{p_0}{2R_0} x^2 \frac{v'(\vartheta)}{v(\vartheta)}. \quad (1.3.13)$$

Equation (1.3.12) can be transformed into

$$v'' + g_x(\vartheta)v = 0. \quad (1.3.14)$$

As we discussed, its solution can be written in the form

$$v = a f_x(\vartheta) \cos[\psi_x + \chi_x(\vartheta)].$$

This yields

$$S_x(x, \psi_x, \vartheta) = \frac{p_0}{2R_0} x^2 \left(\frac{f'_x}{f_x} - \frac{\tan[\psi_x + \chi_x(\vartheta)]}{f_x^2} \right). \quad (1.3.15)$$

The same expression (after obvious substitution of the index z for x) determines the generating function $S_z(z, \psi_z, \vartheta)$. Using Eq. (1.3.15) we can write

$$\begin{aligned} x &= a_x f_x(\vartheta) \cos[\psi_x + \chi_x(\vartheta)], & p_x &= \frac{p_0}{R_0} \frac{dx}{d\vartheta}, \\ z &= a_z f_z(\vartheta) \cos[\psi_z + \chi_z(\vartheta)], & p_z &= \frac{p_0}{R_0} \frac{dz}{d\vartheta}, \end{aligned} \quad (1.3.16)$$

and, respectively,

$$h(\mathbf{I}, \boldsymbol{\psi}) = v_x I_x + v_z I_z = v \mathbf{I},$$

$$I_\alpha = \frac{p_0 a_\alpha^2}{2R_0}, \quad \psi_\alpha = v_\alpha \vartheta + \psi_\alpha^{(0)}, \quad \alpha = x, z. \quad (1.3.17)$$

Equation (1.3.17) evidently shows the conservation of I_α and that the phases ψ_α are cyclic variables. Moreover, it is easy to find that phase trajectories for, say, horizontal oscillations are ellipses:

$$\frac{x^2}{R_0 f_x^2} + R_0 f_x^2 \left(\frac{p_x}{p_0} - \frac{f'_x}{f_x} \frac{x}{R_0} \right)^2 = \frac{2I_x}{p_0}, \quad (1.3.18)$$

with the area $2\pi I_x$. Typically, instead of I_α , the values $J_\alpha = 2I_\alpha/p_0$ are introduced. This transforms Eq. (1.3.16) and Eq. (1.3.18) into

$$\begin{aligned}
 x &= \sqrt{J_x \beta_x} \cos[\psi_x + \chi_x(s)], & p_x &= \frac{p_0}{R_0} \frac{dx}{ds}, \\
 z &= \sqrt{J_z \beta_z} \cos[\psi_z + \chi_z(s)], & p_z &= \frac{p_0}{R_0} \frac{dz}{ds}, \\
 \frac{x^2}{\beta_x} + \beta_x \left(\frac{p_x}{p_0} - \frac{1}{2\beta_x} \frac{d\beta_x}{ds} x \right)^2 &= J_x.
 \end{aligned} \tag{1.3.19}$$

One important property of the variables I_α [$\alpha = x, z$] is their adiabatic conservation. For instance, this causes the variation in the beam sizes to be proportional to $\sqrt{p_0}$ during the beam acceleration or deceleration.

The fact that the motion of particles is described by the hamiltonian equations, generally, simplifies the calculation of tunes, stability conditions and β -functions. For example, for coupled $x-z$ betatron oscillations the hamiltonian equations

$$\dot{x}_\alpha = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathcal{H}}{\partial x_\alpha}, \quad \alpha = x, z$$

can be written in the form

$$\frac{d}{d\vartheta} \hat{X} = -\hat{U} \hat{P} \hat{X}. \tag{1.3.20}$$

Here,

$$\hat{X} = \begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{P}_{\alpha\beta} = \frac{\partial^2 \mathcal{H}}{\partial X_\alpha \partial X_\beta} \Big|_{r=r_0(\vartheta)}$$

If $\hat{X}_\lambda$ [$\lambda = 1, 2, 3, 4$] are the linearly independent solutions of Eq. (1.3.20), then using

$$\frac{d}{d\vartheta} \hat{X}_\lambda^T = \hat{X}_\lambda^T \hat{P}^T \hat{U} = \hat{X}_\lambda^T \hat{P} \hat{U}, \tag{1.3.21}$$

where $\hat{X}^T$ means the transposed matrix, we find that

$$\frac{d}{d\vartheta} (\hat{X}_\lambda, \hat{U} \hat{X}_\lambda) = -\hat{X}_\lambda^T \hat{P} \hat{X}_\lambda + \hat{X}_\lambda^T \hat{P} \hat{X}_\lambda = 0,$$

or, that Eq. (1.3.20) has the following integrals of motion

$$\hat{X}_\lambda^T(\vartheta) \hat{U} \hat{X}_\lambda(\vartheta) = C_{\lambda,\lambda}. \tag{1.3.22}$$

If $\hat{M}(\vartheta, \vartheta_1)$ is the transport matrix

$$\hat{X}(\vartheta) = \hat{M}(\vartheta, \vartheta_1) \hat{X}(\vartheta_1),$$

then, according to Eq. (1.3.22) the matrix $\hat{M}$ obeys the so-called symplectic condition:

$$\hat{M}^T(\vartheta, \vartheta_1) \hat{U} \hat{M}(\vartheta, \vartheta_1) = \hat{U}. \quad (1.3.23)$$

Since $|\hat{U}| = 1$ [here, we define $|\hat{U}| \equiv \det(\hat{U})$] and $|\hat{M}(\vartheta_1, \vartheta_1)| = 1$, Eq. (1.3.23) results in $|\hat{M}(\vartheta, \vartheta_1)| = 1$. Another important consequence of the symplectic condition in Eq. (1.3.23) is that the eigennumbers of the transport matrix $\hat{M}$:

$$\lambda \hat{X} = \hat{M} \hat{X} \quad (1.3.24)$$

can be grouped in the couples of the reciprocally related numbers² [$\lambda_1 \lambda_2 = 1$, $\lambda_3 \lambda_4 = 1$, for a 4×4 matrix].

In order to find the tunes and β -functions we add to the eigenvalue equation (1.3.24) the identical equation

$$1/\lambda \hat{X} = \hat{M}^{-1} \hat{X}.$$

This results in

$$2 \cos \mu \hat{X} = (\hat{M} + \hat{M}^{-1}) \hat{X}, \quad 2 \cos \mu = \lambda + \frac{1}{\lambda}. \quad (1.3.25)$$

From $\hat{U}^2 = -\hat{I}$ and Eq. (1.3.23) we find that, if $\hat{M}$ is a symplectic matrix, its inverse matrix $\hat{M}^{-1}$ is

$$\hat{M}^{-1} = -\hat{U} \hat{M}^T \hat{U} \equiv -\bar{M}. \quad (1.3.26)$$

Therefore, we can rewrite Eq. (1.3.25) in the form

$$2 \cos \mu \hat{X} = (\hat{M} - \bar{M}) \hat{X}. \quad (1.3.27)$$

Now, using

$$\hat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A , B , C and D are 2×2 matrixes, and $A - \bar{A} = \text{Sp}(A)$, we obtain

$$\hat{M} - \bar{M} = \begin{pmatrix} \text{Sp}(A) & B - \bar{C} \\ C - \bar{B} & \text{Sp}(D) \end{pmatrix}, \quad \bar{B} = u B^T u, \quad u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Defining

$$\hat{X} = \begin{pmatrix} x \\ z \end{pmatrix}$$

we can replace Eq. (1.3.27) by the following system of equations

$$\begin{aligned} (2 \cos \mu - \text{Sp}(A))x - (B - \bar{C})z &= 0, \\ (2 \cos \mu - \text{Sp}(D))z - (C - \bar{B})x &= 0, \end{aligned} \quad (1.3.28)$$

Resolving these equations relative to z

$$z = \frac{C - \bar{B}}{2(\cos \mu - \cos \mu_z)} x, \quad \cos \mu_z = \frac{\text{Sp}(D)}{2},$$

and using

$$B\bar{B} - BC - \overline{BC} + \bar{C}C = |B| + |C| + \text{Sp}(CB),$$

we obtain the equation, which replaces Eq. (1.2.22) for the coupled oscillations:

$$\begin{aligned} (\cos \mu - \cos \mu_z)(\cos \mu - \cos \mu_x) - \frac{\mathcal{B}}{4} &= 0, \\ \mathcal{B} &= |B| + |C| + \text{Sp}(BC). \end{aligned} \quad (1.3.29)$$

The roots of this equation read

$$(\cos \mu)_{1,2} = \frac{\cos \mu_x + \cos \mu_z \pm \sqrt{(\cos \mu_x - \cos \mu_z)^2 + \mathcal{B}}}{2}. \quad (1.3.30)$$

In agreement with Eq. (1.2.23), without coupling [$\mathcal{B} = 0$], this equation gives the roots:

$$\cos \mu_1 = \cos \mu_x, \quad \cos \mu_2 = \cos \mu_z.$$

The strength of the coupling of oscillations is determined by the parameter

$$\Lambda = \frac{\mathcal{B}}{(\cos \mu_x - \cos \mu_z)^2} = \frac{\mathcal{B}}{4 \sin^2(\pi[v_x - v_z]/N_B) \sin^2(\pi[v_x + v_z]/N_B)}. \quad (1.3.31)$$

This expression shows that Λ increases, when $v_x \pm v_z$ approaches integer numbers kN_B , [$k = \pm 1, \pm 2, \dots$]. The case $v_x - v_z = kN_B$ corresponds to the so-called differential-type of the coupling resonance, while the case $v_x + v_z = kN_B$ to the sum-type of the coupling resonances. Some features of the motion near these resonances will be discussed in the next section.

The β -functions of coupled oscillations are determined by the eigenvectors of the matrix $\hat{M} = \hat{M}(\vartheta + \vartheta_0, \vartheta)$, where $\vartheta_0 = 2\pi/N_B$ is the lattice period. We define

$$\begin{aligned} e^{i\mu_a} \hat{f}_a &= \hat{M} \hat{f}_a, \\ e^{-i\mu_a} \hat{f}_a^* &= \hat{M} \hat{f}_a^*, \quad a = 1, 2 \end{aligned}$$

and use as a normalization condition $\hat{f}_a^* \hat{U} \hat{f}_b = -2i\delta_{a,b}$. Since any vector $\hat{X}(\vartheta)$ can be expanded in $e^{i\mu_a \vartheta} \hat{f}_a(\vartheta)$:

$$\hat{X}(\vartheta) = \sum_{a=1,2} [C_a e^{i\mu_a \vartheta} \hat{f}_a(\vartheta) + \text{c.c.}], \quad (1.3.32)$$

$$\hat{f}_a(\vartheta + \vartheta_0) = \hat{f}_a(\vartheta),$$

we write

$$\hat{X}(\vartheta_1) = \sum_{a=1,2} [C_a \hat{f}_a(\vartheta_1) + \text{c.c.}].$$

or, $C_a = -\hat{f}_a^*(\vartheta_1) \hat{U} \hat{X}(\vartheta_1) / (2i)$. Substituting these coefficients in Eq. (1.3.32), we obtain the expression for M similar to that, given in Eq. (1.2.32):

$$\hat{M}(\vartheta_1 + \vartheta_0, \vartheta_1) = - \sum_{a=1,2} \text{Im}\{e^{i\mu_a} \hat{f}_a \hat{f}_a^{*T} \hat{U}\}. \quad (1.3.33)$$

This expression enables the calculation of the β -functions at the point $\vartheta = \vartheta_1$. For other points of the lattice β -functions can be calculated using, for instance, Eq. (1.2.37).

1.4. DISPERSION FUNCTION OF A STORAGE RING

As both the position of the closed orbit inside the vacuum chamber of the ring and the focusing strength depend on the particle energy, it seems important to clarify how much the motion of two particles, which have close momenta p and $p + \Delta p$, differs.

If the deviation Δp from the given value is small, the trajectory of the particle with the momentum $p + \Delta p$ can be represented by the power series

$$\mathbf{r}(\vartheta, p + \Delta p) = \mathbf{r}_0(\vartheta) + \mathbf{r}^{(1)}(\vartheta) (\Delta p/p) + \mathbf{r}^{(2)}(\vartheta) (\Delta p/p)^2 + \dots$$

The Hamiltonian describing the first-order corrections can be obtained by the linearization of Eq. (1.3.5) over $\Delta p/p$:

$$h(\mathbf{p}_\perp, \mathbf{r}_\perp, \vartheta, \Delta p/p) = h_0(\mathbf{p}_\perp, \mathbf{r}_\perp, \vartheta) - \frac{R_0}{R(\vartheta)} \Delta p x + \frac{1}{2R_0} (\Delta g_x x^2 + \Delta g_z z^2). \quad (1.4.1)$$

Here, h_0 is determined by Eq. (1.3.9); the linear term in x describes the chromatic distortion of the closed orbit; the quadratic terms in x and z describe the chromatic variations of the focusing properties of the ring.

Both the calculation of the corrections $\Delta g_{x,z}(\vartheta)$ and their influence on the particle motion will be discussed shortly. [Moreover, in a certain sense these items are of second order in $(x \Delta p/p)$.] Let us consider first the influence of the item

$$\Delta h = -\frac{R_0}{R(\vartheta)} \Delta p x.$$

The Hamiltonian $h + \Delta h$ generates the following equations of motion:

$$x'' + g_x(\vartheta)x = \frac{R_0^2}{R(\vartheta)} \frac{\Delta p}{p}, \quad (1.4.2)$$

$$z'' + g_z(\vartheta)z = 0. \quad (1.4.3)$$

Due to the assumed planar geometry of the unperturbed orbit, the equation for the vertical oscillations did not change, while the horizontal one has got the external force proportional to $\Delta p/p$. As is well known, the general solution of such an equation is given by the sum of a solution of the homogeneous equation and of the partial solution of the inhomogeneous equation. As coefficients in Eq. (1.4.2) are periodical functions of ϑ , the same property has the partial solution of Eq. (1.4.2). The last circumstance is however evident—this solution describes the closed orbit of the particle with the momentum $p + \Delta p$.

To find the necessary partial solution $x_0(\vartheta)$ we introduce the so-called dispersion function of the ring,

$$x_0(\vartheta) = \eta(\vartheta) \frac{\Delta p}{p}, \quad (1.4.4)$$

and rewrite Eq. (1.4.2) as

$$\eta'' + g_x(\vartheta)\eta = \frac{R_0^2}{R(\vartheta)}. \quad (1.4.5)$$

Since $\eta(\vartheta) = \eta(\vartheta + \vartheta_0)$, one may find the solution of Eq. (1.4.5) in the form²

$$\eta(\vartheta) = f_x(\vartheta) \sum_{k=-\infty}^{\infty} C_k \exp(ikN_B \hat{\vartheta}), \quad \hat{\vartheta} = \vartheta + \frac{\chi_x(\vartheta)}{v_x}. \quad (1.4.6)$$

Substituting this expression into Eq. (1.4.5) and using the orthogonality condition

$$\int_0^{\vartheta_0} \frac{d\vartheta}{f_x^2(\vartheta)} \exp[i(k-k')N_B \hat{\vartheta}] = \delta_{k,k'} v_x \vartheta_0, \quad (1.4.7)$$

one gets

$$\eta = v_x f_x(\vartheta) \sum_{k=-\infty}^{\infty} \frac{e^{ikN_B \hat{\vartheta}}}{v_x^2 - (kN_B)^2} \int_0^{\vartheta_0} \frac{d\vartheta'}{\vartheta_0} \frac{R_0^2}{R} f_x(\vartheta') e^{ikN_B \hat{\vartheta}'}. \quad (1.4.8)$$

The function $\eta(\vartheta)$ generates one more important parameter of the ring. Let us calculate the perimeter of the orbit for the particle with momentum $p + \Delta p$:

$$\begin{aligned}\Pi(p + \Delta p) &= \oint d\mathbf{r} \tau(\vartheta) = \int_0^{2\pi} d\vartheta R_0 \left(1 + \frac{x_0(\vartheta)}{R(\vartheta)} \right) \\ &= \Pi(p) + \frac{\Delta p}{p} \Pi(p) \int_0^{2\pi} d\vartheta \frac{\eta(\vartheta)}{2\pi R(\vartheta)},\end{aligned}$$

or

$$\Pi(p + \Delta p) = \Pi(p) \left(1 + \alpha \frac{\Delta p}{p} \right), \quad (1.4.9)$$

where

$$\alpha = \int_0^{2\pi} d\vartheta \frac{\eta(\vartheta)}{2\pi R(\vartheta)} = \left(\frac{p}{\Pi} \frac{\partial \Pi}{\partial p} \right)_H \quad (1.4.10)$$

is the so-called momentum compaction factor of the ring. [Here, expression $(\dots)_H$ means that the value inside the brackets is calculated at constant magnetic field H .] The substitution of Eq. (1.4.8) into Eq. (1.4.10) yields one more expression for α :

$$\alpha = \sum_{k=-\infty}^{\infty} \frac{\nu_x}{\nu_x^2 - (kN_B)^2} \left| \int_0^{\vartheta_0} \frac{d\vartheta}{\vartheta_0} \frac{R_0}{R(\vartheta)} f_x(\vartheta) e^{ikN_B \vartheta} \right|^2. \quad (1.4.11)$$

For the rings with the azimuthal symmetry of the guiding field ($R_0 = R$, and $f_x = 1/\sqrt{\nu_x}$) the calculation of both η and α is easy:

$$\eta = R_0 \alpha = R_0 / \nu_x^2. \quad (1.4.12)$$

Equations (1.4.8) and (1.4.11) indicate that both η and α can be estimated by Eq. (1.4.12) for lattices where the modulation of the factor $f_x R_0 / R(\vartheta)$ is not so strong. Note also that $\eta(\vartheta)$ as well as α generally grow provided ν_x tends to the resonant value $\nu_x = kN_B$.

Let us now calculate the adiabatic invariant of betatron oscillations for the particle with the momentum $p + \Delta p$. We shall write the total solution of Eq. (1.4.2) in the form

$$x = x_b + \eta \frac{\Delta p}{p} = a_x f_x \cos(\psi_x + \chi_x) + \eta \frac{\Delta p}{p}. \quad (1.4.13)$$

From Eq. (1.3.16) we have

$$p_x = \frac{p}{R_0} \left(x'_b + \eta' \frac{\Delta p}{p} \right)$$

and, therefore,

$$a_x \sin(\psi_x + \chi_x) = -f_x R_0 \frac{p_x}{p} + f'_x x + \Phi(\vartheta) \frac{\Delta p}{p},$$

where $\Phi(\vartheta) = \eta' f_x - f'_x \eta$. For $I_x(p_x, x, \Delta p/p)$ this yields

$$\frac{2R_0 I_x}{p} = \frac{\left(x - \eta \frac{\Delta p}{p}\right)^2}{f_x^2} + \left(R_0 f_x \frac{p_x}{p} - f'_x x - \Phi \frac{\Delta p}{p}\right)^2. \quad (1.4.14)$$

Using the equation $\partial I_\alpha / \partial \mathbf{p} = \partial \mathbf{r} / \partial \psi_\alpha$ and Eq. (1.4.14), one can easily calculate the part of ϑ describing its moderation by betatron oscillations (the so-called *betatron phase*):

$$\frac{\partial \vartheta}{\partial \psi_x} = \frac{1}{R_0} \frac{\partial I_x}{\partial \Delta p}. \quad (1.4.15)$$

Since

$$\frac{\partial I_x}{\partial \Delta p} = \frac{1}{R_0} \left(\Phi a_x \sin(\psi_x + \chi_x) - \frac{\eta x_b}{f_x^2} \right),$$

the integration of Eq. (1.4.15) over ψ_x yields

$$\vartheta = \psi_{||} + \vartheta_b = \psi_{||} + \frac{1}{R_0^2} \left(\eta \frac{dx_b}{d\vartheta} - x_b \frac{d\eta}{d\vartheta} \right), \quad (1.4.16)$$

where $\psi_{||}$ is the function independent of ψ_x . It can be calculated by the comparison

$$\frac{d\vartheta_b}{d\vartheta} = -\frac{x_b}{R} \quad (1.4.17)$$

and Eq. (1.1.13), written for the particle with momentum $p + \Delta p$:

$$\frac{d\vartheta}{dt} = \omega_0 \left(1 - \frac{x}{R(\vartheta)} \right) + \frac{\omega_0 \Delta p}{\gamma^2 p} = \omega_0 \left[1 + \alpha_p(\vartheta) \left(\frac{\Delta p}{p} - \frac{x_b}{R} \right) \right], \quad (1.4.18)$$

$$\alpha_p(\vartheta) = 1/\gamma^2 - \eta/R.$$

Comparing this formula with Eq. (1.4.17) we can write for $\psi_{||}$:

$$\psi_{||} = \vartheta_s [1 + \alpha_p(\vartheta) \Delta p/p],$$

where $\vartheta_s = \omega_0 t$ is the azimuth of the particle with momentum p . Finally, we can write

$$\vartheta(I_x, \psi_x, \psi_{||}) = \vartheta_s [1 + \alpha_p(\vartheta_s) \Delta p/p] + \vartheta_b(I_x, \psi_x). \quad (1.4.19)$$

Usually the cyclic part of $\psi_{||}$ is separated as

$$\psi_{\parallel} = \vartheta_s \left[1 + \left(\frac{1}{\gamma^2} - \alpha \right) \frac{\Delta p}{p} \right] \cong \omega_0(p + \Delta p)t,$$

where

$$\omega_0(p + \Delta p) = \omega_0(p) \left[1 + \left(\frac{1}{\gamma^2} - \alpha \right) \frac{\Delta p}{p} \right] = \omega_0 + \left(\frac{d\omega_0}{dp} \right) \Delta p.$$

Note that at energy $\gamma = \alpha^{-1/2}$ the derivative $(d\omega_0/dp)_0$ can change sign if $\alpha < 1$. The value $\gamma_{tr} = \alpha^{-1/2}$ determines the so-called transition energy of the ring.

Let us give the final expression for the Hamiltonian of the particle having the momentum $p + \Delta p$ in the action-phase variables:

$$H(I, \Delta p) = \nu_0 \Delta p + \frac{\Delta p^2}{2\mu_{\parallel}(\vartheta)} + \omega_{\perp} I_{\perp}. \quad (1.4.20)$$

Here, $\omega_{\perp} = \omega_0 \nu_{\perp}$ and

$$\mu_{\parallel}^{-1}(\vartheta) = \frac{\alpha_p(\vartheta)}{\gamma m} \quad (1.4.21)$$

is so-called effective mass of the particle longitudinal motion in the storage ring.

1.5. PERTURBED BETATRON OSCILLATIONS

1.5.1. In this section we shall continue the study of betatron oscillations of a particle described by the Hamiltonian (1.4.1). To be more definite, we shall talk about vertical oscillations, when Eq. (1.4.1) can be rewritten as

$$h_z(p_z, z, \Delta p/p, \vartheta) = h_{z,0}(p_z, z, \vartheta) + \Delta h. \quad (1.5.1)$$

Here, $h_{z,0}(p_z, z, \vartheta)$ is given by Eq. (1.3.9), while

$$\Delta h_z(p_z, z, \Delta p/p, \vartheta) = \frac{p_0}{2R_0} \Delta g_z z^2 \quad (1.5.2)$$

describes the perturbation of the betatron oscillations of the particle.

Using action-phase variables one can rewrite h in the form

$$h = \nu I + \frac{p_0}{2R_0} \Delta g_z(\vartheta) z^2(I, \psi, \vartheta). \quad (1.5.3)$$

(omitting the subscript z for short). Then, equations of motion read

$$I' = -\frac{\partial h}{\partial \psi} = \frac{p_0 \Delta g(\vartheta)}{R_0} z \frac{\partial z}{\partial \psi}, \quad (1.5.4)$$

$$\psi' = \frac{\partial h}{\partial I} = \nu + \frac{p_0 \Delta g(\vartheta)}{R_0} z \frac{\partial z}{\partial I}. \quad (1.5.5)$$

Since usually Eqs. (1.5.4) and (1.5.5) cannot be solved in some general way, perturbation theory must be employed for future discussion. Due to the cyclic nature of the particle motion in the ring, the functions contributing into the r.h.s. of Eqs. (1.5.4) and (1.5.5) are periodic in ψ and ϑ . Therefore, this system can be studied using the averaging method.¹⁷

In the first approximation, the systematic variation of I and ψ is calculated by averaging Eqs. (1.5.4) and (1.5.5) over intervals of ϑ sufficiently longer than the period of betatron oscillations. Small oscillations of I and $\psi - \nu \vartheta$, which can be excited by the perturbation, are neglected in the calculations in this step. Expanding $\Delta h(I, \psi, \vartheta)$ into the series

$$\Delta h(I, \psi, \vartheta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \Delta h_{m,n}(I) e^{im\psi - in\vartheta}, \quad (1.5.6)$$

one can see that, after the averaging, at least the term

$$\Delta h_{0,0}(I) = \int_0^{2\pi} \frac{d\vartheta}{2\pi} \int_0^{2\pi} \frac{d\psi}{2\pi} \Delta h(I, \psi, \vartheta) \quad (1.5.7)$$

contributes into the r.h.s of Eqs. (1.5.4) and (1.5.5). The contributions of the other terms generally depend on the tuning of the betatron frequency to resonant values

$$\nu = n/m \quad (1.5.8)$$

(the necessary value of the tuning is determined by the value of the harmonic $\Delta h_{m,n}$). Provided the frequency ν is far away from the resonance (1.5.8), harmonics with $m, n \neq 0$ will make fast oscillations and thus will not contribute into the r.h.s of Eqs. (1.5.4) and (1.5.5).

Therefore, in the nonresonant case, the Hamiltonian in the first approximation of the averaging method has the form

$$h = \nu I + \Delta h_{0,0}(I), \quad (1.5.9)$$

and the perturbation changes the frequencies of betatron oscillations by the value

$$\Delta \nu = \frac{\partial \Delta h_{0,0}}{\partial I}. \quad (1.5.10)$$

Substituting Δh from Eq. (1.5.3) yields

$$\Delta \nu_z = \frac{1}{2} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \Delta g_z(\vartheta) f_z^2(\vartheta),$$

or, using β -function,

$$\Delta\nu_z = \frac{1}{2R_0} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \Delta g_z(\vartheta) \beta_z(\vartheta). \quad (1.5.11)$$

The similar equation with the obvious replacement of subscripts x with z determines the tune shift of the horizontal betatron oscillations.

In particular, the first chromatic corrections to betatron tunes in strong focusing rings ($|n(\vartheta)| \gg 1$) are described by

$$\Delta g_{x,z}(\vartheta) = \mp \frac{\Delta p}{p} \frac{eR_0^2}{pc} \left\{ \frac{\partial H_z}{\partial x} - \eta(\vartheta) \frac{\partial^2 H_z}{\partial x^2} \right\}_{r=r_0(\vartheta)}. \quad (1.5.12)$$

In this case the expansion of the r.h.s. of Eqs. (1.1.14) and (1.1.15) should take into account the quadratic terms in x and z . After the substitution of Eq. (1.5.12) into Eq. (1.5.11) we write

$$\Delta\nu_{x,z}(\Delta p) = \mp \frac{\Delta p}{p} \frac{e}{4\pi pc} \int_0^\pi ds \beta_{x,z}(s) \left(\frac{\partial H_z}{\partial x} - \eta(s) \frac{\partial^2 H_z}{\partial x^2} \right)_{r=r_0(s)}. \quad (1.5.13)$$

The dependence of the chromatic correction $\Delta\nu_{x,z}(\Delta p)$ on the sextupole components of the guiding magnetic field $\partial^2 H_z / \partial x^2$ is typically used to control the natural chromaticity of the ring:

$$\Delta\nu_{x,z}(\Delta p) = \mp \frac{\Delta p}{p} \frac{e}{4\pi pc} \int_0^\pi ds \beta_{x,z}(s) \left(\frac{\partial H_z}{\partial x} \right)_{r=r_0(s)}.$$

Referring the tune shifts the nonlinearity of the guiding field can cause one more important effect—the dependence of betatron tunes on the amplitudes of betatron oscillations. From the symmetry of forces acting on a particle, it is clear that in the lowest approximation of the averaging method such corrections might be caused by the odd powers of z in the expansion of the guiding field (cubic and so on). If, for instance, the octupole is placed on the orbit

$$H_z = \frac{1}{6} \frac{\partial^3 H_z}{\partial x^3} (x^3 - 3xz^2),$$

$$H_x = -\frac{1}{6} \frac{\partial^3 H_z}{\partial x^3} (z^3 - 3zx^2),$$

the correction Δg_z in Eq. (1.5.1) now reads

$$\Delta g_z = -\frac{eR_0^2}{6pc} \frac{\partial^3 H_z}{\partial x^3} (z^2 - 3zx^2),$$

which corresponds to

$$\Delta h = -\frac{1}{12} \frac{eR_0}{c} \frac{\partial^3 H_z}{\partial x^3} (x^4 + z^4 - 6x^2 z^2). \quad (1.5.14)$$

The substitution of Eq. (1.5.14) into Eq. (1.5.10) yields

$$\begin{aligned} \Delta v_z(J_z, J_x) &= \frac{\partial v_z}{\partial J_z} J_z + \frac{\partial v_z}{\partial J_x} J_x, \quad I_{x,z} = pJ_{x,z}/2, \\ \frac{\partial v_z}{\partial J_{z,x}} &= \mp \frac{e}{8\pi pc} \int_0^\Pi ds \beta_z(s) \beta_{z,x}(s) \left(\frac{\partial^3 H_z}{\partial x^3} \right)_{\mathbf{r}=\mathbf{r}_0(s)}. \end{aligned} \quad (1.5.15)$$

The even (sextupole and others) nonlinearities of the guiding field will contribute in $\Delta v(J)$ in the second-order approximation of the averaging method.

1.5.2. Let us now discuss the distortion of the particle motion by resonant perturbations. The simplest case is described by the Hamiltonian (1.5.3), which we shall rewrite in the form

$$h = \nu I + \frac{I}{2} \Delta g(\vartheta) f^2(\vartheta) \{1 + \cos(2[\psi + \chi(\vartheta)])\}. \quad (1.5.16)$$

This expression indicates that close to parametric resonance $\nu \simeq n/2$ except for $\Delta h_{0,0}$ the term

$$\frac{I}{2} |Q_n| \cos(2\psi - n\vartheta + \arg Q_n)$$

becomes a slowly varying function of ϑ . Here,

$$Q_n = \int_0^{2\pi} \frac{d\vartheta}{2\pi} \Delta g(\vartheta) f^2(\vartheta) \exp[in\vartheta + 2i\chi(\vartheta)].$$

Introducing the so-called slow phase

$$\delta\psi = \psi - \frac{n}{2} \vartheta + \frac{1}{2} \arg Q_n \quad (1.5.17)$$

and taking into account in Eq. (1.5.16) only the slowly varying terms, we can write the so-called reduced Hamiltonian

$$h(I, \delta\psi) = \Delta I + \frac{I}{2} |Q_n| \cos(2\delta\psi) \quad (1.5.18)$$

where $\Delta = \nu - n/2$ is the detuning from the resonance.

Since Δh does not depend on the azimuth ϑ itself, this Hamiltonian is the integral of the motion. Thus, we can write for the phase trajectories of the particles

$$I(\delta\psi) = \frac{h}{\Delta + \frac{|Q_n|}{2} \cos(2\delta\psi)}, \quad (1.5.19)$$

which indicates that outside the resonance

$$|Q_n| < 2|\Delta|$$

the amplitudes of betatron oscillations execute the beatings with the depth

$$\frac{\Delta I_{\max}}{\Delta I_{\min}} = \frac{\Delta + \frac{|Q_n|}{2}}{\Delta - \frac{|Q_n|}{2}}. \quad (1.5.20)$$

Inside the region of the resonance

$$|Q_n| \geq 2|\Delta| \quad (1.5.21)$$

the amplitudes of oscillations grow infinitely. Therefore, condition (1.5.21) specifies the width of the resonance stopband near $\nu = n/2$.

One can study the general properties of the motion near a resonance inspecting the behavior of functions⁸

$$h_{\pm} = h[I, \cos(2\delta\psi) = \pm 1] = \Delta I \pm \frac{I}{2} |Q_n|. \quad (1.5.22)$$

Since h is the integral of the motion, in the region $|Q_n| \leq 2|\Delta|$ the particle, starting say from the line $h_+(I)$, moves up to the line $h_-(I)$, and then returns back. [See Fig. 1.1(a).] On the contrary, if the condition (1.5.21) is valid, Hamiltonians $h_{\pm}$ form an open corner [Fig. 1.1(b)] and a particle starting from h_+ (or from h_-) can move to infinity.

As we mentioned before, the cubic nonlinearity of the guiding field causes the dependence of the tunes on the amplitudes of the betatron oscillations. This modifies Eq. (1.5.22) into

$$h_{\pm} = \Delta I + \frac{I^2}{2} \frac{\partial \nu}{\partial I} \pm \frac{I}{2} |Q_n|. \quad (1.5.23)$$

Inspection of Fig. 1.2(a) shows that, in the off-resonant region $|Q_n| \leq 2|\Delta|$ this causes no significant variations in dependencies of amplitudes on the time.

The situation drastically changes inside the resonance region $|Q_n| \geq 2|\Delta|$. First of all, due to nonlinear correction, the infinite growth of amplitudes is now

replaced by the beating. The reason is obvious—the variation of the betatron tune with increase in amplitude detunes the particle from the resonance. Secondly, as seen in Fig. 1.2, the character of the phase trajectories now depends on the sign of h . In the region $h > 0$ the amplitude oscillates between the curves $h_{\pm}(I)$, while the slow phase $\delta\psi$ grows infinitely. In the region $h < 0$ [see Fig. 2(b)] both stop points for I belong to $h_{-}(I)$, which corresponds to oscillation of the slow phase within the interval

$$|\delta\psi - \delta\psi_s| \leq \arccos \left\{ \frac{2}{|Q_n|} \sqrt{2|h|(\partial\nu/\partial I)} + \Delta \right\},$$

$$\delta\psi_s = \pi/2, \quad 3\pi/2 \quad (1.5.24)$$

Figure 1.2(b) shows that the phase trajectory, which separates the finite and infinite motion in $\delta\psi$ (the separatrix), corresponds to $h = 0$. The position of the center of the separatrix (below, we shall also call this curve a bucket) I_s and its width ΔI_c can be found from the equations

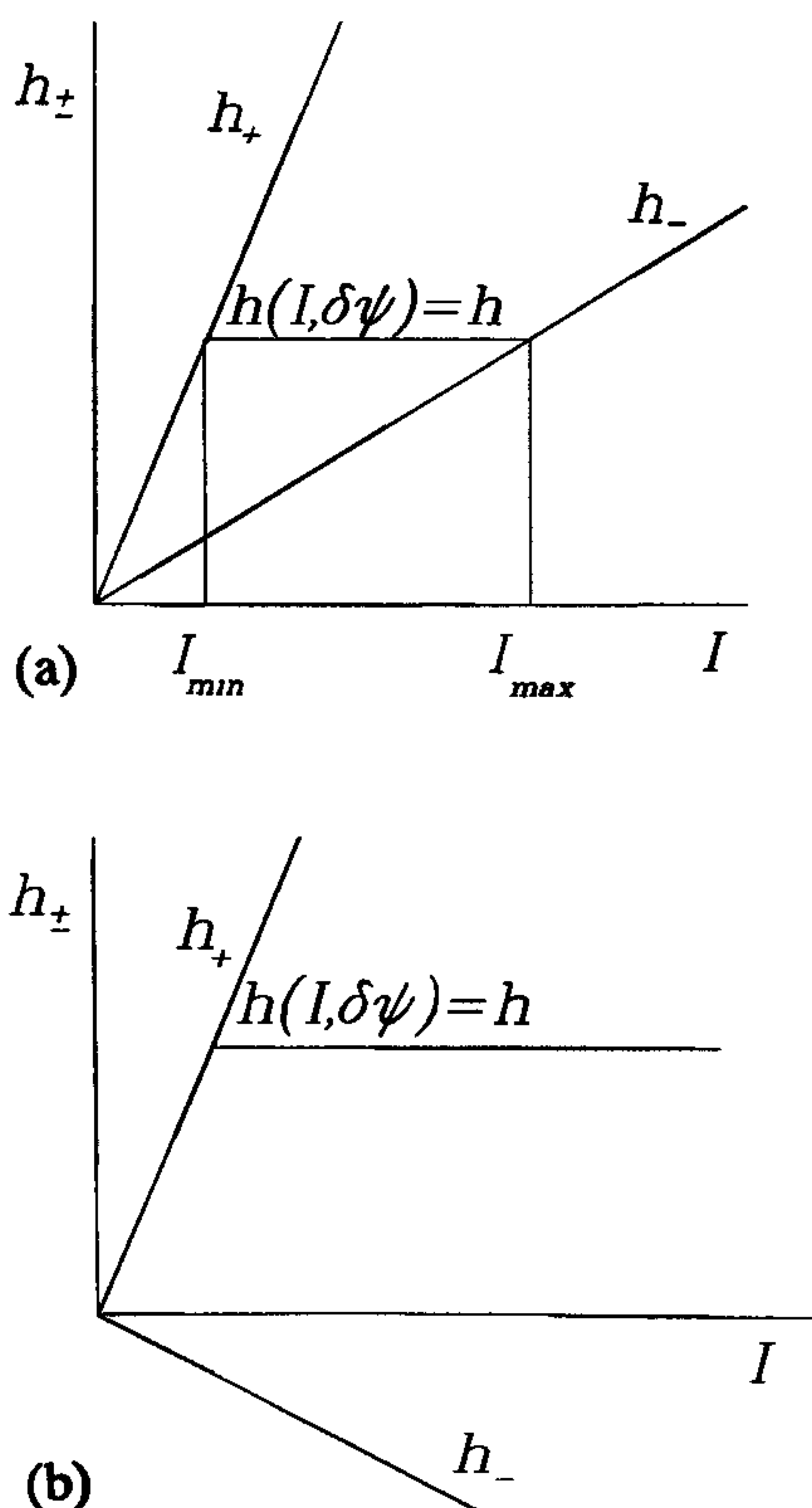


FIG. 1.1. The Hamiltonians $h_{\pm}$ for linear parametric resonances; (a) outside the stopband $|\Delta| > |Q_n|/2$, (b) inside the stopband $|\Delta| < |Q_n|/2$.

$$\frac{\partial h_{-}(I)}{\partial I} = 0, \quad h_{-}(\Delta I) = 0. \quad (1.5.25)$$

Substituting here $h_{-}(I)$ from Eq. (1.5.23) we obtain

$$I_s = \frac{|Q_n| - 2\Delta}{2|\partial v / \partial I|}, \quad \Delta I_c = 2I_s. \quad (1.5.26)$$

These equations indicate that an increase in the coefficient $\partial v / \partial I$ generally decreases the beatings of amplitudes due to the parametric instability. This effect is important since the forming of separatrices in phase space may cause the collection of particles inside them. As the amplitudes of oscillations of such particles will be close to $\sqrt{2I_s R/p}$, the collection of particles in separatrices will stratify the beam and, therefore, dilute its phase space density.

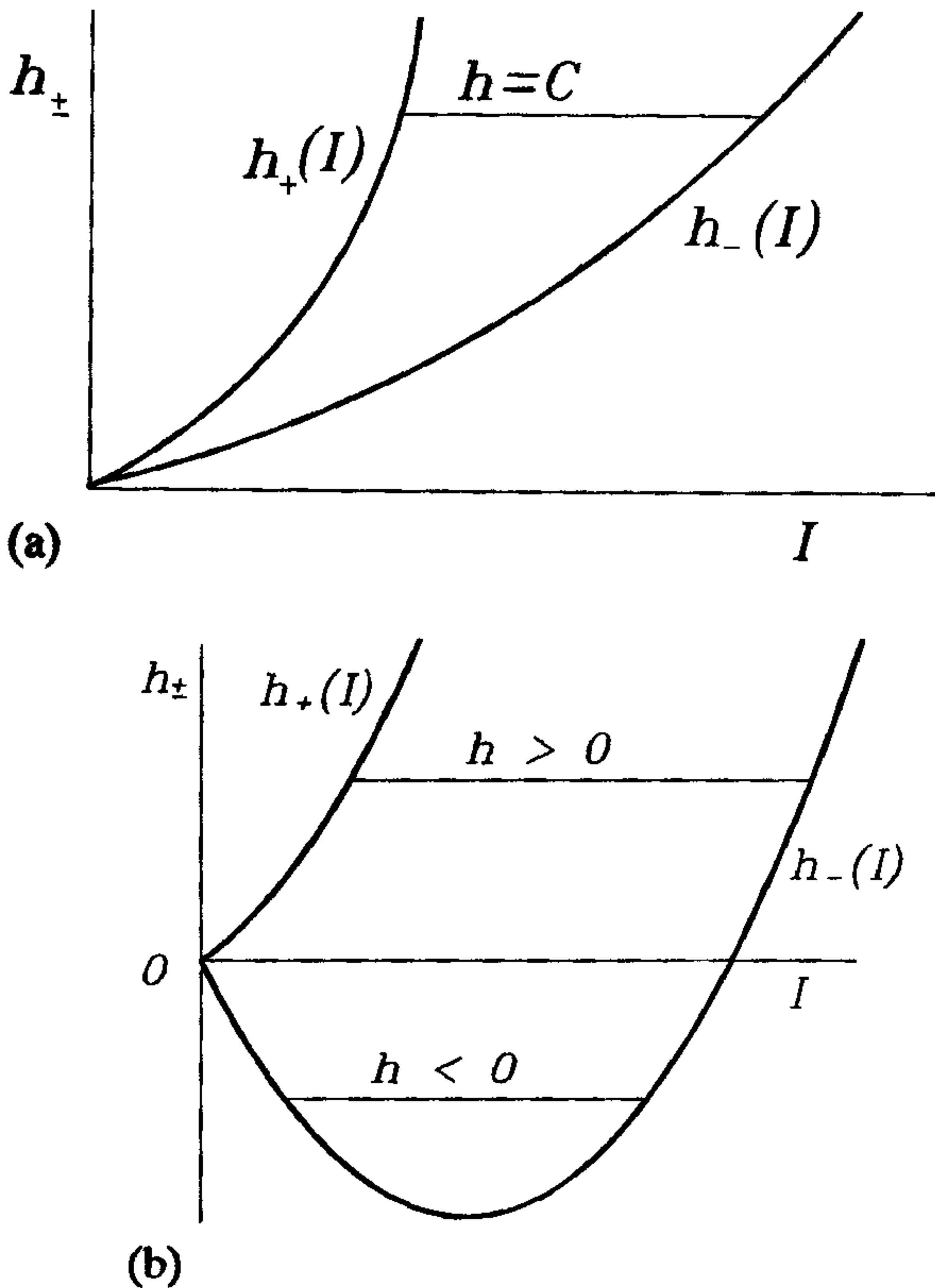


FIG. 1.2. The Hamiltonians $h_{\pm}$ for nonlinear parametric resonances. (a) outside the stopband $|\Delta| > |Q_n|/2$, (b) inside the stopband $|\Delta| < |Q_n|/2$; ($h > 0$ —the motion outside the separatrix, $h < 0$ —inside the separatrix).

1.5.3. The parametric resonance is certainly not the only dangerous resonance in storage rings. In the general case, any perturbation of the Hamiltonian (1.3.9) having the form

$$\Delta h(\mathbf{I}, \psi, \vartheta) = \sum_{\mathbf{m}, n} \Delta h_{\mathbf{m}, n} e^{i\mathbf{m}\psi - in\vartheta},$$

$$\mathbf{m}\psi = m_x \psi_x + m_z \psi_z, \quad (1.5.27)$$

will excite the nonlinear resonance if the combination tune ($\mathbf{m}\nu$) becomes close to the integer

$$m_x \nu_x + m_z \nu_z = n. \quad (1.5.28)$$

These resonances can be either one-dimensional $m_x m_z = 0$ (the parametric resonance previously discussed corresponds to the combination, for instance, $m_x = 2, m_z = 0$) or the coupling resonances $m_x m_z \neq 0$. In the simplest and one of the most important cases, it is the linear coupling $\nu_x \pm \nu_z = n$.

The perturbation of the particle motion due to the particular resonance Eq. (1.5.28) is described by the averaged Hamiltonian

$$h = \nu \mathbf{I} + \Delta h_{0,0}(\mathbf{I}) + 2 |\Delta h_{\mathbf{m},n}(\mathbf{I})| \cos(\mathbf{m}\psi - n\vartheta + \arg \Delta h_{\mathbf{m},n}). \quad (1.5.29)$$

It generates the equations of motion

$$I'_\alpha = -\frac{\partial h}{\partial \psi_\alpha}, \quad \psi'_\alpha = \frac{\partial h}{\partial I_\alpha},$$

which imply the complementary integral of motion

$$\frac{I_x}{m_x} - \frac{I_z}{m_z} = C_m. \quad (1.5.30)$$

Due to the conservation of C_m the characters of the motion near the sum ($m_x m_z > 0$) and differential ($m_x m_z < 0$) types of resonances significantly differ. If $m_x m_z < 0$, the sum of the positive values

$$\frac{I_x}{m_x} + \frac{I_z}{m_z}$$

is conserved. Therefore, the growth of the amplitude in one degree of freedom I_x (or I_z) will be accompanied by a decrease of the amplitude in the other degree—the differential type resonance redistributes the energy between coupled oscillations. On the other hand, for the sum type resonance, the conservation of the difference

$$\frac{I_x}{m_x} - \frac{I_z}{m_z}$$

will cause the blowup of both amplitudes, provided one of them grows.

Once C_m is conserved, the number of variables necessary to describe the motion near the coupling resonance can be reduced to two, say, I_x and $\Delta\psi = m\psi - n\vartheta + \arg\Delta h_{m,n}$, which converts the problem into an equivalent one-dimensional one. At this point the inspection of the Hamiltonians $h_{\pm}(I_x)$

$$h_{\pm}(I_x, C_m) = \Delta_{m,n} I_x + m_x \Delta h_{0,0}(I) \pm 2m_x |\Delta h_{m,n}(I)| \quad (1.5.31)$$

again can simplify the analysis. Here, $\Delta_{m,n} = m\nu - n$ is the detuning from the resonance. If the cubic nonlinearity provides the main contribution to $\Delta h_{0,0}$,

$$\Delta h_{0,0} = \frac{I^2}{2} \frac{\partial \nu}{\partial I},$$

one can see that the Hamiltonian (1.5.31) will describe the finite motion in I , provided $|\Delta h_{m,n}|$ grows with I not faster than I^2 . Otherwise, $h_{\pm}(I)$ can make the open corners (Fig. 1.3) and among the phase trajectories unclosed ones can appear.

1.5.4. In low-energy proton, or ion storage rings and accelerators the Coulomb repulsion of particles can cause additional instability of a particle phase space. Typically this instability poses a serious limitation on the beam intensity in those installations. We consider here the basic reference points of this phenomenon within the framework of a very simplified model. Namely, we assume that the beam is coasting and has a round cross section. For the sake of simplicity we assume also that the curvature of the closed orbit as well as perturbation of a particle motion by a longitudinal electric field of the beam can be neglected. Then, in the cylindrical coordinate system with z -axis directed along the pipe axis, the electric field due to the beam space charge has only radial

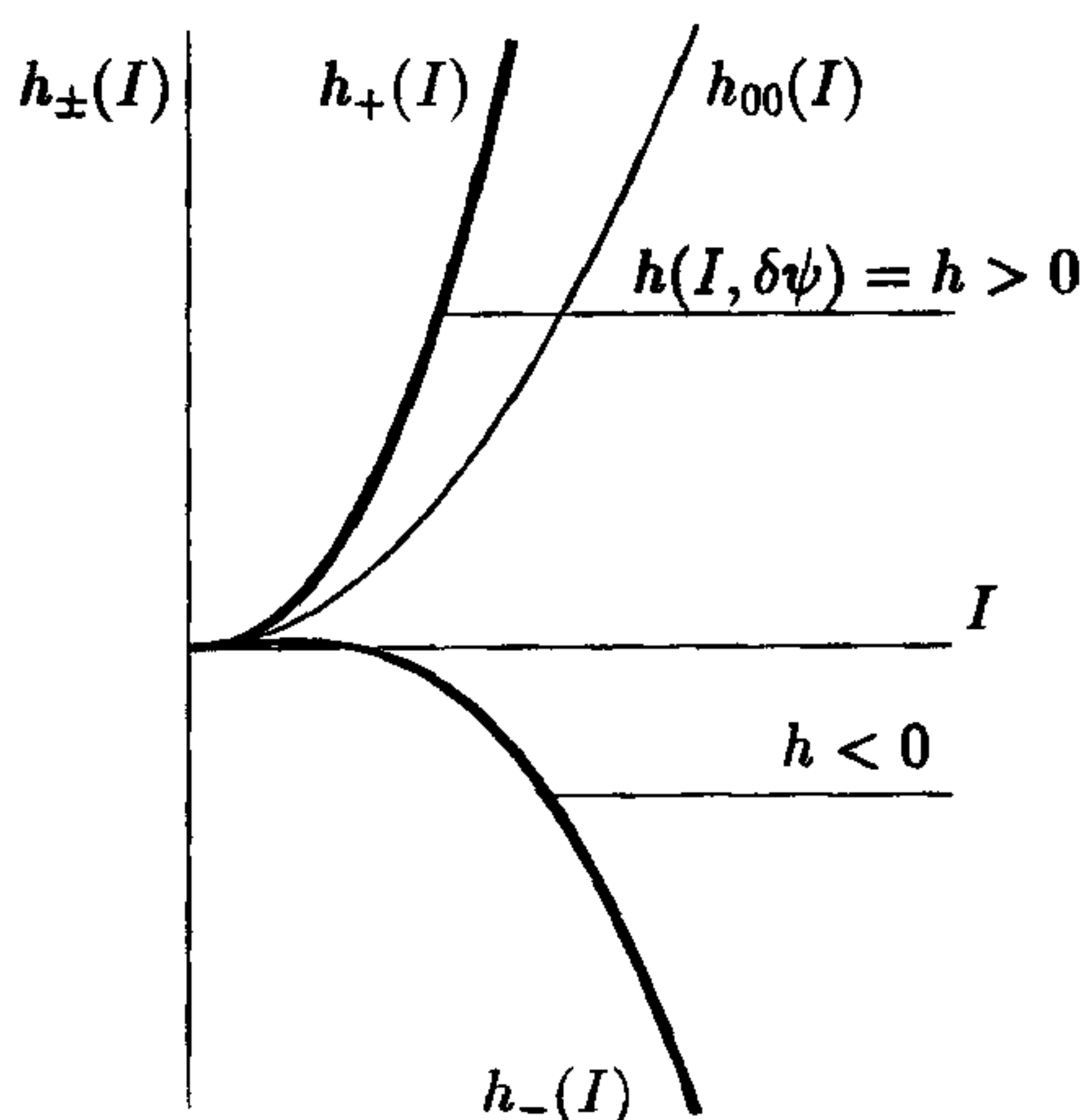


FIG. 1.3. The Hamiltonians $h_{\pm}$ for one-dimensional nonlinear resonances with $m > 4$.

E_r , and magnetic-only azimuthal H_ϕ components. This results in the perturbation of betatron oscillations by the radial force:

$$\mathcal{F}_r = e \left(E_r - \frac{v}{c} H_\phi \right). \quad (1.5.32)$$

With taken assumptions the fields E_r and H_ϕ can be calculated using, for instance, the Gauss and Ampère laws,⁹ yielding

$$H_\phi = \frac{v}{c} E_r$$

and

$$2\pi r E_r = 4\pi e 2\pi \int_0^r dr_1 r_1 \rho(r_1), \quad (1.5.33)$$

where ρ is the beam density. Taking as ρ the Gaussian distribution over transverse coordinates of particles

$$\rho(r_1) = \frac{N}{\Pi 2\pi\sigma^2} \exp\left(-\frac{r_1^2}{2\sigma^2}\right), \quad (1.5.34)$$

we rewrite Eq. (1.5.33) in the following form:

$$E_r = \frac{2Ne}{\Pi r} [1 - \exp(-r_1^2/2\sigma^2)].$$

This results in

$$\mathcal{F}_r = \frac{eE_r}{\gamma^2} = \frac{2Ne^2}{\Pi\gamma^2 r_1} [1 - \exp(-r_1^2/2\sigma^2)],$$

or, in the vector form,

$$\mathcal{F} = \frac{2Ne^2}{\Pi\gamma^2} \frac{[1 - \exp(-r_1^2/2\sigma^2)]}{r_1^2} \mathbf{r}_1. \quad (1.5.35)$$

The value σ in these equations is related to the beam emittance ε and the lattice β -function:

$$\sigma^2 = \varepsilon\beta(s). \quad (1.5.36)$$

Within the very simplified description adopted here we may discuss only very crude space charge effects. Let us consider first the perturbations of small betatron oscillations ($r_1 \ll \sigma$), when $\mathcal{F}$ can be linearized in r_1 :

$$\mathcal{F} = \frac{Ne^2}{\Pi\gamma^2\sigma^2} \mathbf{r}_1. \quad (1.5.37)$$

This defocusing force, acting along the whole orbit, can change betatron tunes of particles and lattice β -functions, or the beam envelope. In the first approximation, calculation of the tune shifts is easy. Since

$$\dot{\psi}_\alpha = \omega_\alpha + \frac{\partial \psi_\alpha}{\partial \mathbf{p}} \cdot \vec{\mathcal{F}},$$

the tune shift is defined by

$$\begin{aligned} \Delta \nu_\alpha &= \frac{\langle \dot{\psi}_\alpha \rangle - \nu_\alpha \omega_0}{\omega_0} \\ &= -\frac{2R_0}{pv} \left\langle \frac{\partial r_1}{\partial J_\alpha} \mathcal{F} \right\rangle. \end{aligned} \quad (1.5.38)$$

The substitution here of Eq. (1.5.37) and the averaging over betatron phases and s yield the so-called Coulomb or Laslett tune shift^{10,11}:

$$\Delta \nu_{\alpha,L} = -\frac{Ne^2}{4\pi p v \gamma^2 \varepsilon}. \quad (1.5.39)$$

In application to proton accelerators in view of adiabatic conservation of $\gamma(v/c)\varepsilon = \varepsilon_{inv}$, Eq. (1.5.39) is also written in the form

$$\Delta \nu_\alpha = -\frac{Nr_p}{4\pi(v/c)\gamma^2 \varepsilon_{inv}}, \quad r_p = \frac{e^2}{mc^2}. \quad (1.5.40)$$

As could be expected beforehand, due to Coulomb repulsion of particles $\Delta \nu$ is negative.

Another important issue of the linear space charge perturbation is the modification of β -functions of the ring. (See also, for instance, Refs. 12 and 13.) The action on particles of the force from Eq. (1.5.37) changes the envelope equation (1.2.30):

$$f'' + g(\vartheta)f(\vartheta) = \frac{1}{f^3(\vartheta)} + \frac{Ne^2}{\Pi p v \gamma^2 \sigma^2} f(\vartheta).$$

Taking as a new variable $\sigma = f(\vartheta) \sqrt{R_0 \varepsilon}$, we can rewrite this equation in the form

$$\frac{d^2 \sigma}{ds^2} + k(s)\sigma = \frac{\varepsilon^2}{\sigma^3} + \frac{2|\Delta \nu_L| \varepsilon}{R_0 \sigma(s)}. \quad (1.5.41)$$

Even in simple cases this equation must be solved numerically. As the illustration we consider the change of the beam envelope in a straight section of the ring, when $k(s) = 0$. Using here $p = d\sigma/ds$, we can write down the first integral of Eq. (1.5.41):

$$h(p, \sigma) = \frac{p^2}{2} + \frac{\varepsilon^2}{2\sigma^2} + \frac{2|\Delta v_L|\varepsilon}{R_0} \ln \sigma. \quad (1.5.42)$$

Assuming that the β -function has the minimum in the middle of the straight section $p(s=0) = 0$ and $\beta(0) = \beta_0$, we can express h in terms of β_0 only. This results in

$$2\frac{s}{\beta_0} = \int_1^{\beta(s)/\beta_0} \frac{du}{\sqrt{u(1+\Lambda \ln u) - 1}}, \quad \Lambda = \frac{4|\Delta v_L|\beta_0}{R_0}. \quad (1.5.43)$$

Numerical integration in Eq. (1.5.43) (see Fig. 1.4) indicates the enhancement of the modulation of $\beta(s)$, as the space charge parameter

$$\Lambda = 4 \frac{|\Delta v_L|}{v} \frac{\beta_0}{\bar{\beta}}, \quad \bar{\beta} = \frac{R_0}{v} \quad (1.5.44)$$

increases.

Apart from lattice variations, the space charge instability of the beam can manifest itself in numerous nonlinear phenomena. If, for instance, we substitute in Eq. (1.5.38) the total force from Eq. (1.5.35), then for one-dimensional (say, vertical $J_x=0$, $J_z=J$) oscillations simple calculations yield

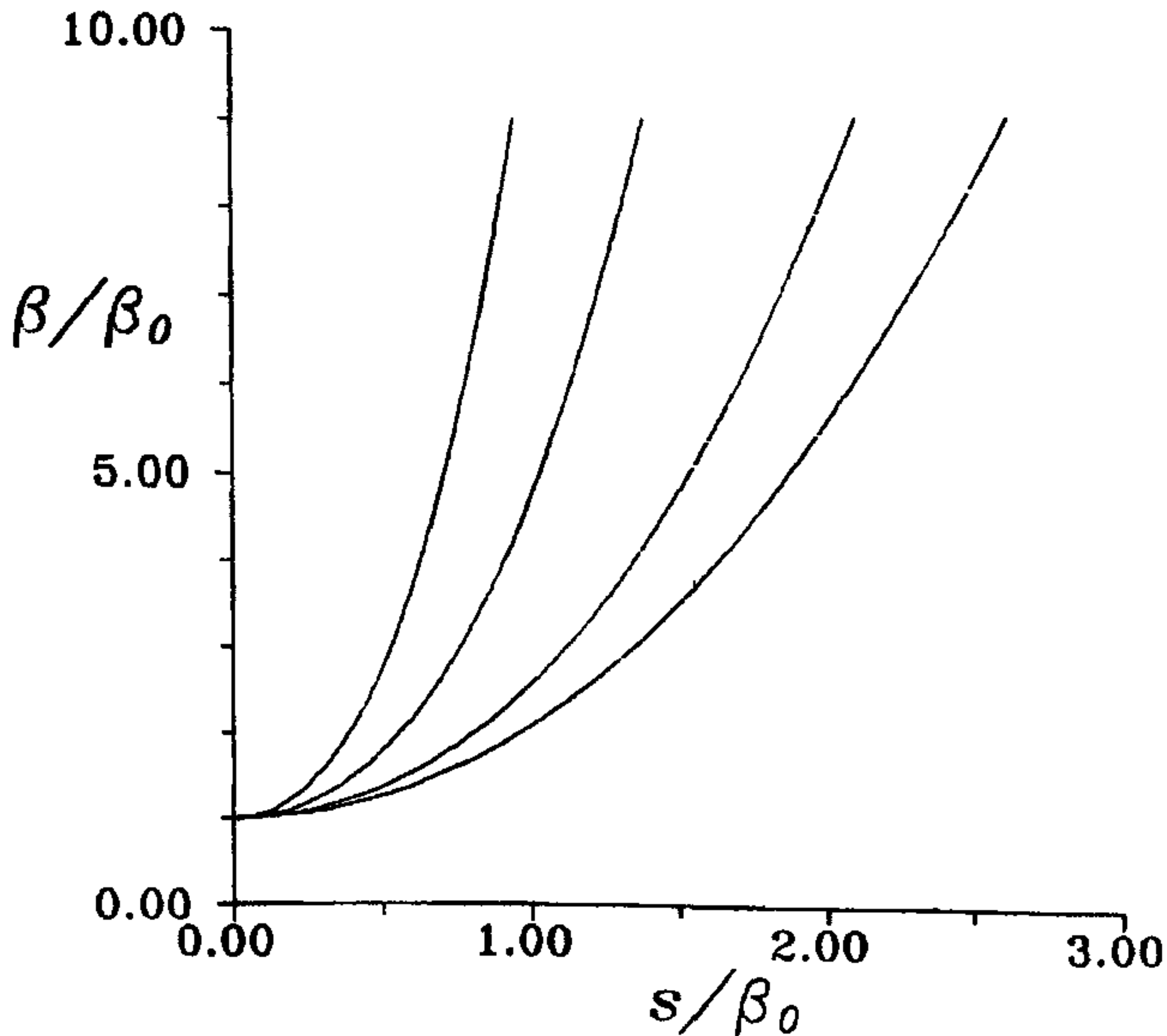


FIG. 1.4. The influence of the beam space charge on the dependence of β on s in a straight section (due to assumed mirror symmetry only the right half of the figure is plotted); from top to bottom the parameter $\Lambda = 4|\Delta v_L|\beta_0/R_0$ is: 5, 2, 0.5, 0.1.

$$\Delta v_z = \Delta v_L \frac{2\varepsilon}{J} \left[1 - \exp\left(-\frac{J}{2\varepsilon}\right) \right]. \quad (1.5.45)$$

This formula shows that due to nonlinearity of the space charge force, instead of the one working point, the beam covers in the plane (v_x, v_z) the whole spot with the width defined by $|\Delta v_L|$. It is obvious that to avoid the dilution of the beam phase space density, the machine resonances due to lattice imperfections should be well suppressed inside this spot.

Since σ in Eq. (1.5.35) generally depends on s , in the linear lattice nonlinear resonances of betatron oscillations can be excited by the space charge itself. For one-dimensional oscillations it can take place if the fractional part of tune ν occurs close to a rational number n/m . Before making particular calculations, we note that the force in Eq. (1.5.35) has the potential

$$\mathcal{F} = -\frac{\partial U}{\partial \mathbf{r}_\perp},$$

$$U(\mathbf{r}_\perp) = -\frac{Ne^2}{\gamma^2 \Pi} \int \frac{d^2 k_\perp}{\pi k_\perp^2} \exp\left(i \mathbf{k}_\perp \cdot \mathbf{r}_\perp - \frac{k_\perp^2 \sigma^2}{2}\right). \quad (1.5.46)$$

Perturbed one-dimensional oscillations are described by the Hamiltonian:

$$h = \nu J + \frac{2R_0}{pv} U[z(J, \psi, \vartheta), \vartheta]. \quad (1.5.47)$$

As was already mentioned, near the resonance $\nu \simeq n/m$ only the zeroth and m th harmonics of h in betatron phases are significant resulting in the Hamiltonian in slow variables J and $\delta\psi = \psi - n\vartheta/m$:

$$h = \Delta_0 J + h_{0,0}(J) + h_{m,n}(J) \cos(m\delta\psi), \quad \Delta_0 = \nu - n/m. \quad (1.5.48)$$

The function $h_{0,0}$ indeed was calculated in Eq. (1.5.45). Thus, we can write

$$h_{0,0}(J) = -2|\Delta v_L| \varepsilon \int_0^{J/2\varepsilon} \frac{dx}{x} (1 - e^{-x}). \quad (1.5.49)$$

The calculation of $h_{m,n}$ is simplified using the expansion

$$\exp(ix \cos \psi) = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\psi}, \quad (1.5.50)$$

where $J_m(x)$ is Bessel function of the m th order.¹⁴ Direct calculations yield

$$h_{m,n} = -4|\Delta v_L| \varepsilon V_m(J/2\varepsilon) Y_{m/2}, \quad (1.5.51)$$

where

$$Y_q = \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp[i2q\phi(\vartheta)], \quad m=2q, \quad \phi(\vartheta) = v\vartheta + \chi(\vartheta) \quad (1.5.52)$$

and

$$V_m = \int_{-\infty}^{\infty} \frac{dk}{k} J_m(k\sqrt{J/2\varepsilon}) e^{-k^2/2},$$

$$= \begin{cases} 0, & m=2q+1, \\ \left(\frac{J}{2\varepsilon}\right)^q \frac{(q-1)!}{(2q)!} F\left(q, 2q+1, -\frac{J}{2\varepsilon}\right), & m=2q. \end{cases} \quad (1.5.53)$$

Here, $F(\alpha, \gamma, x)$ is the confluent hypergeometric function.¹⁴ Using identities,¹⁴

$$F(q, 2q+1, -x) = e^{-x} F(q+1, 2q+1, x) = 2e^{-x} \frac{d}{dx} F(q, 2q, x),$$

and

$$F(q, 2q, x) = \Gamma(q+1/2) e^{x/2} \left(\frac{4}{x}\right)^{(q-1/2)} I_{q-1/2}(x/2),$$

where $I_m(x)$ is the Bessel function of imaginary argument, we also can present V_{2q} in the form

$$V_{2q} = \frac{1}{2q} \sqrt{\frac{\pi J}{2\varepsilon}} \exp\left(-\frac{J}{4\varepsilon}\right) \left[I_{q-1/2}\left(\frac{J}{4\varepsilon}\right) + I_{q+1/2}\left(\frac{J}{4\varepsilon}\right) \right]. \quad (1.5.54)$$

As seen from Eq. (1.5.51), apart from V_m the strengths of space charge resonances are proportional to Y_q , describing the azimuthal modulation of deflecting force. Since

$$\frac{d\phi}{ds} = \frac{1}{\beta(s)},$$

after integration in Eq. (1.5.52) by parts, we can rewrite Y_q like

$$Y_q = \frac{i}{2q} \int_0^\pi \frac{ds}{\Pi} \frac{d\beta(s)}{ds} \exp[2qi\phi(s)]. \quad (1.5.55)$$

This equation clearly indicates that the strengths ($|h_{m,n}|$) of space charge resonances are significantly related to Floquet modulation of betatron functions. For instance, in a round, weak focusing machine one has $\chi = 0$, and hence, the space charge resonances will not be excited. In a strong focusing machine the value of Y_q generally depends on the distribution of lattice functions around the ring.

As an illustration we consider the model ring, where the focusing of particles is performed by rather short triplet blocks, separated by straight sections (or

dipoles) of length L . For the sake of simplicity, we neglect the distortion of β functions due to space charge in these straight sections. Then, with the proper choice of the triplets parameters, β_x and β_z can be equal inside the straight section:

$$\beta_x(s) = \beta_z(s) = \beta_0 + \frac{s^2}{\beta_0}. \quad (1.5.56)$$

Inside triplets blocks β_x and β_z will certainly differ. But, since we assumed that these blocks are very short, we may neglect the contribution from these pieces of trajectory into Y_q . With these assumptions Eq. (1.5.52) can be rewritten in the form

$$Y_q = \int_{-1}^1 \frac{dx}{2} \left(\frac{2\beta_0/L + ix}{2\beta_0/L - ix} \right)^q. \quad (1.5.57)$$

Results of numerical integration in Eq. (1.5.57) show (see Fig. 1.5) that any space charge resonance can be eliminated or substantially suppressed by a proper choice of the ratio $2\beta_0/L$. The prediction of similar suppression of the strengths of a group of resonances generally demands more careful calculations.

As was mentioned, many specific features of phase trajectories can be studied by inspecting dependences of the Hamiltonians $h_{\pm}$ on J :

$$h_{\pm} = \Delta_0 J + h_{0,0}(J/2\varepsilon) \mp 4|\Delta\nu_L|\varepsilon Y_q V_{2q}(J/2\varepsilon). \quad (1.5.58)$$

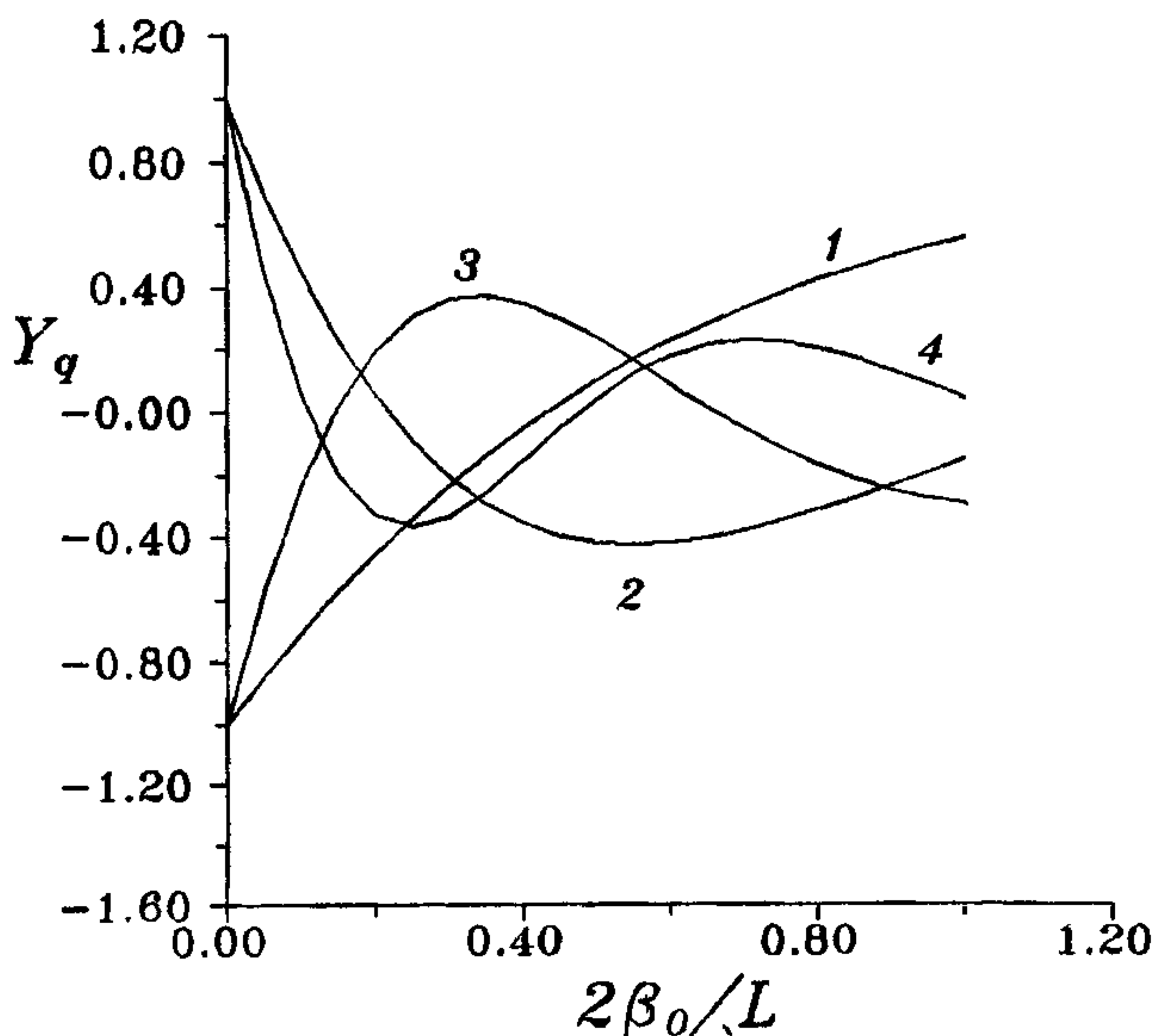


FIG. 1.5. Schematic behavior of the suppression factors Y_q for initial resonances; numbers near curves denote the value of q .

Since at small amplitudes

$$h_{0,0} \rightarrow \Delta \nu_L J |_{J \ll 2\varepsilon}$$

and

$$V_{2q} \rightarrow \begin{cases} \frac{J}{4\varepsilon}, & q=1, \\ (J/2\varepsilon)^q \frac{(q-1)!}{(2q)!}, & q \neq 1 \end{cases} \quad (1.5.59)$$

we shall consider separately the cases of the parametric ($q = 1$) and of higher-order resonances. For parametric resonance at small amplitudes we can write

$$h_{\pm} = [\Delta_0 - |\Delta \nu_L| (1 \pm Y_1)] J, \quad J \ll \varepsilon.$$

This expression shows that oscillations with small amplitudes are unstable within the following stopband

$$|\Delta \nu_L| (1 - Y_1) \leq \Delta_0 \leq |\Delta \nu_L| (1 + Y_1). \quad (1.5.60)$$

At large amplitudes the nonlinearity of the space charge force saturates the instability (see Fig. 1.6) and forms four buckets, separated by the trajectory, corresponding to $h_{\pm} = 0$ —two inside the beam and two well outside. [See Fig. 1.6(a)]. Since in the bigger bucket ($h < 0$) particles are transported to abnormally large amplitudes, they definitely will be lost in real rings. Besides dilutions of the phase space density of the beam, this will also limit its lifetime. The size of the central stable region decreases, so the situation becomes worse when $\Delta_0 \ll |\Delta \nu_L|$. The decrease in Y_1 decreases the sizes of both stable regions [see Fig. 1.6(b)] and generally causes their better separation. This can cure the instability, provided Y_1 is small enough. Similarly, the tuning of the working point ν_x, ν_z toward the upper side of the stopband, defined by Eq. (1.5.60), generally kills this resonance. Due to strong distortion by the space charge instability of betatron oscillations, designing the proton or ion machines one must avoid the vicinity of the parametric resonance.

If the order of the resonance is high enough, one can see from Eq. (1.5.59) that functions $h_{\pm}$ will coincide in the region of small amplitudes. This means that buckets of these resonances occur at large amplitudes. Since asymptotically $V_m \rightarrow \text{const}$, when $J \gg \varepsilon$, positions of centers of these buckets can be found from a true resonant condition:

$$\Delta_0 + \Delta \nu(J) = 0.$$

In the region $\Delta_0 \ll |\Delta \nu_L|$ this equation yields

$$J_s \simeq 2\varepsilon \frac{|\Delta \nu_L|}{\Delta_0}. \quad (1.5.61)$$

Describing the dilution of the beam density in the phase space by the ratio of excited amplitudes to unperturbed beam emittance (J/ε), we see that qualitatively it can be characterized by some acceptable or threshold value of $|\Delta\nu_L|$.

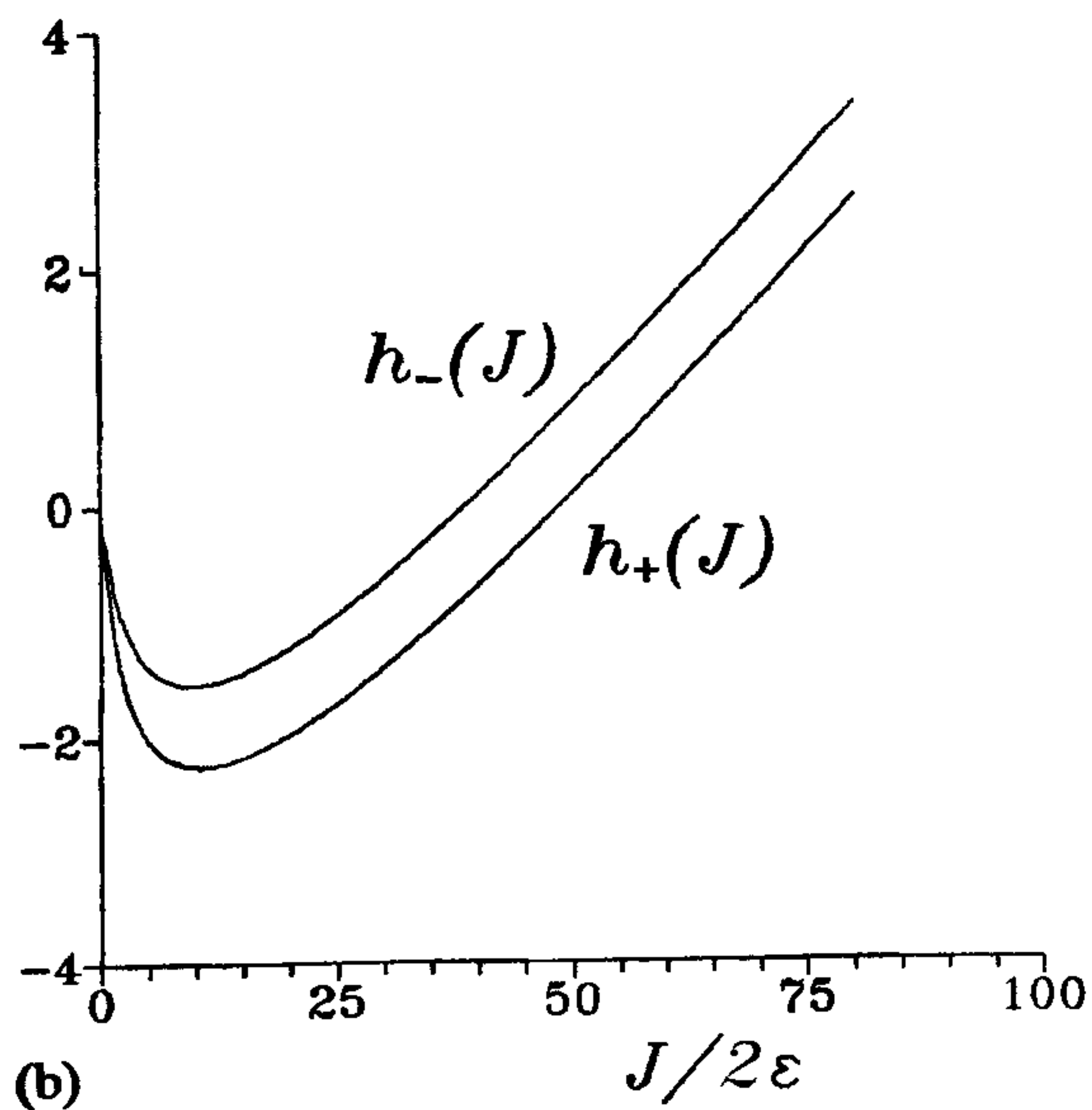
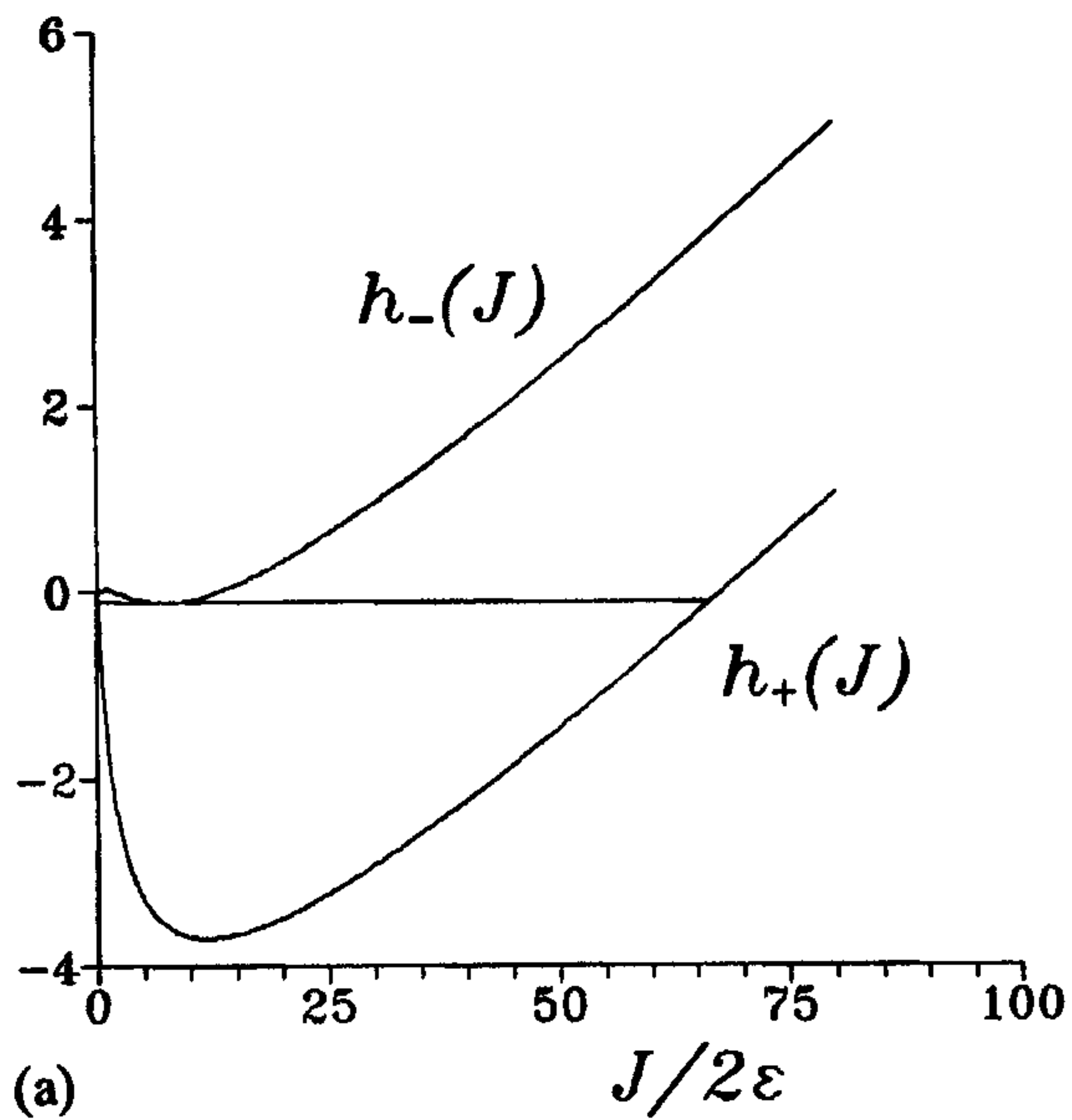


FIG. 1.6. The dependence of the Hamiltonians $h_{\pm}$ on J for the parametric space charge resonance; $q=1$, $\Delta_0=0.1$; (a) $Y_1=1$, (b) $Y_1=0.2$.

This threshold generally may differ from one ring to another, but it seems that its “average over the world” value can be found in the range between 0.1 (for storage rings) and 0.4 (for fast boosters). This fact typically is used for the engineering calculations related to space charge limitations when designing new rings.

1.6. SYNCHROTRON OSCILLATIONS

Let us now consider the action on the particles of the radio frequency (RF) electromagnetic fields. In storage rings these can be used for the acceleration (deceleration) of particles, for the compensation of the energy losses, to bunch the beam, and for other purposes. We shall assume the typical case, where the time dependence of the RF field is harmonic

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r}) + \delta\mathbf{A}(\mathbf{r})\exp\left(-i \int dt \dot{\omega}_s(t)\right) + \text{c.c.} \quad (1.6.1)$$

Here, $\mathbf{A}_0(\mathbf{r})$ describes the guiding magnetic field and $\omega_s(t)$ is the frequency of the RF field. The fields with the vector-potential Eq. (1.6.1) vary the energy of a particle, which is described by the equation

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = e\mathbf{v}\mathbf{E}(\mathbf{r},t), \quad \mathbf{E}(\mathbf{r},t) = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \quad (1.6.2)$$

Typically, for circular accelerators and storage rings the variation of the energy due to the RF field $\mathbf{E}(\mathbf{r},t)$ occurs in the time intervals which are much longer than both the revolution period and the periods of betatron oscillations. Thus, $\mathbf{r}$ and $\mathbf{v}$ in Eq. (1.6.2) can be replaced by their instantaneous values

$$\mathbf{r} = \mathbf{r}_0(t) + \boldsymbol{\rho}(t), \quad \mathbf{v} = \mathbf{v}_0(t) + \dot{\boldsymbol{\rho}}(t). \quad (1.6.3)$$

Here, $\mathbf{r}_0(t)$ and $\boldsymbol{\rho}(t)$ describe respectively the instantaneous orbit and betatron oscillations. After the substitution of Eq. (1.6.3) into Eq. (1.6.2) it is easy to see that the r.h.s. of this equation contains harmonics of the combination frequencies $\omega_0(n + \mathbf{m}\mathbf{v})$. If the working point of the ring $\mathbf{v}$ is detuned from the machine resonances, the systematic variations of $\mathcal{E}$ are described by the equation obtained by averaging Eq. (1.6.2) over the revolution period and phases of betatron oscillations:

$$\frac{d\mathcal{E}}{dt} = \overline{e\mathbf{v}\mathbf{E}[\mathbf{r}(t),t]}. \quad (1.6.4)$$

Neglecting here the quadratic and higher powers of $\boldsymbol{\rho}$, we can also rewrite this equation in the form

$$\frac{d\mathcal{E}}{dt} = \overline{evE_{\vartheta}[\mathbf{r}_0(t), t]}, \quad (1.6.5)$$

$$v(\mathcal{E}) = \tau \frac{d\mathbf{r}_0}{dt}, \quad E_{\vartheta} = \tau \mathbf{E}[\mathbf{r}_0(t), t].$$

The electric field on the orbit E_{ϑ} depends only on the azimuth of the particle ϑ and on the time. Expanding $E_{\vartheta}(\vartheta, t)$ in a Fourier series in ϑ ,

$$E_{\vartheta}(\vartheta, t) = \frac{1}{2} \sum_{q=-\infty}^{\infty} E_q \exp\left(iq\vartheta - i \int \omega_s dt\right) + \text{c.c.}$$

one can verify that the r.h.s. of Eq. (1.6.5) differs from zero, provided that holds the resonant condition:

$$q\omega_0(p_s, t) = \omega_s(t). \quad (1.6.6)$$

The condition (1.6.6) specifies the closed orbit of the so-called synchronous particle, which by definition is always in exact resonance with the accelerating field. Below, all values referring to this trajectory will be marked with the subscript s . For the synchronous particle the variation of the energy is

$$\frac{d\mathcal{E}_s}{dt} = \frac{ev_s E_q}{2} e^{i\varphi_s} + \text{c.c.} = \frac{eV\omega_0(p_s)}{2\pi} \cos\varphi_s, \quad (1.6.7)$$

where

$$\varphi_s = q\vartheta_s(t) - \int dt \omega_s(t) \quad (1.6.8)$$

is the phase of the synchronous particle and

$$V = \left| \oint ds \mathbf{E}(\vartheta) e^{-iq\vartheta} \right|$$

is the RF voltage and q is harmonic number of the RF system.

The variation of the momentum (or of the energy) of the synchronous particle is controlled by the resonance condition Eq. (1.6.6):

$$\frac{\alpha_p}{p_s} \frac{dp_s}{dt} = \frac{1}{\omega_s} \frac{d\omega_s}{dt} - \frac{1}{\omega_0} \frac{\partial \omega_0}{\partial t} \quad (1.6.9)$$

or, using $\omega_0 = v(p_s)/R_0$ and

$$\left(\frac{\partial \ln R_0}{\partial \ln H_z} \right)_p = - \left(\frac{\partial \ln R_0}{\partial \ln p} \right)_H = -\alpha,$$

$$\frac{\alpha_p}{p_s} \frac{dp_s}{dt} = \frac{1}{\omega_s} \frac{d\omega_s}{dt} - \alpha \frac{\partial \ln \bar{H}_z}{\partial t}. \quad (1.6.10)$$

In particular, in synchrotrons (and most storage rings) the position of the orbit inside the vacuum chamber is kept constant during acceleration. As can be seen from Eq. (1.6.10) this corresponds to

$$\frac{\partial \ln \bar{H}_z}{\partial t} = \frac{\gamma^2}{\omega_s} \frac{d\omega_s}{dt}, \quad (1.6.11)$$

or to the constant RF frequency with the relativistic ($\gamma \gg 1$) accuracy.

The combination of Eq. (1.6.7) and Eq. (1.6.10) indicates that generally, during acceleration, the amplitude and the synchronous phase of the RF have to vary according to the equation

$$eV \cos \varphi_s = \frac{p_s \Pi}{\alpha_p} \left(\frac{d \ln \omega_s}{dt} - \alpha \frac{\partial \ln \bar{H}_z}{\partial t} \right). \quad (1.6.12)$$

Typically the particles are stored at constant energy, when $\omega_s = \text{const}$, $\bar{H}_z = \text{const}$ and without the energy losses $\varphi_s = \pi/2$. If particles lose energy with the power P_s (say due to synchrotron radiation), it must be compensated by the voltage

$$eV \cos \varphi_s = \frac{2\pi}{\omega_s} P_s. \quad (1.6.13)$$

According to Eqs. (1.6.5) and (1.6.7), the variation of the momentum of the off-synchronous particle can be described by the following equation

$$\frac{dp}{dt} = \frac{1}{v} \frac{d\mathcal{E}}{dt} = \frac{eV \cos \varphi}{\Pi}.$$

Subtracting from this equation dp_s/dt , we obtain the equation for $\Delta p = p - p_s$:

$$\frac{d\Delta p}{dt} = \frac{eV}{\Pi} [\cos \varphi - \cos \varphi_s],$$

which together with

$$\frac{d\varphi}{dt} = \omega_0(p) - \omega_s \simeq q \frac{d\omega_0}{dp} \Delta p,$$

describes the so-called synchrotron oscillations of a particle. These equations can be obtained, using the Hamiltonian:

$$h(\Delta p, \varphi) = q \frac{\Delta p^2}{2\mu_{\parallel} R_0} - \frac{eV}{\Pi} (\sin \varphi - \varphi \cos \varphi_s). \quad (1.6.14)$$

Let us discuss in more detail the case when $\varphi_s = \pi/2$. [For convenience we shall replace φ by $\varphi - \pi/2$ in Eq. (1.6.14).] Then we can write

$$h(\Delta p, \varphi) = q \frac{\Delta p^2}{2\mu_{\parallel} R_0} - \frac{eV}{\Pi} \cos \varphi. \quad (1.6.15)$$

This Hamiltonian yields the phase trajectories

$$\Delta p(\varphi) = \pm \left\{ \frac{2\mu_{\parallel} R_0}{q} \left[h + \frac{eV}{\Pi} \cos \varphi \right] \right\}^{1/2}, \quad (1.6.16)$$

which depending on h can describe either finite or infinite motion in φ . These trajectories are separated by the separatrix, which corresponds to the special value of $h = eV/\Pi$:

$$\Delta p_c(\varphi) = \pm \left\{ \frac{2\mu_{\parallel} R_0 eV}{q\Pi} (1 + \cos \varphi) \right\}^{1/2}.$$

It has sizes

$$\Delta p_c = \pm 2 \left\{ \frac{\mu_{\parallel} R_0 eV}{q\Pi} \right\}^{1/2}, \quad \Delta \varphi_c = \pm \pi.$$

With harmonic number q there can be q separatrices along the orbit and thus the beam can contain q bunches.

In many important cases the bunch occupies a small area inside the separatrix. For small synchrotron oscillations the Hamiltonian (1.6.15) can be expanded in a power series in φ :

$$h(\Delta p, \varphi) = q \frac{\Delta p^2}{2\mu_{\parallel} R_0} + \frac{eV}{\Pi} \frac{\varphi^2}{2}. \quad (1.6.17)$$

This describes harmonic oscillations of the phase

$$\varphi = \varphi_c \cos \psi_c, \quad g\Delta p = -\mu_{\parallel} \omega_c R_0 \varphi_c \sin \psi_c \quad (1.6.18)$$

with the frequency

$$\omega_c = \omega_s \sqrt{q \frac{eV |\alpha_p|}{2\pi p_s v_s}}, \quad \omega_s = \omega_0(p_s). \quad (1.6.19)$$

Equations (1.6.18) generate canonical transformation from the variables $(\Delta p, \varphi)$ to the action-phase variables of synchrotron oscillations:

$$I_c = R_0^2 \varphi_c^2 \frac{|\mu_{\parallel}| \omega_c}{2q}, \quad \dot{\psi}_c = \omega_c \operatorname{sgn}(\omega'_0), \quad \omega'_0 = (d\omega_0/dp)_s.$$

In these variables the Hamiltonian in Eq. (1.6.17) reads

$$h = \text{sgn}(\omega'_0) \omega_c I_c / R_0. \quad (1.6.20)$$

Synchrotron oscillations of particles generally enlarge the class of resonances perturbing the motion of particles in storage rings. In addition to the resonances described by Eq. (1.5.28) synchrotron coupling resonances can appear:

$$m_x \omega_x + m_z \omega_z + m_c \omega_c = n. \quad (1.6.21)$$

Except for the azimuthal modulation of perturbations, these resonances can be caused by the dependence of betatron frequencies $\omega_{\perp}$ on the particle energy as well as on the radial dependence of perturbing fields.

* * *

Let us summarize the main results of this chapter. Without perturbations, the motion of a particle in a storage ring is the combination of the motion along a given closed orbit and oscillations about this orbit. For typical storage rings with planar closed orbits, these oscillations are described by the following equations:

$$x = x_b + \eta(\vartheta) \frac{\Delta p}{p}, \quad \mathbf{p}_{\perp} = \frac{p_s}{R_0} \frac{d\mathbf{r}_{\perp}}{d\vartheta}, \quad \Delta p = p - p_s,$$

$$(x_b, z) = a f(\vartheta) \cos[\psi + \chi(\vartheta)]|_{\alpha}, \quad \frac{d\chi_{\alpha}}{d\vartheta} + \nu_{\alpha} = \frac{1}{f_{\alpha}^2}, \quad \alpha = x, z,$$

$$\dot{\psi}_{\alpha} = \omega_{\alpha} = \omega_0(p) \nu_{\alpha}(p, a^2), \quad \vartheta = \vartheta_s + \varphi + \vartheta_b,$$

$$\vartheta_b = \frac{1}{R_0^2} \left(\eta \frac{dx_b}{d\vartheta} - x_b \frac{d\eta}{d\vartheta} \right),$$

$$\dot{\varphi} = \omega'_0 \Delta p = \frac{\omega_s \alpha_p}{p_s} \Delta p, \quad \alpha_p = 1/\gamma^2 - 1/\gamma_{tr}^2,$$

$$I_{\alpha} = \frac{p_s}{2R_0} a_{\alpha}^2. \quad (1.6.22)$$

For bunched beams Eqs. (1.6.22) must be supplemented by equations describing synchrotron oscillations. For the simplest case of one bunch per beam, and for

small amplitudes the synchrotron oscillations of the particle are described by the formulas

$$\begin{aligned}\varphi &= \varphi_c \cos \psi_c, \quad \Delta p = -\frac{\omega_c}{|\omega'_0|} \varphi_c \sin \psi_c, \\ \dot{\psi}_c &= \omega_c \operatorname{sgn}(\omega'_0), \quad I_c = R_0 \frac{\omega_c \varphi_c^2}{2|\omega'_0|}.\end{aligned}\quad (1.6.23)$$

Equations (1.6.22) and (1.6.23) generate a canonical transformation from variables $\mathbf{P}, \mathbf{r}$ to the action-phase variables $I_\alpha = I_\alpha(\mathbf{P}, \mathbf{r}, t)$, $\psi_\alpha = \psi_\alpha(\mathbf{P}, \mathbf{r}, t)$ of the unperturbed oscillations. After this transformation the Hamiltonian H_0 does not depend on the phase variables ψ_α :

$$\dot{I}_\alpha = -\frac{\partial \mathcal{H}_0}{\partial \psi_\alpha} \equiv 0, \quad \dot{\psi}_\alpha = \frac{\partial \mathcal{H}_0}{\partial I_\alpha} = \omega_\alpha. \quad (1.6.24)$$

Typically the distortions of the orbit due to errors in the guiding fields of the ring, or due to the fields, which are induced by the beam, are small. If perturbing fields are described by the vector $\mathbf{A}(\mathbf{r}, t)$ and scalar $A_0(\mathbf{r}, t)$ potentials, the Hamiltonian $\mathcal{H}$ can be expanded in a power series of $e\mathbf{A}/\mathcal{E}$ and $eA_0/\mathcal{E}$. The first-order correction reads

$$\mathcal{H} = \mathcal{H}_0 - L(\mathbf{I}, \psi, t), \quad (1.6.25)$$

where

$$L = \frac{e}{c} \mathbf{v} \mathbf{A}(\mathbf{r}, t) - eA_0(\mathbf{r}, t)$$

is the Lagrangian of the interaction of particles with perturbing fields. The second-order corrections are

$$\frac{e^2}{\mathcal{E}} \left\{ \frac{\mathbf{A}^2}{\gamma^2} + \left(\frac{\mathbf{v}}{c} \times \mathbf{A} \right)^2 \right\}.$$

These are negligible, provided $|e\mathbf{A}| \ll \mathcal{E}$. Taking into account Eq. (1.6.24), the Hamiltonian in Eq. (1.6.25) yields the equations of motion

$$\dot{I}_\alpha = \frac{\partial L}{\partial \psi_\alpha}, \quad \dot{\psi}_\alpha = \omega_\alpha - \frac{\partial L}{\partial I_\alpha}. \quad (1.6.26)$$

Due to the periodicity of the particle motion along the closed orbit the right-hand sides of these equations are periodic functions of both ϑ and phases ψ_α . Therefore the averaging method can be employed as the basis for the perturbation theory.

CHAPTER 2

Statistical Description of a Beam in a Storage Ring

In many cases the analysis of beam intensity limitations consists of the study of variations in different collective or kinetic processes when the beam intensity increases. The solution of these generally technical problems sometimes needs rather special methods of investigation. In this chapter we discuss the application of statistical mechanics to the description of the beam in a storage ring. The nontriviality of this problem can have numerous causes. First of all, the beam in the storage ring, strictly speaking, presents the system with a discrete spectrum of frequencies. Secondly, usually the interaction of particles in storage rings is a long-range one with macroscopic large radii of the interaction. Finally, very often the interaction between particles is caused by fields induced in the dissipative elements of the vacuum chamber. The consideration of these and other details brings important features into the description of both collective and kinetic effects on stored beams. Therefore, construction of the general theory for such phenomena can be interesting both for technical applications and from the general point of view.

2.1. BEAM AS A MANY-BODY SYSTEM

The state of the beam in a storage ring will be completely determined provided that the instantaneous positions for all its particles are given in the phase space:

$$\mathbf{r}=\mathbf{r}_a(t), \quad \mathbf{p}=\mathbf{p}_a(t), \quad a=1,2,\dots,N. \quad (2.1.1)$$

Here, N is the number of particles in the beam. Such detailed description demands knowledge of $6N$ functions of time and initial conditions

$$\mathbf{r}=\mathbf{r}_a(0), \quad \mathbf{p}=\mathbf{p}_a(0), \quad a=1,2,\dots,N. \quad (2.1.2)$$

Since the initial values of particle coordinates and momenta are known to only a certain degree of accuracy, such a complete description cannot be realized. In these circumstances, since N is very large, it is more productive to describe the beam statistically using a small amount of thermodynamic variables. Nevertheless, some important results can be obtained using the dynamic description of the beam.

As mentioned in Chapter 1, if stability conditions are valid, the motion of particles is finite. This means that in phase space the beam occupies the finite volume

$$\Delta\Gamma(N) = \int d\Gamma_1 \dots d\Gamma_N, \quad (2.1.3)$$

where $d\Gamma_a$ is the element of the phase space volume of particle a . The volume $\Delta\Gamma(N)$ is the parameter of the beam as the whole (though being the dynamic variable). Equation (2.1.1) carries the transformation of the initial point (2.1.2) (in $6N$ -dimensional phase space) into the instantaneous point (2.1.1). If initial values of the coordinates of particles in the phase space are used as independent variables, Eq. (2.1.3) reads

$$\Delta\Gamma(N,t) = \int d\Gamma_1^{(0)} \dots d\Gamma_N^{(0)} D(t), \quad (2.1.4)$$

where

$$D(t) = \frac{\partial[\mathbf{r}_1(t), \dots, \mathbf{r}_N(t); \mathbf{p}_1(t), \dots, \mathbf{p}_N(t)]}{\partial[\mathbf{r}_1(0), \dots, \mathbf{r}_N(0); \mathbf{p}_1(0), \dots, \mathbf{p}_N(0)]} \quad (2.1.5)$$

is the Jacobian of this transformation. Without dissipative forces on the orbit Eq. (2.1.1) represents the canonic transformation. Then, as is well known,¹⁵ $D(t) = 1$ and

$$\Delta\Gamma(N) = \Delta\Gamma^{(0)}(N) = \text{const.} \quad (2.1.6)$$

Eq. (2.1.6) yields the contents of Liouville's theorem, which plays an important role in the theory of stored beams.

The effect of the dissipative forces on particles causes the nonconservation of the beam volume in the phase space. In one of the most important cases such forces can be written as

$$\mathcal{F}_a^a = -\Lambda_{a,\beta}^a p_\beta^a - \sum_{b \neq a}^N \Lambda_{a,\beta}^b p_\beta^b. \quad (2.1.7)$$

If the equations of motion can be written in the form

$$\dot{\mathbf{r}}_a = \frac{\partial \mathcal{H}}{\partial \mathbf{p}_a}, \quad \dot{\mathbf{p}}_a = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}_a} + \mathcal{F}_a^a \quad (2.1.8)$$

the Jacoby theorem⁵ is valid:

$$\frac{dD}{dt} = -\left(\sum_{a=1}^N \text{Sp} \hat{\Lambda}^a \right) D(t). \quad (2.1.9)$$

After the differentiation of Eq. (2.1.4) this equation yields

$$\frac{d\Delta\Gamma(N)}{dt} = \int d\Gamma_1^{(0)} \dots d\Gamma_N^{(0)} \frac{dD}{dt} = - \sum_{a=1}^N \text{Sp}\hat{\Lambda}^a \int d\Gamma_1^{(0)} \dots d\Gamma_N^{(0)} D(t), \quad (2.1.10)$$

or

$$\frac{d\ln\Delta\Gamma(N)}{dt} = - \sum_{a=1}^N \text{Sp}\hat{\Lambda}^a. \quad (2.1.11)$$

The values $\text{Sp}(\hat{\Lambda}^a)$ in Eq. (2.1.11) are the sums of decrements of the beam particles. As the trace of the matrix is the invariant relative to the orthogonal transformations, the sums of decrements $\text{Sp}(\hat{\Lambda}^a)$ are independent of the coupling between the degrees of freedom of the particle. Therefore, to damp the oscillations of the particles by forces given in Eq. (2.1.7) it is necessary to have

$$\text{Sp}(\hat{\Lambda}^a) > 0. \quad (2.1.12)$$

Since the rate of change of the total phase space volume of the beam is determined by the sum of decrements for all particles of the beam [Eq. (2.1.11)], $\Delta\Gamma(N)$ decreases provided that the weaker condition holds:

$$\sum_{a=1}^N \text{Sp}\hat{\Lambda}^a > 0. \quad (2.1.13)$$

Practically, it turns out to be more convenient to place the phase trajectories of all particles of the beam in the same six-dimensional phase space (the so-called μ -space). Once the number of particles in the beam is a certain number and their motion is finite, the beam at any time occupies the finite volume Γ in μ -space. For instance, if one can neglect the interaction between particles, the border of Γ is determined by the phase trajectory of the utmost particle. In contrast with $\Delta\Gamma(N)$, the value Γ determines the effective phase space volume of the beam (i.e., the volume containing most of the beam's particles). Without the interaction between particles the value of $\Delta\Gamma(N)$ is

$$\Delta\Gamma(N) = \prod_{a=1}^N \Delta\Gamma_a. \quad (2.1.14)$$

The conservation of $\Delta\Gamma(N)$ follows from the conservation of $\Delta\Gamma_a$, which also yields the conservation of Γ . This narrower fact is usually assumed in the analysis of particle accumulation problems.

Positions of obstacles inside the vacuum chamber and the specification of the ring lattice yield the volume in phase space Γ_A , where without perturbations particles can move forever. Without dissipative forces on the orbit, the phase space volume of the injected beam Γ_{inj} is conserved. This means that the possible number of injections during the accumulation is limited by the ratio $\Gamma_A/\Gamma_{\text{inj}}$. The use of the dissipative forces (or beam cooling) cancels this limitation due to the reduction of the phase space volume of the injected beam.

The dynamic state of the beam is determined, if in μ -space the so-called microscopic phase space density is given

$$F(\mathbf{p}, \mathbf{r}, t) = \sum_{a=1}^N \delta[\mathbf{p} - \mathbf{p}_a(t)] \delta[\mathbf{r} - \mathbf{r}_a(t)]. \quad (2.1.15)$$

The physical sense of this function is in determination of the instantaneous number of particles $\Delta N(t, \Delta\Gamma)$, populating the given phase space volume:

$$\Delta N(t, \Delta\Gamma) = \int_{\Delta\Gamma} \frac{d^3p d^3r}{(2\pi)^3} F(\mathbf{p}, \mathbf{r}, t). \quad (2.1.16)$$

In particular, the integration over the total volume yields

$$\int d\Gamma F(\mathbf{p}, \mathbf{r}, t) = N, \quad d\Gamma = \frac{d^3p d^3r}{(2\pi)^3}. \quad (2.1.17)$$

The density F enables one to calculate any parameter of the beam. If $A(\mathbf{p}, \mathbf{r})$ is any physical value, one can write

$$A(t) = \int d\Gamma A(\mathbf{p}, \mathbf{r}) F(\mathbf{p}, \mathbf{r}, t) = \sum_{a=1}^N A[\mathbf{r}_a(t), \mathbf{p}_a(t)]. \quad (2.1.18)$$

For instance, for microscopic charge and current densities of the beam this yields

$$\rho(\mathbf{r}, t) = e \int d^3p F(\mathbf{p}, \mathbf{r}, t) = e \sum_{a=1}^N \delta[\mathbf{r} - \mathbf{r}_a(t)], \quad (2.1.19)$$

$$\mathbf{j}(\mathbf{r}, t) = e \int d^3p \mathbf{v} F(\mathbf{p}, \mathbf{r}, t) = e \sum_{a=1}^N \mathbf{v}_a \delta[\mathbf{r} - \mathbf{r}_a(t)]. \quad (2.1.20)$$

Using this rule as well as Eqs. (1.3.18) and (1.3.19) we can construct the value

$$\begin{aligned} \varepsilon_x &= \frac{1}{N} \int d\Gamma \left(\frac{x^2}{\beta_x} + \beta_x \left[x' - \frac{\beta'}{2\beta} x \right]^2 \right) F \\ &= \frac{\langle x^2 \rangle}{\beta_x} \left(1 + \frac{(\beta')^2}{4} \right) + \beta_x \langle (x')^2 \rangle - \beta'_x \langle x x' \rangle, \end{aligned} \quad (2.1.21)$$

which usually is called as the beam emittance. Here,

$$\langle \dots \rangle = \frac{1}{N} \sum_{a=1}^N (\dots). \quad (2.1.22)$$

If the motion of particles is described by the Hamiltonian $\mathcal{H}$, the function $F(\mathbf{p}, \mathbf{r}, t)$ obeys the equation

$$\frac{\partial F}{\partial t} + [\mathcal{H}, F] = 0, \quad (2.1.23)$$

where

$$[\mathcal{H},] = \frac{\partial \mathcal{H}}{\partial P_\alpha} \frac{\partial}{\partial r_\alpha} - \frac{\partial \mathcal{H}}{\partial r_\alpha} \frac{\partial}{\partial P_\alpha} \quad (2.1.24)$$

is the Poisson bracket. As the beam consists of charged particles, Eq. (2.1.23) must be supplemented with Maxwell equations:

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 4\pi e \int d^3p F(\mathbf{p}, \mathbf{r}, t), & \operatorname{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \operatorname{div} \mathbf{H} &= 0, & \operatorname{rot} \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi e}{c} \int d^3p \mathbf{v} F(\mathbf{p}, \mathbf{r}, t) \end{aligned} \quad (2.1.25)$$

with suitable boundary conditions. The system of Eqs. (2.1.25) results in the energy balance equation

$$\frac{\partial W_0}{\partial t} + \operatorname{div} \mathbf{W} = -\mathbf{E} \mathbf{j}, \quad (2.1.26)$$

where

$$W_0 = \frac{E^2 + H^2}{8\pi}, \quad \mathbf{W} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \quad (2.1.27)$$

are respectively the density of the energy and the Poynting vector of the induced field. The integration of Eq. (2.1.26) over the volume of the vacuum chamber yields

$$\frac{d\mathcal{E}_f}{dt} + \oint d\Sigma \mathbf{W} = -e \sum_{a=1}^N \mathbf{v}_a(t) \mathbf{E}[\mathbf{r}_a(t), t], \quad (2.1.28)$$

where

$$\mathcal{E}_f = \int d^3r W_0(\mathbf{r}, t) \quad (2.1.29)$$

is the energy of the field and the surface integral must be calculated over the surface of the vacuum chamber. According to Eq. (1.6.2) the energy of the particle a varies like

$$\frac{d\mathcal{E}_a}{dt} = e \mathbf{v}_a(t) \mathbf{E}[\mathbf{r}_a(t), t]. \quad (2.1.30)$$

For this reason, we can rewrite Eq. (2.1.28) as the energy conservation law

$$\frac{d}{dt} \left[\mathcal{E}_f + \sum_{a=1}^N \mathcal{E}_a \right] = - \oint d\Sigma \mathbf{W}. \quad (2.1.31)$$

Without an external feeding of energy, the surface integral in Eq. (2.1.31) describes the dissipation of electromagnetic energy by the walls of the vacuum chamber. These losses can be due to either wall resistivity or radiation from holes in the chamber.

Returning to Eqs. (2.1.23) and (2.1.25) we have to emphasize that these equations yield the most complete description of the beam in the storage ring. But due to this completeness, it turns out to be practically impossible to get their solutions. Nevertheless, this system gives a useful base for the construction of statistical mechanics. In the last case Eqs. (2.1.23) and (2.1.25) must allow for the simplifications associated with the dominance of particular effects.

2.2. THE BEAM PERTURBED BY WHITE NOISE

Let us start with the description of the motion of the beam affected by white noise. Without the beam cooling such a problem is usually associated with the calculation of the beam lifetime. However, here we would like to emphasize its heuristic worth to illustrate the transition from microscopic to macroscopic description of the beam.

For the sake of simplicity, we consider the special case when at any point of the orbit the motion of particles in a coasting beam is perturbed by a random force, which is directed along τ . Using instead of Δp a new variable, $\omega = \omega'_0 \Delta p$, we can write the equations of motion for the particle a in the form

$$\dot{\omega}_a = V(t), \quad \dot{\vartheta}_a = \omega_a. \quad (2.2.1)$$

Here, $V(t)$ is a random function of time with the following properties:

$$\langle V(t_1) \dots V(t_{2k+1}) \rangle = 0, \quad k=1,2,\dots, \quad (2.2.2)$$

$$\langle V(t_1) \dots V(t_{2k}) \rangle = \sum_P \langle V(t_1) V(t_2) \rangle \dots \langle V(t_{2k-1}) V(t_{2k}) \rangle, \quad (2.2.3)$$

$$\langle V(t_1) V(t_2) \rangle = \langle \Delta V^2 \rangle \delta(t_1 - t_2), \quad (2.2.4)$$

The brackets $\langle \dots \rangle$ mean the statistical average over the fluctuations of V .

The formal solution of Eq. (2.2.1) reads

$$\omega_a(t) = \omega_a^{(0)} + \int_0^t dt_1 V(t_1), \quad (2.2.5)$$

$$\vartheta_a(t) = \vartheta_a^{(0)} + \omega_a^{(0)} t + \int_0^t dt_1 \int_0^{t_1} dt_2 V(t_2), \quad (2.2.6)$$

where $\vartheta_a^{(0)}, \omega_a^{(0)}$ are initial coordinates of the particle a . The substitution of these solutions into Eq. (2.1.15) yields the microscopic density

$$F(\omega, \vartheta, t) = \sum_{a=1}^N \delta[\omega - \omega_a(t)] \delta[\vartheta - \vartheta_a(t)]. \quad (2.2.7)$$

Due to the periodicity of the particle motion, Eq. (2.2.7) can be represented by the expansion

$$F(\omega, \vartheta, t) = \sum_{n=-\infty}^{\infty} F_n(\omega, t) e^{in\vartheta}, \quad (2.2.8)$$

$$F_n(\omega, t) = \sum_{a=1}^N \frac{e^{-in\vartheta_a(t)}}{2\pi} \delta[\omega - \omega_a(t)]. \quad (2.2.9)$$

Since $\omega_a(t)$ and $\vartheta_a(t)$ are the random functions of time,

$$\langle \omega_a(t) \rangle = \omega_a^{(0)}, \quad \langle (\vartheta_a(t) - \vartheta_a^{(0)} - \omega_a^{(0)} t) \rangle = 0, \quad (2.2.10)$$

the amplitudes $F_n(\omega, t)$ are the random functions of time as well. For this reason, we write

$$F_n(\omega, t) = \langle F_n(\omega, t) \rangle + \delta F_n(\omega, t), \quad \langle \delta F_n(\omega, t) \rangle = 0. \quad (2.2.11)$$

The random function $\delta F_n(\omega, t)$ describes the fluctuations of the beam phase space density that are generated by the random force $V(t)$.

The physical sense has the combinations of harmonics $F_n(\omega, t)$ averaged over the fluctuations. Let us calculate, for instance, $\langle F_n(\omega, t) \rangle$. Since the terms of the sum in Eq. (2.2.9) are independent, we have to calculate only one:

$$F_n^a = \langle \delta[\omega - \omega_a(t)] e^{-in\vartheta_a(t)} \rangle. \quad (2.2.12)$$

Using the integral representation for the δ -function

$$\delta(\omega) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i\omega u}, \quad (2.2.13)$$

we can write

$$F_n^a = \int_{-\infty}^{\infty} \frac{du}{2\pi} K(u, t) \exp[iu(\omega - \omega_a^{(0)}) - in\vartheta_a^{(0)} - in\omega_a^{(0)} t], \quad (2.2.14)$$

where

$$K(u, t) = \left\langle \exp \left[-i \int_0^t dt_1 \omega(t - t_1) V(t_1) \right] \right\rangle, \quad (2.2.15)$$

and $w = u + n(t - t_1)$. Expanding the exponent in Eq. (2.2.15) in the Taylor series,

$$\begin{aligned} K(u, t) &= \sum_{l=0}^{\infty} \frac{(-i)^l}{l!} \left\langle \left(\int_0^t dt_1 w(t - t_1) V(t_1) \right)^l \right\rangle, \\ &= \sum_{l=0}^{\infty} \frac{1}{(2l)!} \left\langle \left(\int_0^t dt_1 w(t - t_1) V(t_1) \right)^{2l} \right\rangle \end{aligned}$$

and using Eqs. (2.2.2)–(2.2.4) results in

$$\begin{aligned} K(u, t) &= \exp \left\{ -\frac{\langle \Delta V^2 \rangle}{2} \int_0^t dt_1 w^2(t - t_1) \right\} \\ &= \exp \left\{ -\frac{\langle \Delta V^2 \rangle}{2} \left(ut - nut^2 + \frac{n^2 t^3}{3} \right) \right\}. \end{aligned} \quad (2.2.16)$$

Substituting this expression for $K(u, t)$ into Eq. (2.2.14), after the integration over u we find

$$F_n^a = \frac{1}{\sqrt{2\pi} \sigma(t)} \exp \left\{ -\frac{(\omega - \omega_a^{(0)})^2}{2\sigma^2(t)} - in\vartheta_a^{(0)} + in\omega t - \frac{\langle \Delta V^2 \rangle n^2 t^3}{24} \right\}, \quad (2.2.17)$$

$$\sigma^2(t) = \langle \Delta V^2 \rangle t. \quad (2.2.18)$$

Equation (2.2.17) indicates two significant aspects in the beam relaxation. First, it shows that due to the action of the noise, harmonics F_n^a decay and the lifetime of these harmonics in $\langle F(\omega, \vartheta, t) \rangle$ depends on the harmonic number rather strongly. After the time interval of the order of

$$\tau_n = \left(\frac{24}{\langle \Delta V^2 \rangle n^2} \right)^{1/3} \quad (2.2.19)$$

$\langle F(\omega, \vartheta, t) \rangle$ will contain only one (the slowest) harmonic with $n = 0$:

$$F_0(\omega, t) = \frac{1}{\sqrt{2\pi} \sigma(t)} \sum_{a=1}^N \exp \left[\frac{(\omega - \omega_a^{(0)})^2}{2\sigma^2(t)} \right]. \quad (2.2.20)$$

Secondly, it shows that after the averaging, the harmonics in Eq. (2.2.17) become nonsingular functions. Their widths $\sigma(t)$ increase with time following the diffusion law. Therefore, if $\Delta\omega_\Sigma$ is the initial interval of frequencies, which is occupied by the beam, after the time interval

$$t_{\Sigma} \gg \frac{1}{\langle \Delta V^2 \rangle} \left(\frac{\Delta \omega_{\Sigma}}{N} \right)^2 \quad (2.2.21)$$

the widths $\sigma(t)$ will exceed the average distance between neighboring maxima $\Delta \omega_{\Sigma}/N$ and, with sufficient accuracy, Eq. (2.2.20) can be replaced by

$$F_0(\omega, t) \simeq \frac{N}{\sqrt{2\pi} \sigma(t)} \exp \left\{ -\frac{\omega^2}{2\sigma^2(t)} \right\}. \quad (2.2.22)$$

In agreement with Eq. (2.1.16), this expression determines the average number of particles populating the frequency interval $d\omega$, that is, the beam distribution function in ω :

$$\frac{\Delta N}{N} = f_0(\omega, t) d\omega = \frac{d\omega}{\sqrt{2\pi} \sigma(t)} \exp \left\{ -\frac{\omega^2}{2\sigma^2(t)} \right\}. \quad (2.2.23)$$

On time intervals smaller than τ_n but still larger than t_{Σ} from Eq. (2.2.21)

$$\frac{1}{\langle \Delta V^2 \rangle} \left(\frac{\Delta \omega_{\Sigma}}{N} \right)^2 \ll t \ll \left(\frac{24}{\langle \Delta V^2 \rangle n^2} \right)^{1/3}, \quad (2.2.24)$$

the distribution function also conserves the azimuthal harmonics. In this respect Eq. (2.2.11) can be rewritten in the form

$$F(\omega, \vartheta, t) = N f(\omega, \vartheta, t) + \delta F(\omega, \vartheta, t), \quad (2.2.25)$$

or

$$N f(\omega, \vartheta, t) = \langle F(\omega, \vartheta, t) \rangle. \quad (2.2.26)$$

By direct calculation it is easy to verify that the distribution function in Eq. (2.2.26) satisfies the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \omega \frac{\partial f}{\partial \vartheta} = \frac{\langle \Delta V^2 \rangle}{2} \frac{\partial^2 f}{\partial \omega^2}. \quad (2.2.27)$$

The irreversibility of the evolution of f , which is described by Eq. (2.2.27), could be expected from the very beginning as the beam is perturbed by the random force.

As mentioned before, the importance of this problem can be associated with the calculation of the beam lifetime. If $\Delta \omega_A$ is the frequency spread corresponding to the aperture of the vacuum chamber, σ_{in} is initial frequency spread of the beam and $\Delta \omega_A \gg \sigma_{in}$, then the rate of the particle losses from the beam is

$$\begin{aligned} \frac{d \ln N}{dt} &= \langle \Delta V^2 \rangle \frac{\partial f}{\partial \omega} \Big|_{\Delta \omega_A} \\ &\simeq -\frac{1}{\sqrt{2\pi}} \frac{\Delta \omega_A \langle \Delta V^2 \rangle}{[\sigma^2(t) + \sigma_{\text{in}}^2]^{3/2}} \exp\left(-\frac{\Delta \omega_A^2}{2[\sigma^2(t) + \sigma_{\text{in}}^2]}\right). \end{aligned} \quad (2.2.28)$$

Denoting now

$$\tau = \Delta \omega_A^2 / \langle \Delta V^2 \rangle \quad (2.2.29)$$

and integrating Eq. (2.2.28), we obtain

$$N(t) = N_0 \exp \left\{ -\frac{1}{\sqrt{2\pi}} \int_0^{t/\tau} \frac{du}{\left[u + \left(\frac{\sigma_{\text{in}}}{\Delta \omega_A} \right)^2 \right]^{3/2}} \exp \left(-\frac{1/2}{u + \left(\frac{\sigma_{\text{in}}}{\Delta \omega_A} \right)^2} \right) \right\}, \quad (2.2.30)$$

where N_0 is initial number of particles in the beam.

2.3. THE BEAM WITH COHERENT OSCILLATIONS

For one more heuristic problem we consider the model where, at any point of the orbit, the beam is affected by a force that is proportional to the dipole momentum of the beam:

$$\mathcal{F} = k \sum_{a=1}^N \mathbf{r}_a \equiv Nk \langle \mathbf{r} \rangle, \quad (2.3.1)$$

$$\langle \mathbf{r} \rangle = \frac{1}{N} \sum_{a=1}^N \mathbf{r}_a. \quad (2.3.2)$$

Assuming, for the sake of simplicity, the smoothed focusing approximation, we can write for, say, horizontal oscillations

$$\ddot{x}_a + \omega_a^2 x_a = \frac{1}{\gamma m} \mathcal{F}_x, \quad a = 1, 2, \dots, N. \quad (2.3.3)$$

The coefficient k contributing to Eq. (2.3.1) can be calculated using the solution of the appropriate electrodynamic problem. Keeping in mind the interaction of the beam with the surrounding electrodes, and to avoid here the calculation of induced fields, we assume that k can be represented by the expression

$$k = -e^2 \frac{g}{\Pi b^2}, \quad (2.3.4)$$

where b is the characteristic transverse dimension of the electrodes and g is the form factor, which must be calculated from the electrodynamic problem. The substitution of Eqs. (2.3.1) and (2.3.4) into Eq. (2.3.3) yields

$$\ddot{x}_a + \omega_a^2 x_a = -\Omega^2 x_a, \quad (2.3.5)$$

$$\Omega^2 = \frac{Ne^2 g}{\gamma m \Pi b^2}. \quad (2.3.6)$$

Equation (2.3.5) can be easily solved using a Fourier transformation. Using

$$x_a(\omega) = \int_0^\infty dt \exp(i\omega t) \dot{x}_a(t), \quad (2.3.7)$$

we can write

$$x_a(\omega) = \frac{1}{\omega_a^2 - \omega^2} \{ -\Omega^2 \langle x \rangle_\omega - [\dot{x}_a(t) - i\omega x_a(t)] e^{i\omega t} \big|_0^\infty \}. \quad (2.3.8)$$

To make sense, the values $x_a(t)$ in this expression should not grow faster than the exponent $\exp(\delta_a t)$ and, therefore, the values of the complex variable ω must be chosen from the region $\text{Im}\omega > \max\{\delta_a\}$, where solutions are calculated in terms of initial conditions:

$$x_a(\omega) = \frac{1}{\omega_a^2 - \omega^2} \{ -\Omega^2 \langle x \rangle_\omega + [\dot{x}_a(0) - i\omega x_a(0)] \}. \quad (2.3.9)$$

This equation shows that the solutions $[x_a(t)]$ will be found after the calculation of $\langle x \rangle_\omega$. The summation of Eq. (2.3.9) yields

$$\langle x \rangle_\omega = -\frac{X(\omega)}{\epsilon(\omega)}, \quad X(\omega) = \frac{1}{N} \sum_{a=1}^N \frac{\dot{x}_a(0) - i\omega x_a(0)}{\omega^2 - \omega_a^2}, \quad (2.3.10)$$

$$\epsilon(\omega) = 1 - \frac{\Omega^2}{N} \sum_{a=1}^N \frac{1}{\omega^2 - \omega_a^2}. \quad (2.3.11)$$

Substituting these equations into Eq. (2.3.9) and using the inverse Fourier transformation, we find

$$\begin{aligned} x_a(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} x_a(\omega), \\ &= - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 - \omega_a^2} \left(\frac{\Omega^2 X(\omega)}{\epsilon(\omega)} + [\dot{x}_a(0) - i\omega x_a(0)] \right), \quad \text{Im}\omega > 0. \end{aligned} \quad (2.3.12)$$

This expression shows that the oscillations of a particle consists of two parts. The second and third terms in the integrand of Eq. (2.3.12) obviously describe free oscillations due to the initial deflection. On the contrary, the time dependence of the first term in Eq. (2.3.12) is determined by the position of the roots of the dispersion equation:

$$\epsilon(\omega) = 0 \quad (2.3.13)$$

or

$$1 = \frac{\Omega^2}{N} \sum_{a=1}^N \frac{1}{\omega^2 - \omega_a^2} = \Omega^2 \Phi(\omega), \quad (2.3.14)$$

which essentially depends on the beam intensity, on the distribution of frequencies in the beam, and on the interaction between particles. Even if only one particle is initially deflected, it will excite the oscillations of others. Thus, it describes collective behavior of the beam, which generally can be described by one parameter—the dipole moment of the beam $\langle x(t) \rangle$:

$$\langle x(t) \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \langle x \rangle_{\omega}, \quad \text{Im}\omega > 0. \quad (2.3.15)$$

In particular, Eq. (2.3.12) indicates that oscillations of particular particles will not grow with time infinitely if the oscillations of the dipole moment $\langle x(t) \rangle$ of the beam do not grow or if the collective oscillations of the beam are stable. This means that all roots of the dispersion equation must occur in the lower half-plane of ω .

Eq. (2.3.14) contains a polynomial of the power $2N$. For this reason, it must be solved numerically. Nevertheless, since the coefficients of this polynomial are real numbers, the stability condition can be easily found computing how many branches of $\Phi(\omega)$ are crossed by the horizontal line $\Phi = 1/\Omega^2$. (See Fig. 2.1.) Figure 2.1 indicates that in the region $\Omega^2 > 0$, the line $1/\Omega^2$ crosses $2N$ branches of the function $\Phi(\omega)$, and therefore, this equation has $2N$ real roots. Thus, if $\Omega^2 > 0$, collective oscillations of the beam are stable. Typically, the frequencies in the beam are distributed around the central frequency $\bar{\omega}$ with a relatively small frequency spread $\delta\omega$. If we can neglect this frequency spread ($\bar{\omega} \gg \delta\omega$), the roots of Eq. (2.3.14) are determined by the simple expression

$$\omega_{1,2} = \pm \sqrt{\bar{\omega}^2 + \Omega^2}, \quad (2.3.16)$$

which differ from particle frequencies by a value that depends on the interaction of particles. Provided that $\Omega \ll \bar{\omega}$, the frequency shift due to collective interaction linearly depends on the beam intensity

$$\Delta\omega = \frac{\Omega^2}{2\bar{\omega}}. \quad (2.3.17)$$

In the region $\Omega^2 < 0$, one branch of $\Phi(\omega)$ remains to be uncrossed (see Fig. 2.1) if

$$\frac{|\Omega^2|}{N} \sum_{a=1}^N \frac{1}{\omega_a^2} > 1. \quad (2.3.18)$$

This means that Eq. (2.3.14) has two complex conjugated roots, and hence, one of the modes of collective oscillations becomes unstable. Neglecting in Eq. (2.3.18) the frequency spread, we can write the stability condition in the form

$$|\Omega| < \bar{\omega}, \quad (2.3.19)$$

which means that the interaction should not compensate for the rigidity of oscillations due to the guiding field.

Note that for this particular problem the frequency spread of the beam

$$\delta\omega^2 = \frac{1}{N} \sum_{a=1}^N (\omega_a - \bar{\omega})^2 \quad (2.3.20)$$

decreases the threshold current of the beam ($|\Omega^2|$). According to criterion (2.3.18) we write

$$\frac{1}{|\Omega^2|_{th}} \approx \frac{1}{N} \sum_{a=1}^N \frac{1}{\bar{\omega}^2} \left(1 + \frac{\delta\omega_a^2}{\bar{\omega}^2} \right), \quad \delta\omega_a^2 = (\omega_a - \bar{\omega})^2, \quad (2.3.21)$$

or

$$|\Omega^2|_{th} \approx \bar{\omega}^2 - \delta\omega^2. \quad (2.3.22)$$

This phenomenon is specific for the systems with discrete spectrum.

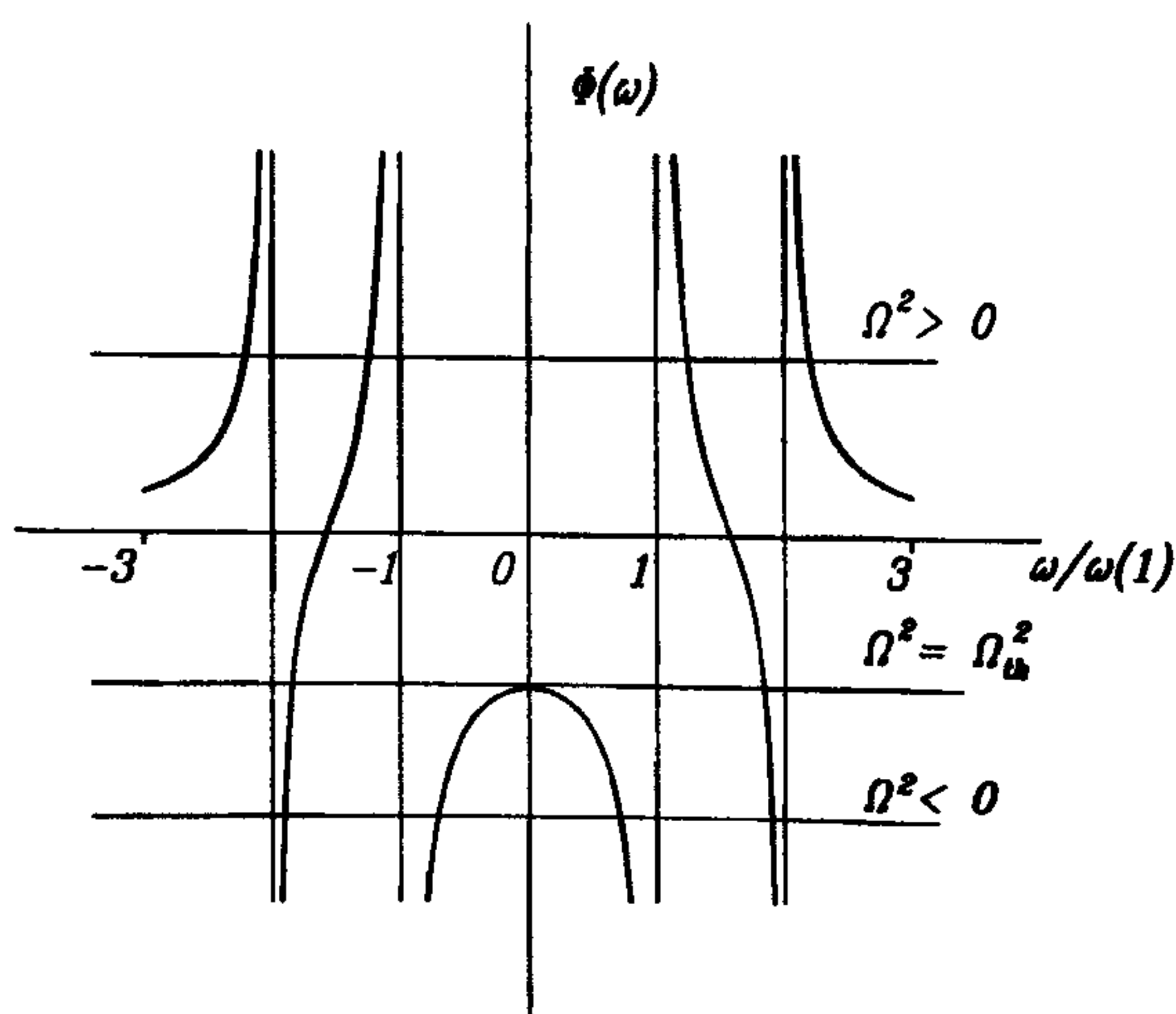


FIG. 2.1. Schematic plot of the function $\Phi(\omega)$ from Eq. (2.3.14); ($N=2$ for simplicity).

As seen from Eq. (2.3.10), the amplitude of the dipole momentum ($\langle x \rangle_\omega$) strongly depends on the distribution of initial values $x_a(0)$ and $\dot{x}_a(0)$ as well as on the distribution of frequencies in the beam. Even in the case when

$$\sum_{a=1}^N \dot{x}_a = 0, \quad \sum_{a=1}^N x_a = 0,$$

so that

$$\langle x(0) \rangle = 0, \quad \langle \dot{x}(0) \rangle = 0$$

the amplitude $\langle x \rangle_\omega$ can differ from zero due to finite frequency spread. This phenomenon is also specific for the systems with the finite number of particles (or with the discrete spectrum).

The function $\epsilon(\omega)$, which was defined in Eq. (2.3.10) can be interpreted as the dielectric constant of the beam. Using Eqs. (2.3.9) and (2.3.10), we can rewrite Eq. (2.3.1) in the form

$$\mathcal{F}_x = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \mathcal{F}_x(\omega), \quad \mathcal{F}_x(\omega) = \frac{NkX(\omega)}{\epsilon(\omega)}. \quad (2.3.23)$$

This formula shows that the collective interaction of particles can strongly affect the spectral properties of the force, acting between these particles, provided that $|\epsilon(\omega)|$ strongly differs from 1.

2.4. DYNAMIC AND KINETIC EQUATIONS

Let us proceed to the description of the beam using methods of statistical mechanics. This means that we want to replace the dynamic equation by a kinetic one to describe the state of the beam by a rather small number of variables. Technically this can be done using different approaches. In this book we shall use the so-called method of the microscopic phase space density developed by U. L. Klimontovich.¹⁶

Within the framework of this method the most fundamental value is the microscopic density of the beam in phase space. If the trajectories of particles in the phase space are known, this density is defined by Eq. (2.1.15). If the oscillations of particles are described by the action-phase variables of the unperturbed motion $(\mathbf{I}, \psi)$, we can write F in the form

$$F(\mathbf{I}, \psi, t) = \sum_{a=1}^N \delta[\mathbf{I} - \mathbf{I}_a(t)] \delta[\psi - \psi_a(t)]. \quad (2.4.1)$$

Due to the interaction of particles their coordinates in phase space $\mathbf{I}_a(t), \psi_a(t)$ cannot be calculated exactly. That is why, in the method of microscopic phase space density, the value $F(\mathbf{I}, \psi, t)$ is assumed to be a random function of time. As additional sources of the randomization of F , the action on the particles of

random forces as well as the possible dynamic stochastization can be pointed out. Subtracting from F the systematic part we write

$$F(\mathbf{I}, \psi, t) = \langle F(\mathbf{I}, \psi, t) \rangle + \delta F(\mathbf{I}, \psi, t). \quad (2.4.2)$$

The brackets $\langle \dots \rangle$ mean the statistical averaging over the beam fluctuation. As far as the first term in Eq. (2.4.2) gives the average number of particles inside the phase space volume $d\Gamma$, according to Eq. (2.2.11) we shall identify this term as the single-particle distribution function of the beam

$$Nf(\mathbf{I}, \psi, t) = \langle F(\mathbf{I}, \psi, t) \rangle. \quad (2.4.3)$$

The many particles distribution functions can be constructed in a similar way. For instance, the averaging of the product $F(1, t)F(2, t)$ yields $[(1) \equiv (\mathbf{I}, \psi)_1, (2) \equiv (\mathbf{I}, \psi)_2]$

$$\begin{aligned} \langle F(1, t)F(2, t) \rangle &= \sum_{a, b=1}^N \langle \delta[\mathbf{I}_1 - \mathbf{I}_a(t)] \delta[\psi_1 - \psi_a(t)] \\ &\quad \times \delta[\mathbf{I}_2 - \mathbf{I}_b(t)] \delta[\psi_2 - \psi_b(t)] \rangle. \end{aligned} \quad (2.4.4)$$

Separating here the diagonal ($a = b$) and off-diagonal ($a \neq b$) terms, we can rewrite this expression as

$$\begin{aligned} \langle F(1, t)F(2, t) \rangle &= N\delta(1-2)f(1, t) + \sum_{a \neq b}^N \langle \delta[\mathbf{I}_1 - \mathbf{I}_a(t)] \delta[\psi_1 - \psi_a(t)] \\ &\quad \times \delta[\mathbf{I}_2 - \mathbf{I}_b(t)] \delta[\psi_2 - \psi_b(t)] \rangle \end{aligned} \quad (2.4.5)$$

or, neglecting the terms of the order of $1/N$,

$$N^2 f(1, 2, t) = \langle F(1, t)F(2, t) \rangle - N\delta(1-2)f(1, t). \quad (2.4.6)$$

Substituting F from Eq. (2.4.2) into Eq. (2.4.6), one can get one more useful relation:

$$\langle \delta F(1, t) \delta F(2, t) \rangle = N\delta(1-2)f(1, t) + g(1, 2, t), \quad (2.4.7)$$

where

$$g(1, 2, t) = N^2 [f(1, 2, t) - f(1, t)f(2, t)] \quad (2.4.8)$$

is the simultaneous correlation function.

Generally, the averaging in Eqs. (2.4.2) through (2.4.7) must be carried out over the physically small time intervals

$$\langle \dots \rangle = \int_0^T \frac{dt}{T} (\dots). \quad (2.4.9)$$

The possibility to define such an interval implies that the problem should provide for time hierarchy between macroscopic and microscopic processes. Namely, the time intervals specific for macroscopic processes τ_r (for instance, a relaxation time of the distribution function) must be essentially longer than that, specific for microscopic processes τ_k . In this case, we can choose the averaging time interval T from the region

$$\tau_k \ll T \ll \tau_r. \quad (2.4.10)$$

For problems with the time hierarchy, the averaging over time in Eq. (2.4.9) can be replaced by the averaging over the ensemble. The question of how to specify the time interval, which can be chosen as τ_k , must be decided for a particular problem with some care.

Once we want to describe the beam in the storage ring by the distribution function $f(1, t)$, we need the kinetic equation, describing its evolution. Within the method of microscopic density, it can be calculated by the subtraction from Eq. (2.1.23) of the fluctuating part. Assuming that the unperturbed motion is described by the Hamiltonian $\mathcal{H}_0$, while the perturbations are described by the Lagrangian L , we write

$$F = Nf + \delta F, \quad L = \langle L \rangle + \delta L, \quad \langle \mathcal{H} \rangle = \mathcal{H}_0 - \langle L \rangle, \quad (2.4.11)$$

where δF and δL are fluctuating parts of the beam phase space density and of the Lagrangian respectively. The substitution of these formulas into Eq. (2.1.23) and its averaging over fluctuations yield the system of equations

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [\langle \mathcal{H} \rangle; f] = \frac{1}{N} \langle [\delta L; \delta F] \rangle, \quad (2.4.12)$$

$$\frac{\partial \delta F}{\partial t} + [\langle \mathcal{H} \rangle; \delta F] - [\delta L; Nf] = \delta([\delta L; \delta F]). \quad (2.4.13)$$

As might be expected, the evolution of the beam density in phase space is caused by fluctuating processes (or, which is the same, by collisions). Without fluctuations, which in the formal sense correspond in Eq. (2.4.12) to the limit $N \rightarrow \infty$, the phase space density of the beam is conserved, while Eq. (2.4.12) is converted into the well-known Vlasov's equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [\langle \mathcal{H} \rangle; f] = 0, \quad (2.4.14)$$

describing dynamic processes in the beam (i.e., the processes that proceed with the conservation of the beam phase space volume). Though, through $\langle L \rangle$, the coefficients in Eq. (2.4.14) depend on the fields induced by the beam, these fields are considered to be external by this equation. Equation (2.4.14) is the fundamental one for studying collective, collisionless processes.

The effect from the fluctuating fields causes the nonconservation of f . However, before discussing how to calculate the collision integral

$$\text{St} = \frac{1}{N} \langle [\delta L; \delta F] \rangle, \quad (2.4.15)$$

we have to mention the following. Since the motion of particles in the machine is cyclic, the functions entering into Eq. (2.4.14) are periodic functions of phases and therefore can be represented by Fourier series

$$X(\mathbf{I}, \psi, t) = \sum_{\mathbf{m}} X_{\mathbf{m}}(\mathbf{I}, t) e^{im\psi}, \quad (2.4.16)$$

where X is any function from $f, \langle \mathcal{H} \rangle$, and St . Making use of this expansion splits Eq. (2.4.14) into two equations:

$$\frac{\partial f_0}{\partial t} - [\langle L \rangle; f]_0 = \text{St}(\mathbf{I}, t)_0, \quad (2.4.17)$$

$$\frac{\partial f_{\mathbf{m}}}{\partial t} + im\omega f_{\mathbf{m}} - [\langle L \rangle; f]_{\mathbf{m}} = \text{St}(\mathbf{I}, t)_{\mathbf{m}}. \quad (2.4.18)$$

The inhomogeneity of the distribution function in phases, which is described by the harmonics $f_{\mathbf{m}}$, indicates the excitation in the beam of collective oscillations. The development of these oscillations in the most general way is described by Eq. (2.4.18), which, in contrast with Eq. (2.4.14), takes into account the influence of collisions on collective processes. Below we shall see that in some cases collisions can significantly modify specific features of collective motion.

Depending on forces acting between particles, the coherent oscillations of the beam will either grow (in the case of instability) or decay. Both these processes can be characterized by a definite scale of time τ_k . Due to reasons to be discussed we assume now that the collective oscillations of the beam decay during the time intervals on the order of τ_k . Then after the time $\sim \tau_k$ the distribution function becomes independent of phase variables ψ_α and a new inhomogeneity of f can appear only due to external kicks. Provided that such kicks are excluded, after time intervals $\sim \tau_k$, one can replace Eqs. (2.4.17) and (2.4.18) by the equation

$$\frac{\partial f_0}{\partial t} = \text{St}(\mathbf{I}, t)_0, \quad (2.4.19)$$

describing the evolution of the beam without coherent oscillations.

Let us now return to the analysis of Eqs. (2.4.12) and (2.4.13). These equations are as complicated as the original Eq. (2.1.23). For example, to solve Eq. (2.4.12) we have to calculate the simultaneous correlation function $\langle [\delta L; \delta F] \rangle$. However, since the fluctuations of L and F are coupled by Eqs.

(2.1.25), the necessary correlation function is determined by the nonsimultaneous correlation function

$$K^{(2)}(1,2) = \langle \delta F(1,t_1) \delta F(2,t_2) \rangle. \quad (2.4.20)$$

Using Eq. (2.4.13) one can easily find the equation for $K^{(2)}$; however due to the term $\delta([\delta L; \delta F])$ in the r.h.s. of Eq. (2.4.13) the equation for $K^{(2)}$ will contain the triple correlation function

$$K^{(3)} = \langle \delta F(1,t_1) \delta F(2,t_2) \delta F(3,t_3) \rangle, \quad (2.4.21)$$

which must be calculated before the calculation of $K^{(2)}$. It is well known that the repetition of this scheme leads to the system of coupled equations, which is analogous to the so-called BBGKI set of equations,^{*} which take into account the possibility of the correlation between more and more particles. However, in real cases, the necessity to calculate the correlation functions of higher orders occurs very rarely. If, for instance, the radius of the force action r_0 is defined in the problem, which almost always is true, then the m -particle correlation function generally will be proportional to the probability for m particles to be collected in the volume r_0^3 . Without special conditions the probability of this event sharply decays as m increases. Respectively, this decreases the contributions of these correlation functions in the variation of f . Therefore, on some step one can put $K^{(m)} = 0$, and thus cut the set of equations for correlation functions. The remaining system of equations becomes closed and in principle can be solved. The possibility of such cutting requires the time hierarchy in the problem,¹⁷ which enables one to separate the microscopic and macroscopic effects. In particular, specific time intervals for the variations of f must sufficiently exceed that for δF .

Usually, the interaction between particles in storage rings is caused by long-range forces. For this reason, the correlations in particle motions may have a two-fold origin. First, these are correlations due to collisions of the particles, when the values describing an elementary process vary during the so-called collision time τ_{st} . A typical value of τ_{st} is determined by the ratio of the interaction radius to the velocity spread in the beam. Second, these correlations can be due to various collective motions of the beam. In this case, specific time interval τ_{col} generally depends on the beam intensity and on the frequency spread in the beam. The identification of either τ_{st} or τ_{col} as τ_k generally yields different kinetic equations. In this Chapter, we assume that

$$\tau_k = (\tau_{st} \ll \tau_{col}). \quad (2.4.22)$$

If collective oscillations of the beam do not grow, collective correlations are described by the regular part of the linearized Eq. (2.4.13). For times $t \gg \tau_k$ the

^{*}This is an abbreviation for Bogolubov-Born-Green-Kirkwood-Ivon.

separation in Eq. (2.4.13) of regular and singular parts

$$\delta F = \delta F_R + \delta F_S, \quad \delta L = \delta L_R + \delta L_S, \quad \delta L_S = \delta L(\delta F_S) \quad (2.4.23)$$

yields the following system of equations:

$$\left(\frac{\partial}{\partial t} + [\mathcal{H}_0;] \right) \delta F_R = [\delta L_R; Nf], \quad (2.4.24)$$

$$\left(\frac{\partial}{\partial t} + [\mathcal{H}_0;] \right) \delta F_S = [\delta L_S; Nf] + [\delta L; \delta F]_S. \quad (2.4.25)$$

The nonsingular function $\delta F_R(\mathbf{I}, \psi, t)$ in these equations in fact presents the fluctuating correction to the distribution function $f(\mathbf{I}, t)$. It describes coherent oscillations of the beam, which are excited by the thermal motion of particles. Collisions of particles in these equations are described by the singular part of δF , which satisfies Eq. (2.4.25). The simplification of this equation is essentially associated with the possibility to cut the system of equations for higher-order correlation functions, which has been discussed above. At this step some perturbation procedure must be employed. Typically, the pair correlations approximation is quite sufficient to describe beam behavior in storage rings. As can be seen from Eq. (2.4.25), this approximation corresponds to formal linearization of Eq. (2.4.25) and yields δF_S independent of δF_R :

$$\left(\frac{\partial}{\partial t} + [\mathcal{H}_0;] \right) \delta F_S = [\delta L_S; Nf], \quad (2.4.26)$$

as might be expected for collisions of individual pairs of particles. On the other hand, Eq. (2.4.25) indicates that higher-order correlation functions generally lose this feature—the collective motion can more strongly affect collisions of higher number of particles. Within the pair collisions approximation Eqs. (2.4.12), (2.4.24), and (2.4.26) yield the closed system of equations, that couples f and the random functions δF_S and δF_R . Using the formal analogy between Eq. (2.4.24) and Eq. (2.4.26), for particular calculations it can be more convenient to unify both these equations and to work directly with δF without separation of its regular and singular parts. The resulting system of equations has the form

$$\frac{df_0}{dt} = \frac{1}{N} \langle [\delta L, \delta F] \rangle, \quad (2.4.27)$$

$$\left(\frac{\partial}{\partial t} + [\mathcal{H}_0;] \right) \delta F = [\delta L; Nf_0]. \quad (2.4.28)$$

Due to the assumed time hierarchy, the solution of Eq. (2.4.28) and therefore the contribution into the r.h.s. of Eq. (2.4.27) of the correlation function can be

expressed in terms of the distribution function f_0 , which transforms this equation into a kinetic one.

One can note a remarkable analogy between Eq. (2.4.28) and the linearized Vlasov's Eq. (2.4.14). This is not so surprising—both equations describe collisionless processes and therefore are dynamic. Provided the theory of collective oscillations is well developed, this fact simplifies the calculation of the correlation function entering into the r.h.s. of Eq. (2.4.27). Nevertheless, between δF and the oscillating part of the distribution function f , there is a significant difference. As we mentioned before, due to the finite number of particles in the beam, the fluctuations δF are generated by the thermal motion of particles. That is why, even without collective instabilities, the beam always has a fluctuating coherent background. On the contrary, the oscillations of f can be regenerated only in beams with collective instabilities.

As was said before, to calculate the collision integral one has to solve Eq. (2.4.28). In the action-phase variables it has the form

$$\frac{\partial \delta F}{\partial t} + \omega \frac{\partial \delta F}{\partial \psi} + N \frac{\partial \delta L}{\partial \psi} \frac{\partial f_0}{\partial \mathbf{I}} = 0. \quad (2.4.29)$$

Due to the cyclic motion of particles in the ring, the function δL in Eq. (2.4.29) depends periodically on the azimuth ϑ . Therefore, one may use the expansion

$$\delta L = \sum_{n=-\infty}^{\infty} \delta L_n(\mathbf{r}, \mathbf{v}, t) e^{in\vartheta}. \quad (2.4.30)$$

According to Eqs. (1.6.3), (1.6.22), and (1.6.23) the arguments in δL can be written in the form

$$\mathbf{r} = \mathbf{r}_0(t) + \boldsymbol{\rho}(\mathbf{I}, \psi, t), \quad (2.4.31)$$

where the vector $\boldsymbol{\rho}(\mathbf{I}, \psi, t)$ describes the oscillations of the particle. Therefore, the azimuth ϑ in Eq. (2.4.30) can be replaced by $\omega_s t + \Delta\vartheta(\mathbf{I}, \psi)$ and Eq. (2.4.30) can be rewritten in the form

$$\delta L = \sum_{n=-\infty}^{\infty} \delta L_n(\mathbf{I}, \psi, t) \exp[in\omega_s t + in\Delta\vartheta(\mathbf{I}, \psi)]. \quad (2.4.32)$$

This means that the solution of Eq. (2.4.29), except for the systematic part, may have a periodic dependence on time caused by the periodicity of the particle motion along the closed orbit:

$$\delta F(\mathbf{I}, \psi, t) = \sum_{n=-\infty}^{\infty} \delta F_n(\mathbf{I}, \psi, t) \exp(in\omega_s t). \quad (2.4.33)$$

The substitution of Eq. (2.4.33) into Eq. (2.4.29) yields

$$\left(\frac{\partial}{\partial t} + in\omega_s + \omega \frac{\partial}{\partial \psi}\right) \delta F_n(\mathbf{I}, \psi, t) + N \frac{\partial f_0}{\partial \mathbf{I}} \frac{\partial}{\partial \psi} \{ \delta L_n(\mathbf{I}, \psi, t) \exp[in\Delta\vartheta(\mathbf{I}, \psi)] \} = 0. \quad (2.4.34)$$

Using the Fourier expansions

$$\delta F_n(\mathbf{I}, \psi, t) = \sum_{\mathbf{m}=-\infty}^{\infty} \delta F_{\mathbf{m},n}(\mathbf{I}, t) \exp(im\psi), \quad (2.4.35)$$

$$\delta F_{\mathbf{m},n}(\mathbf{I}, \omega) = \int_0^{\infty} dt \exp(i\omega t) \delta F_{\mathbf{m},n}(\mathbf{I}, t), \quad \text{Im}(\omega) > 0, \quad (2.4.36)$$

one can find for Fourier amplitudes $\delta F_{\mathbf{m},n}(\mathbf{I}, \omega)$

$$\delta F_{\mathbf{m},n}(\mathbf{I}, \omega) = \frac{i\delta_{n,0}\delta F_{\mathbf{m}}^{(0)}(\mathbf{I})}{\omega - \omega_{\mathbf{m},n}} + \frac{N\delta L_{\mathbf{m},n}}{\omega - \omega_{\mathbf{m},n}} \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right). \quad (2.4.37)$$

Here

$$\delta L_{\mathbf{m},n} = \{ \delta L_n(\mathbf{I}, \psi, \omega) \exp[in\Delta\vartheta(\mathbf{I}, \psi)] \}_{\mathbf{m}}, \quad (2.4.38)$$

$\omega_{\mathbf{m},n} = \mathbf{m}\omega + n\omega_s$, and $\delta F_{\mathbf{m}}^{(0)}(\mathbf{I}) = \delta F_{\mathbf{m}}(\mathbf{I}, t=0)$ is the initial value of the harmonic $\delta F_{\mathbf{m}}$. Moreover, Eq. (2.4.37) takes into account that harmonics $\delta F_{\mathbf{m},n}$ with $n \neq 0$ are excited by the perturbation δL , while $\delta F^{(0)}$ has no such modulation:

$$\delta F(\mathbf{I}, \psi, t=0) = \sum_{\mathbf{m}=-\infty}^{\infty} \delta F_{\mathbf{m}}(\mathbf{I}, t=0) \exp(im\psi). \quad (2.4.39)$$

If the perturbation δL is caused by the beam induced fields, δL is related to δF through Maxwell's equations. Therefore, Eq. (2.4.37) is an inhomogeneous integral equation, generally, of the Fredholm type. Solutions of such an equation can be constructed using, for instance, the eigenfunctions and the eigenfrequencies of the homogeneous problem. In the most general case these eigenfrequencies can be written in the form

$$\omega = \omega_{\mathbf{m},n} + \Omega_{\mathbf{m},n}, \quad (2.4.40)$$

where the deviation from the unperturbed frequencies $\Omega_{\mathbf{m},n}$ describes the collective interaction and is usually called the coherent frequency shift.

2.5. THE PAIR COLLISIONS INTEGRAL

Close inspection of Eqs. (2.4.28) and (2.4.37) shows that the time intervals specific for the variation of $\delta F_{\mathbf{m}}$ can be determined either by the frequency spread in the beam $\delta\omega_{\mathbf{m}}$ or by the collective interaction of particles. In the last case, the time scale is specified by the value of the coherent frequency shift $\Omega_{\mathbf{m},n}$, which we introduced in the previous section:

$$\delta F_{m,n}(\mathbf{I}, t) = \delta F_{m,n}(\mathbf{I}) \exp[-i(\omega_{m,n} + \Omega_{m,n})t].$$

The calculation of $\Omega_{m,n}$ will be discussed in the next chapter. Here we note only that generally, provided that

$$|\Omega_{m,n}| \ll \delta\omega_{m,n}, \quad (2.5.1)$$

one may expect a weak influence of collective processes on the behavior of δF and therefore choose $\tau_k \simeq \delta\omega_{m,n}$. In this region of parameters the variations of f are mainly caused by collisions when distances between colliding particles are smaller than the screening radius due to collective reaction of the beam. For particles in storage ring this screening radius can be estimated using the ratio $|\Omega_{m,n}|/\delta\omega_{m,n}$. In a selected region of parameters the collision integral can be calculated with a more or less general assumption about the interaction between particles.

Suppose we know the Green functions of the electrodynamic problem. Then solutions of Eq. (2.1.25) can be written in the form

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla A_0, \quad \mathbf{H}(\mathbf{r}, t) = \text{rot} \mathbf{A}(\mathbf{r}, t), \quad (2.5.2)$$

$$A_\alpha(\mathbf{r}, t) = e \int d\Gamma_2 \int_0^\infty d\tau \frac{v_\beta(2)}{c} G_{\alpha\beta}(1, 2, \tau) F(2, t - \tau), \quad (2.5.3)$$

$$A_0(\mathbf{r}, t) = e \int d\Gamma_2 \int_0^\infty d\tau G_{00}(1, 2, \tau) F(2, t - \tau). \quad (2.5.4)$$

With the Lorentz gauge condition

$$\text{div} \mathbf{A}(\mathbf{r}, t) + \frac{1}{c} \frac{\partial A_0}{\partial t} = 0 \quad (2.5.5)$$

Green functions $G_{\alpha,\beta}$ and G_{00} satisfy the equations

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G_{\alpha,\beta}(\mathbf{r}, \mathbf{r}', t - t') = -4\pi \delta_{\alpha,\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.5.6)$$

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G_{00}(\mathbf{r}, \mathbf{r}', t - t') = -4\pi \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (2.5.7)$$

and relevant boundary conditions. This yields for δL

$$\delta L(1, t) = e^2 \int d\Gamma_2 \int_0^\infty d\tau G(1, 2, \tau) F(2, t - \tau), \quad (2.5.8)$$

where

$$G(1,2,\tau) = \frac{v_\alpha(1)v_\beta(2)}{c^2} G_{\alpha,\beta}(1,2,\tau) - G_{00}(1,2,\tau). \quad (2.5.9)$$

These equations specify the kernel of the integral equation (2.4.37) though it is still too complicated for straightforward calculations. Here, we expand the solution in the perturbation series in powers of the parameter $(\Omega_{\mathbf{m},n}/\delta\omega_{\mathbf{m},n})$. The first two terms of this expansion can be written in the form

$$\begin{aligned} \delta F_{\mathbf{m},n}(\omega) &= \frac{i\delta_{n,0}F^{(0)}}{\omega - \mathbf{m}\omega} + \frac{iNe^2}{\omega - \mathbf{m}\omega} \left(\frac{\partial f_0}{\partial \mathbf{I}} \right) \partial L_{\mathbf{m},n}^{(1)}, \\ \delta L_{\mathbf{m},n}^{(1)} &= \sum_{n_1, \mathbf{m}_2} \int d\Gamma_2 \frac{G_{n_1, n_1 - n}^{\mathbf{m}, \mathbf{m}_2}(1, 2\omega + n_1\omega_s) \delta F_{\mathbf{m}_2}^{(0)}(\mathbf{I}_2)}{\omega - \mathbf{m}_2\omega(2) + n\omega_s}, \end{aligned} \quad (2.5.10)$$

where

$$G_{n_1, n_2}^{\mathbf{m}_1, \mathbf{m}_2} = \int_0^{2\pi} \frac{d^3\psi_1}{(2\pi)^3} \frac{d^3\psi_2}{(2\pi)^3} G_{n_1, n_2} \exp\{in_1\Delta\vartheta(1) - in_2\Delta\vartheta(2) - i\mathbf{m}_1\psi_1 + i\mathbf{m}_2\psi_2\} \quad (2.5.11)$$

and

$$G_{n_1, n_2} = \int_0^{2\pi} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} G(1, 2, \tau) e^{-in_1\vartheta_1 + in_2\vartheta_2}. \quad (2.5.12)$$

Reconstructing from Eq. (2.5.10) $\delta F(\mathbf{I}, \psi, t)$

$$\delta F_{\mathbf{m}}(\mathbf{I}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta F_{\mathbf{m}}(\mathbf{I}, \omega) e^{-i\omega t}, \quad \text{Im}(\omega) > 0, \quad (2.5.13)$$

and substituting the resulting expression into Eq. (2.4.27) one can easily verify that the r.h.s. of this equation contains terms of two kinds. The first describe the systematic evolution of f , while the second describe the beatings with combination frequencies $\omega_{\mathbf{m},q}$. If the working point of the machine is far enough from resonances $\mathbf{m}\omega = q\omega_s$, the last terms will not contribute to the evolution of f . To separate the systematic variations of f we can average Eq. (2.4.27) over the time interval T from the region

$$\{\min(\omega_s, |\mathbf{m}\nu + q|)\}^{-1} \ll T \ll \tau_r. \quad (2.5.14)$$

After this averaging the equation for the zeroth harmonic of the distribution function f_0 takes the form

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \mathbf{I}} \mathbf{j}(\mathbf{I}, t), \quad (2.5.15)$$

with

$$\mathbf{j}(\mathbf{I}, t) = \frac{1}{N} \int_0^T \frac{dt}{T} \left\langle \frac{\partial \delta L}{\partial \psi} \delta F(\mathbf{I}, \psi, t) \right\rangle. \quad (2.5.16)$$

As might be expected beforehand, the collision integral in Eq. (2.5.15) is presented by the divergence of the current from Eq. (2.5.16) in the space of the integrals of motion $\mathbf{I}$.

Typically, the difference between τ_r and τ_k is so high that, making calculations with Eq. (2.5.16), we may adopt $T \rightarrow \infty$. This yields

$$\mathbf{j}(\mathbf{I}, t) = \frac{1}{N} \text{Re} \left\{ -i \sum_{\mathbf{m}} \mathbf{m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\delta L_{-\mathbf{m}} \delta F_{\mathbf{m}})_{\omega} \right\}, \quad (2.5.17)$$

where

$$(AB)_{\omega} = \lim_{\Delta \rightarrow 0} 2\Delta \langle A^*(\omega + i\Delta) B(\omega + i\Delta) \rangle \quad (2.5.18)$$

is the spectral density of the correlation function $\langle A(t) B(t - \tau) \rangle$. Substituting here solutions from Eq. (2.5.10) yields

$$\begin{aligned} \mathbf{j} = \sum_{\mathbf{m}} \mathbf{m} \text{Re} \left[\frac{-i}{N} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \lim_{\Delta \rightarrow 0} 2\Delta \left\langle \delta L_{-\mathbf{m}, -\omega_m} \frac{i\delta F_{\mathbf{m}}^{(0)}}{\omega - \omega_m + i\Delta} \right\rangle \right] \\ - \pi \mathbf{m} \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) (\delta L_{\mathbf{m}})_{\omega_m}^2, \quad \omega_m = \mathbf{m}\omega. \end{aligned} \quad (2.5.19)$$

In subsequent calculations we have to find the following spectral densities:

$$K_n(\omega) = \lim_{\Delta \rightarrow 0} 2\Delta \frac{\langle \delta F_{\mathbf{m}_1}^{(0)}(1) \delta F_{\mathbf{m}_2}^{(0)*}(2) \rangle}{[\omega - \omega(1) - i\Delta][\omega - \omega(2) - n\omega_s + i\Delta]}. \quad (2.5.20)$$

Using Eq. (2.4.7) written for harmonics of F , one can get

$$\langle \delta F_{\mathbf{m}_1}^*(1, t) \delta F_{\mathbf{m}_2}(2, t) \rangle = N \delta_{\mathbf{m}_1, \mathbf{m}_2} \delta(\mathbf{I}_1 - \mathbf{I}_2) f_0(\mathbf{I}_1, t) + g_{\mathbf{m}_1, \mathbf{m}_2}(1, 2, t). \quad (2.5.21)$$

Thus, Eq. (2.5.20) can be transformed into

$$\begin{aligned} K_n(\omega) = 2\pi N \delta(1 - 2) \delta[\omega(1) - \omega(2)] \delta_{n,0} f_0 \\ + \lim_{\Delta \rightarrow 0} 2\Delta \frac{g_{\mathbf{m}_1, \mathbf{m}_2}(1, 2)}{[\omega - \omega(1) - i\Delta][\omega - \omega(2) - n\omega_s + i\Delta]}. \end{aligned} \quad (2.5.22)$$

For states of the beam that are close to the equilibrium, the simultaneous correlation function $g(1, 2)$ is a nonsingular function of its arguments. For this reason the second term in Eq. (2.5.21) tends to zero when $\Delta \rightarrow 0$. As a result instead of Eq. (2.4.16) we can write

$$K_n(\omega) = 2\pi N \delta(1-2) \delta[\omega(1) - \omega(2)] \delta_{n,0} f_0(1, t), \quad (2.5.23)$$

which describes the statistically independent generation of simultaneous fluctuations in the beam:

$$\langle \delta F(1) \delta F(2) \rangle = N \delta(1-2) f_0(1). \quad (2.5.24)$$

This yields for $\mathbf{j}(\mathbf{I}, t)$ the expressions

$$\mathbf{j}(\mathbf{I}, t) = \mathbf{j}^{(1)} + \mathbf{j}^{(2)} + \mathbf{j}^{(3)}, \quad (2.5.25)$$

where

$$\mathbf{j}^{(1)} = -e^2 f_0(1) \sum_{\mathbf{m}} \mathbf{m} \text{Im} V_0(1, 1), \quad (2.5.26)$$

$$\begin{aligned} \mathbf{j}^{(2)} = & \pi N e^4 \sum_{\mathbf{m}_1, \mathbf{m}_2} \int_{-\infty}^{\infty} d(2) \left\{ f(1) \left(\mathbf{m}_2 \frac{\partial f(2)}{\partial \mathbf{I}_2} \right) - f(2) \left(\mathbf{m}_1 \frac{\partial f(1)}{\partial \mathbf{I}_1} \right) \right\} \\ & \times \sum_n |V_n(1, 2)|^2 \delta(\omega(1) - \omega(2) + n\omega_s), \end{aligned} \quad (2.5.27)$$

$$\begin{aligned} \mathbf{j}^{(3)} = & -2f_0(1) \sum_{\mathbf{m}} \mathbf{m} N e^4 \sum_{\mathbf{m}_2, n'} \text{Re} \left\{ \int_{-\infty}^{\infty} \frac{d(2)}{[\omega(1) - \omega(2) + n'\omega_s + i\Delta]} \right. \\ & \times \left. \left(\mathbf{m}_2 \frac{\partial f(2)}{\partial \mathbf{I}_2} \right) V_{n'}(1, 2) \text{Im} V_{n'}(2, 1) \right\}, \end{aligned} \quad (2.5.28)$$

and

$$V_n(1, 2) = \sum_{n_1} G_{n_1, n-n_1}^{\mathbf{m}_1, \mathbf{m}_2} [1, 2, \omega(1) + n_1\omega_s] \quad (2.5.29)$$

is the Fourier harmonic of the interaction energy of particles 1 and 2. Previously, the collision integral for the interaction between particles of a general type had been calculated in Ref. 18. This result differs from that obtained in Ref. 18 by the contribution in Eq. (2.5.26) and (2.5.28) from off-diagonal in n harmonics of $V_{n \neq 0}$, which describe the azimuthal asymmetry of the particles interaction. As will be seen in §2.6, these terms in the current $\mathbf{j}(\mathbf{I}, t)$ generally cause the self-heating of the beam and therefore can be important for the analysis of the beam kinetics. Provided that the condition (2.5.1) is valid, the diagonal terms $V_{n=0}$ in Eq. (2.5.26) and (2.5.28) coincide with the results of calculations in Ref. 18.

Before the analysis of Eq. (2.5.25) let us make one more comment. During previous calculations it has been assumed that the only source of the field fluctuations is the Schottky noise of the beam. If there are other sources of noise these must be taken into account when solving Eq. (2.4.28), and thus will

contribute to the current $\mathbf{j}(\mathbf{I}, t)$. In the simplest case this can be the thermal noise of the field in the system, which transfers the interaction between particles. If this system is assumed to be at statistical equilibrium, the spectral density of this noise is determined by the fluctuation-dissipative theorem,¹⁹ which reads

$$\langle \delta A_\alpha(1) \delta A_\beta(2) \rangle = \frac{T_0}{\pi \omega} \text{Im} G_{\alpha\beta}(1, 2, \omega), \quad \hbar \omega \ll T_0, \quad (2.5.30)$$

$2\pi\hbar$ is the Planck constant and T_0 is the temperature of the system. The substitution of this expression into Eq. (2.5.19) results in the following addition to $\mathbf{j}(\mathbf{I}, t)$:

$$\mathbf{j}^{(4)}(\mathbf{I}, t) = -e^2 x T_0 \sum_{\mathbf{m}, n} \left(\mathbf{m} \frac{\partial f}{\partial \mathbf{I}} \right) \mathbf{m} \frac{\text{Im} G(1, 2, \omega_m + n\omega_s)}{\omega_m + n\omega_s}. \quad (2.5.31)$$

Thus, in the region of parameters specified by condition (2.5.1), the collision integral for the interaction of the most general type is determined by the current

$$\mathbf{j}(\mathbf{I}, t) = \mathbf{j}^{(1)} + \mathbf{j}^{(2)} + \mathbf{j}^{(3)} + \mathbf{j}^{(4)}, \quad (2.5.32)$$

where the components $\mathbf{j}^{(a)}(\mathbf{I}, t)$ are given by Eqs. (2.5.26) and (2.5.31). Even the structure of the obtained expressions indicates that the terms of the first and second order in e^2 describe different kinetic processes. The terms of first order describe the heat exchange between particles and the external system, while the terms of the second order describe the mutual influence of beam particles during this heat exchange. In the most general case this mutual influence can manifest itself in a dual way: via the intrabeam scattering (IBS) and via the polarization (or screening) phenomena, when the field induced by a particle is reduced by the collective reaction of the beam. In Eq. (2.5.32) the last phenomenon is described by the component $\mathbf{j}^{(3)}$. Combining in Eq. (2.5.32) the components $\mathbf{j}^{(1)}$ and $\mathbf{j}^{(3)}$ one can get an expression of the same type as Eq. (2.5.26), but with $\text{Im} V_0(1, 1)$ replaced by

$$\begin{aligned} \text{Im} V_0^{(2)}(1, 1) = & \text{Im} V_0(1, 1) + 2Ne \sum_{\mathbf{m}_2, n_2} \text{Re} \left\{ \int_{-\infty}^{\infty} d(2) \text{Im} V_{n_2}(2, 1) \right. \\ & \left. \times V_{n_2}(1, 2) \frac{[\mathbf{m}_2 \partial f(2) / \partial \mathbf{I}_2]}{\omega(1) - \omega(2) - n_2 \omega_s + i\Delta} \right\}, \end{aligned} \quad (2.5.33)$$

which is the imaginary part of the Fourier harmonic of the interaction energy in the second order of the perturbation theory. This expression indicates that $\mathbf{j}^{(3)}$ yields the first-order correction in Ne^2 to the solution of Eq. (2.4.28). As will be shown shortly, calculation of higher-order corrections of this type leads only to renormalization of V by the collective fields of the beam. Since such a renormalization can be important only in a case when condition (2.5.1) is violated, we may omit item $\mathbf{j}^{(3)}$ in Eq. (2.5.32).

To clarify the physical sense of the remaining terms in Eq. (2.5.32) let us calculate the rate of the beam entropy during the relaxation. Since, by definition¹⁹

$$S = - \int d(1) f \ln f, \quad (2.5.34)$$

the integration of Eq. (2.5.15) with $(-\ln f)$ yields

$$\frac{dS}{dt} = \frac{dS^1}{dt} + \frac{dS^2}{dt} + \frac{dS^4}{dt}, \quad (2.5.35)$$

$$\frac{dS^1}{dt} = -e^2 \sum_{\mathbf{m}, n} \int d(1) f(1) \left(\mathbf{m} \frac{\partial}{\partial \mathbf{I}} \text{Im} G(1, 1, \omega_{\mathbf{m}, n}) \right), \quad (2.5.36)$$

where

$$\begin{aligned} \frac{dS^2}{dt} = & \frac{\pi N e^4}{2} \sum_{\mathbf{m}_1, \mathbf{m}_2} \int_{-\infty}^{\infty} d(1) d(2) f(1) f(2) \left\{ \left(\mathbf{m}_1 \frac{\partial \ln f(1)}{\partial \mathbf{I}_1} \right) \right. \\ & \left. - \left(\mathbf{m}_2 \frac{\partial \ln f(2)}{\partial \mathbf{I}_2} \right) \right\}^2 \sum_n |V_n(1, 2)|^2 \delta[\omega(1) - \omega(2) + n\omega_s], \end{aligned} \quad (2.5.37)$$

$$\frac{dS^4}{dt} = e^2 T_0 \sum_{\mathbf{m}, n} \int \frac{d(1)}{f(1)} \left(\mathbf{m} \frac{\partial \ln f(1)}{\partial \mathbf{I}_1} \right)^2 \frac{\text{Im} G(1, 1, \omega_m + n\omega_s)}{\omega_m + n\omega_s}. \quad (2.5.38)$$

Since, by definition,¹⁹ S is connected with the beam emittance, Eq. (2.5.35) yields the instantaneous increment (or decrement) of the total beam phase space volume. If (dS/dt) is negative, the interaction decreases the beam emittance; that is, the interaction with such systems cools the beam. Closer inspection shows that in Eq. (2.5.35) only the term (dS^1/dt) can be negative. [The term (dS^4/dt) is positive since $\text{Im} G(\omega)/\omega > 0$.] This means that the cooling of the beams is always associated with the dissipation of the particle energy by an external system. Eq. (2.5.36) can also be rewritten in the form

$$\frac{dS^1}{dt} = \langle \partial \dot{I}_\alpha / \partial I_\alpha \rangle \quad (2.5.39)$$

or, calculating $\dot{I}_\alpha$ via the force of the radiative reaction $\mathcal{F}(\mathbf{p}, \mathbf{r}, t)$ of a particle,

$$\dot{I}_\alpha = [I_\alpha; \mathbf{r}] \mathcal{F}, \quad (2.5.40)$$

one can find the formula relating this term in dS/dt to the average sum of particle decrements

$$\frac{dS^{(1)}}{dt} = \left\langle \frac{\partial}{\partial I^\alpha} ([I_\alpha; \mathbf{r}] \mathcal{F}) \right\rangle = \left\langle \frac{\partial}{\partial p_\lambda} \mathcal{F}_\lambda(\mathbf{p}, \mathbf{r}, t) \right\rangle. \quad (2.5.41)$$

The terms $(dS^{(2)}/dt)$ and $(dS^{(4)}/dt)$ describe the beam heating due to, respectively, IBS and the scattering of particles by the thermal fluctuations of the field in the system. As is indicated by Eq. (2.5.35), to cool the beam the modulation of the friction force $\mathcal{F}(\mathbf{p}, \mathbf{r}, t)$ must be limited from below:

$$-\langle |\operatorname{div}_{\mathbf{p}} \mathcal{F}| \rangle > \left(\frac{dS^2}{dt} + \frac{dS^4}{dt} \right). \quad (2.5.42)$$

2.6. HEATING OF THE BEAM DUE TO INTRABEAM SCATTERING

Let us apply the general equations obtained in the previous section to the study of the Coulomb IBS. Though such a simplified approach does not give the complete description of this phenomenon, even the calculations within the approximation (2.5.1) can yield important practical predictions. The importance of Coulomb collisions is due to the following fact: The cross section of the Coulomb collisions increases roughly as T^{-2} as the beam temperature T decreases. The average collision frequency is determined by the combination $N/(T^{3/2}a^2)$, where a is the beam radius. Therefore, one may expect the amplification of the influence of the IBS on the beam parameters as its density in the phase space increases.

In the simplest situation the IBS leads to the redistribution of the partial emittances of the beam and so can cause an undesirable dilution of the beam in the phase space. In storage rings, due to special arrangement of the particle motion, the Coulomb relaxation can be accompanied by some complementary effects. In particular, sometimes IBS can heat the beam as a whole^{20,21,22,23} (i.e., it can increase the partial beam emittances simultaneously).

The reason for this self-heating can be explained by a simple model.²¹ Let us consider the collision of two particles that in the comoving reference system initially have equal and oppositely directed radial momenta p , which after scattering at a 90° angle are transferred to the longitudinal motion. Due to such collisions, the total energy of radial betatron oscillations of particles

$$\mathcal{E}_x = \frac{v_x \omega_p p_s}{2R_0} [a_x^2(1) + a_x^2(2)] \quad (2.6.1)$$

in the laboratory system, on the average along the orbit, will change by the value

$$\Delta \mathcal{E}_x = \frac{v_s}{p_s} p^2 \left[\frac{\gamma^2}{R_0^2} \left(\frac{\eta^2}{f_x^2} + \frac{\Phi^2}{f_x^4} \right) - 1 \right], \quad (2.6.2)$$

where $\Phi = \eta' f_x - f'_x \eta$. This is accompanied by the following variation of the energy of the longitudinal motion:

$$\Delta \mathcal{E}_{\parallel} = \frac{v_s}{p_s} p^2 (1 - \alpha \gamma^2). \quad (2.6.3)$$

These equations can be written in an especially simple form for the beam in the ring with azimuthal symmetric focusing:

$$\Delta \mathcal{E}_x = \frac{p^2}{\gamma m} \left(\frac{\gamma^2}{v_x^2} - 1 \right), \quad \Delta \mathcal{E}_{\parallel} = \frac{p^2}{\gamma m} \left(1 - \frac{\gamma^2}{v_x^2} \right), \quad (2.6.4)$$

$$\eta/R_0 = \alpha = 1/v_x^2, \quad f_x = v_x^{-1/2}. \quad (2.6.5)$$

One can see that above the transition energy $\gamma \geq v_x$ collisions cause simultaneous growth of both $\mathcal{E}_x$ and the beam momentum spread. The total energy of the thermal motion in this case conserves $\Delta \mathcal{E}_t = \Delta \mathcal{E}_x + \Delta \mathcal{E}_{\parallel} = 0$.

Strictly speaking, the ability to work below or above the transition energy assumes an azimuthal asymmetry of the ring lattice. For this more general case, except for the simultaneous growth of $\Delta \mathcal{E}_x$ and $\langle (\Delta p)^2 \rangle$, Eqs. (2.6.2) and (2.6.3) indicate the growth of the total energy $\Delta \mathcal{E}_t$ due to collisions

$$\Delta \mathcal{E}_t = \Delta \mathcal{E}_x + \Delta \mathcal{E}_{\parallel} = 2\gamma \frac{p^2}{m} \overline{\left(\frac{\Phi^2}{f_x^2} \right)}. \quad (2.6.6)$$

Moreover, the nonconservation of the energy of the thermal motion is caused precisely by the Floquet modulation of betatron oscillations. The total energy of colliding particles

$$\mathcal{E}(1,2) = c \sqrt{p_1^2 + m^2 c^2} + c \sqrt{p_2^2 + m^2 c^2} \quad (2.6.7)$$

obviously is conserved. This means that, in strong focusing lattices, collisions couple the thermal motion and the motion along the orbit. For distribution functions describing the beam $f_0(\mathcal{E}_t)$ close to equilibrium the rate of $\langle \mathcal{E}_t \rangle$ is connected with the rate of the beam entropy

$$\frac{dS}{dt} = \frac{d}{dt} \ln \langle \mathcal{E}_t \rangle. \quad (2.6.8)$$

Thus, Eqs. (2.6.6) and (2.6.8) mean that independent of the relative position of particle energies and the transition energy of the ring one has $(dS/dt) > 0$, which describes the growth of the total phase space volume of the beam.²² This result agrees with the general theorem which states¹⁹ that at statistical equilibrium the distribution functions can depend only on the additive integrals of motion. In the strong focusing lattices the only integral of this kind is energy. The effect of the beam self-heating and other specific features of IBS can be studied using kinetic Eq. (2.6.11). Except for the use of the systematic approach,

this enables the calculation of the family of the equilibrium distribution functions as well as to the estimation of the relaxation times. The reason for a general discussion of this subject is enhanced by the fact²⁴ that external fields can strongly modify the collision integral if the collision of a pair lasts longer than the periods of the particle oscillations.

In this section we discuss the effect of the beam self-heating due to the Coulomb IBS within a simplest pair collision approximation, with the emphasis on the main features specific for storage rings. For the Coulomb interaction of particles we can write

$$\delta L = - \int d(2) V(1,2) \delta F(2,t), \quad (2.6.9)$$

where $V(1,2) = e^2/|\mathbf{r}_1 - \mathbf{r}_2|$ in the beam comoving system. Therefore, the kinetic equation (2.5.15) can be written in the form

$$\frac{\partial f_0}{\partial t} = - \frac{\partial}{\partial \mathbf{I}} \mathbf{j}^{(2)}(\mathbf{I}, t), \quad (2.6.10)$$

with

$$\begin{aligned} j_\alpha^{(2)} = & -\pi N \sum_{\mathbf{m}_1, \mathbf{m}_2} m_{1\alpha} \int_{-\infty}^{\infty} d(2) \delta[\omega(1) - \omega(2)] |V(1,2)|^2 \\ & \times \left\{ \left(\mathbf{m}_1 \frac{\partial f(1)}{\partial \mathbf{I}_1} \right) f(2) - \left(\mathbf{m}_2 \frac{\partial f(2)}{\partial \mathbf{I}_2} \right) f(1) \right\}. \end{aligned} \quad (2.6.11)$$

Here, $\mathbf{m} = \{\mathbf{m}_1, n\}$, $\omega(1) = \mathbf{m}_1 \omega(1)$, and $\omega(2) = \mathbf{m}_2 \omega(2)$,

$$V(1,2) = \int_0^{2\pi} \frac{d^3 \psi_1 d^3 \psi_2}{(2\pi)^6} V(\mathbf{r}_1 - \mathbf{r}_2) e^{-im_1 \psi_1 + im_2 \psi_2} \quad (2.6.12)$$

is the Fourier-harmonic of V over phases ψ_1 and ψ_2 . Since the interaction here is conservative, the Hamiltonian $\mathcal{H}_0$ of the unperturbed motion is among invariants of Eq. (2.6.10):

$$\begin{aligned} \frac{d\langle \mathcal{H}_0 \rangle}{dt} = & \pi N \sum_{\mathbf{m}_1, \mathbf{m}_2} \int_{-\infty}^{\infty} d(1) d(2) \delta[\omega(1) - \omega(2)] |V(1,2)|^2 \\ & \times \left\{ \omega(1) f(2) \left(\mathbf{m}_1 \frac{\partial f(1)}{\partial \mathbf{I}_1} \right) - \omega(2) f(1) \left(\mathbf{m}_2 \frac{\partial f(2)}{\partial \mathbf{I}_2} \right) \right\} \equiv 0, \end{aligned} \quad (2.6.13)$$

$$\langle \mathcal{H}_0 \rangle = \int d(1) \mathcal{H}_0 f(1,t). \quad (2.6.14)$$

Let us note also that, since the collision integral was calculated assuming the domination of the beam frequency spread over the coherent tune shift, the beam is assumed to be far away from the Coulomb tune shift limit.

Previously the equation analogous to Eq. (2.6.10) had been found by S. T. Belaev²⁴ by the averaging of Bogolubov's equation over fast variables. The transformation of Eq. (2.6.10) into the equation from Ref. 24 can be traced directly if we replace in Eq. (2.5.15) the δ -function by its integral representation,

$$\delta(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}, \quad (2.6.15)$$

and execute the summation over $\mathbf{m}_1$ and $\mathbf{m}_2$. As a result, we have to calculate the values of the following type (only phase variables are specified below):

$$W_{\alpha\beta}^{1,1} = \int_{-\infty}^{\infty} d\tau \sum_{\mathbf{m}_1, \mathbf{m}_2} m_{1\alpha} m_{1\beta} \int_{-\infty}^{\infty} d(1) d(2) d(3) d(4) V(1,2) V(3,4) \\ \times \exp\{-i\mathbf{m}_1[\psi_1 - \psi_3 - \omega(1)\tau] + i\mathbf{m}_2[\psi_2 - \psi_4 - \omega(2)\tau]\}. \quad (2.6.16)$$

Using

$$m_\alpha \exp(-i\mathbf{m}\psi) = i \frac{\partial}{\partial \psi_\alpha} \exp(-i\mathbf{m}\psi) \quad (2.6.17)$$

and the summation formula

$$\sum_{\mathbf{m}=-\infty}^{\infty} e^{i\mathbf{m}\psi} = (2\pi)^3 \sum_{\mathbf{k}=-\infty}^{\infty} \delta(\psi - 2\pi\mathbf{k}), \quad (2.6.18)$$

after the integration over ψ_3 and ψ_4 we find

$$W_{\alpha\beta}^{1,1} = \int_{-\infty}^{\infty} d\tau \sum_{\mathbf{k}_1, \mathbf{k}_2} \int_{-\infty}^{\infty} d(1) d(2) \frac{\partial}{\partial \psi_{1\alpha}} V(\psi_1, \psi_2) \\ \times \frac{\partial}{\partial \psi_{1\beta}} V[\psi_1 - \omega(1)\tau - 2\pi\mathbf{k}_1, \psi_2 - \omega(2)\tau - 2\pi\mathbf{k}_2]. \quad (2.6.19)$$

In contrast with collisions of free particles, Eq. (2.6.19) describes so-called multiple collisions. The corresponding terms with nonzero $\mathbf{k}_1$ and $\mathbf{k}_2$ describe the possibility for a couple of particles to repeat the collision after the periods of unperturbed motion $\Delta t = 2\pi/\omega_\alpha$, which is specific for collisions in the focusing external fields. Sometimes this can be very important due to the resonance energy exchange between colliding particles. But, to describe real processes by Eq. (2.6.19), one has to assume that between successive collisions $\propto 2\pi/\Delta\omega_\alpha$ ($\Delta\omega_\alpha$ are partial frequency spreads) colliding particles do not suffer significant scattering from third particles of the beam. As we initially assumed a big enough frequency spread, it seems that multiple IBS cannot be described in

any straightforward way within a binary collision approximation. Neglecting in Eq. (2.6.19) the terms with $\mathbf{k}_1, \mathbf{k}_2 \neq 0$, we can write

$$j_\alpha = -\frac{N}{2} \int d\mathbf{I}_2 \left\{ W_{\alpha\beta}^{1,1} f(2) \frac{\partial f(1)}{\partial I_{1\beta}} + W_{\alpha\beta}^{1,2} f(1) \frac{\partial f(2)}{\partial I_{2\beta}} \right\}, \quad (2.6.20)$$

$$W_{\alpha\beta}^{a,b} = \int_0^{2\pi} \frac{d^3\psi_1 d^3\psi_2}{(2\pi)^6} \frac{\partial I_\alpha^a}{\partial p_\lambda^a} \frac{\partial V}{\partial r_\lambda^a} \int_{-\infty}^{\infty} d\tau \left(\frac{\partial I_\beta^b}{\partial p_\sigma^b} \frac{\partial V}{\partial r_\sigma^b} \right)_\tau. \quad (2.6.21)$$

Here $d\mathbf{I} = d^2 I_\perp dI_\parallel$, $X_\tau = X(\psi_\alpha - \omega_\alpha \tau)$.

Using Eqs. (2.6.20) and (2.6.21) one can easily prove that Eq. (2.6.10) will be converted into the Landau equation provided at least two conditions are valid.²⁴ First, the scale for the spatial variations of f must significantly exceed the impact parameters, which give the main contribution in Eq. (2.6.11). Secondly, these collisions must be fast (i.e., their collision time τ_{st} must be significantly shorter than the periods of particle oscillations). In fact, for fast collisions $\tau_{st}\omega_\alpha \ll 1$, the value $(\partial \mathbf{I} / \partial \mathbf{p})_\tau$ in Eq. (2.6.21) is a slow function of τ . Therefore, it can be factored out of the integral over τ in Eq. (2.6.21). Moreover, for potentials $V = V(\mathbf{r}_1 - \mathbf{r}_2)$ one has

$$W_{\alpha\beta}^{1,1} = -W_{\alpha\beta}^{1,2}. \quad (2.6.22)$$

As a result this yields for j_α

$$j_\alpha = \frac{\partial I_\alpha^{(1)}}{\partial p_\lambda} j_\lambda(\mathbf{p}, t), \quad (2.6.23)$$

$$j_\lambda(\mathbf{p}, t) = N \int d\mathbf{p}' w_{\lambda\sigma} \left(\frac{\partial f(2)}{\partial p'_\sigma} f(1) - \frac{\partial f(1)}{\partial p_\sigma} f(2) \right), \quad (2.6.24)$$

$$w_{\alpha\beta} = \frac{1}{2} \int d\mathbf{r}' \frac{\partial V}{\partial r_\alpha} \int_{-\infty}^{\infty} d\tau \left(\frac{\partial V}{\partial r_\beta} \right)_\tau. \quad (2.6.25)$$

For Coulomb scattering the calculation of the collision tensor $w_{\alpha\beta}$ in the center of a mass reference system of colliding particles yields

$$w_{\alpha\beta} = \frac{2e^4}{u} \int d^2b \frac{b_\alpha b_\beta}{b^4} = \frac{2\pi e^4}{u^3} (\delta_{\alpha\beta} u^2 - u_\alpha u_\beta) \int \frac{db}{b}. \quad (2.6.26)$$

Here $\mathbf{u}$ is the relative velocity of colliding particles, and b is the impact parameter

$$b_\alpha = \frac{(\delta_{\alpha\beta} u^2 - u_\alpha u_\beta)}{u^2} (\mathbf{r}_1 - \mathbf{r}_2)_\beta. \quad (2.6.27)$$

The logarithmic divergence of the integral over b , which is specific for the Coulomb interaction, is typically regularized by the integration from

$b=b_{\min}=e^2/mu^2$, which corresponds to scattering at large angles up to

$$b=b_{\max}=\min\{r_D, u/\omega_{\perp}, a_p\}, \quad (2.6.28)$$

where r_D is the beam Debye radius and a_p is the beam radius.

Let us also discuss the contribution from adiabatic collisions $\tau_{\text{st}}\omega_{\alpha} \gg 1$. In this case in Eq. (2.6.21) the slow function of τ is $(\partial V/\partial r_{\alpha})_{\tau}$, while $(\partial \mathbf{I}/\partial \mathbf{p})_{\tau}$ makes fast oscillations. Taking into account that $\tau_{\text{st}} \simeq b/u$, after the integration over τ one can estimate

$$W_{\alpha\beta} \propto \exp(-b\omega_{\alpha}/u). \quad (2.6.29)$$

This limits the region of impact parameters $b \leq b_{\max}$, which yields a significant contribution in the collision relaxation of the beam. For beams, where for any reason the region $b \leq b_{\max}$ is prohibited for particles, one may expect the suppression of the IBS. (See Chapter 8 for more details.)

Let us now present more calculations concerning the beam self-heating due to IBS. We shall analyze the formulas for the rate of the beam entropy, which was introduced in § 2.5 and gives directly the rate of the total beam phase space volume

$$\tau_r^{-1} = \frac{d \ln \Delta \Gamma}{dt} = \frac{dS}{dt} = -\frac{d}{dt} \int d\Gamma f \ln f. \quad (2.6.30)$$

Once IBS is caused by the conservative interaction, we can use Eq. (2.5.37) for (dS/dt) directly. It reads

$$\begin{aligned} \frac{dS}{dt} = \frac{\pi N}{2} \sum_{\mathbf{m}_1, \mathbf{m}_2} \int_{-\infty}^{\infty} d(1) d(2) \delta[\omega(1) - \omega(2)] |V(1,2)|^2 f(1) f(2) \\ \times \left\{ \mathbf{m}_1 \frac{\partial \ln f(1)}{\partial \mathbf{I}_1} - \mathbf{m}_2 \frac{\partial \ln f(2)}{\partial \mathbf{I}_2} \right\}^2. \end{aligned} \quad (2.6.31)$$

As might be expected beforehand, Eq. (2.6.31) yields the contents of Boltzman's theorem $(dS/dt) \geq 0$ for IBS relaxation in a storage ring. The growth of the beam's entropy in off-equilibrium states describes that of the beam phase space volume and, therefore, its heating. Once the phase space volume of the beam increases during relaxation, one may expect an increase in the instantaneous relaxation time τ_r , when the beam tends to equilibrium. Therefore, only initial relaxation time has practical importance.

Let us now find the equilibrium distribution function,

$$\frac{dS}{dt} = 0. \quad (2.6.32)$$

We shall start with the simplest case, when the ring focusing has azimuthal symmetry. Since the modulation of $V(\mathbf{r}_1 - \mathbf{r}_2)$ can take place only due to Floquet

modulation of particle positions $\mathbf{r}=\mathbf{r}(\psi,\vartheta)$, in such a ring, harmonics of V are diagonal in the azimuthal harmonic number

$$V_{\mathbf{m},\mathbf{m}'}(1,2)=\delta_{n,n'}V_{\mathbf{m}_1,\mathbf{m}'_1,n}(1,2). \quad (2.6.33)$$

As can be easily seen from Eqs. (2.6.31) and (2.6.32) the equilibrium distribution function must have the form

$$f \propto \exp\left\{-\frac{\omega_{\perp} \mathbf{I}_{\perp} + R_0 \omega'_0 \Delta p^2 / 2}{T}\right\}, \quad (2.6.34)$$

where T is the beam equilibrium temperature. But, for rings with azimuthal symmetric focusing, one always has $\omega'_0 < 0$. This means that the distribution function (2.6.34) cannot be normalized and, thus, cannot describe any physical distribution—due to the particle motion along closed orbits IBS always heats the beam in such a machine. As was mentioned before, if $\omega'_0 < 0$, scattering with momentum transfers from transversal to longitudinal motion increases the total energy of betatron oscillations of the colliding particles.

If we choose as f the distribution function that is specific for a beam,

$$f(\mathbf{I}_{\perp}, \Delta p) \propto \exp\left\{-\frac{\omega_{\perp} \mathbf{I}_{\perp}}{T} - \frac{\Delta p^2}{2\sigma_{\parallel}^2}\right\}, \quad (2.6.35)$$

then the initial relaxation rate $\Lambda_0 = (\tau_r^{-1})_0$ is determined by the expression

$$\begin{aligned} \Lambda_0 = & \frac{\pi N}{2R_0^2 \sigma_{\parallel}^2} \left(1 - \frac{R_0 \omega'_0 \sigma_{\parallel}^2}{T_{\perp}}\right) \int d(1) d(2) f(1) f(2) \left(\frac{\Delta p_1 - \Delta p_2}{\sigma_{\parallel}}\right)^2 \\ & \times \sum_n n^2 \sum_{\mathbf{m}_1, \mathbf{m}'_1} |V(1,2)|^2 \delta[\omega(1) - \omega(2)]. \end{aligned} \quad (2.6.36)$$

One can see that if $\omega'_0 < 0$, the value Λ_0 never vanishes.

Let us mention one more result following from Eq. (2.6.31). If the betatron oscillations are linear ($\partial\omega_{\alpha}/\partial I_{\beta}=0$, $\alpha, \beta = x, z$), the growth of the phase space volume is caused by the off-diagonal in $\mathbf{m}, \mathbf{m}'$ harmonics of V . Provided that we neglect the contributions of these harmonics, Eqs. (2.6.10) and (2.6.32) allow as the equilibrium distribution function any distribution of the form

$$f(\mathbf{I}_{\perp}, \Delta p) \propto \exp\left(-\frac{\omega_{\perp} \mathbf{I}_{\perp}}{T_{\perp}}\right) f_1(\Delta p), \quad (2.6.37)$$

where the momentum distribution function (f_1) is an arbitrary function of Δp . In this case IBS thermalizes the transverse degrees of freedom without changing the momentum distribution of the beam. The motivation for this effect is simple. For potentials $V(\psi_1 - \psi_2)$ the longitudinal degree of freedom in fact does not contribute to the resonant equation $\omega(1) = \omega(2)$, which is responsible for the

redistribution of partial phase space volumes. Nevertheless, we have to mention that the selection of particular harmonics by the interaction generally suggests that collisions are not fast.

Within the smoothed focusing approximation these features of the equilibrium distribution functions will be true for beams moving in the strong focusing ring. But in contrast with the previous case, in strong focusing machines the value $(d\omega_0/dp)_s$ changes sign at the transition energy of the ring γ_{tr} . Below transition ($\gamma < \gamma_{tr}$) one has $\omega'_0 > 0$ and therefore the equilibrium distribution function is determined by Eq. (2.6.34), which describes the equalization of the beam longitudinal and transversal temperatures due to IBS. Above transition ($\gamma > \gamma_{tr}$), IBS causes the self-heating of the beam.^{20,21}

Strictly speaking in the strong focusing ring due to Floquet modulation

$$\mathbf{r}_1 = \mathbf{r}_1(\psi, \vartheta), \quad \mathbf{p}_1 = \mathbf{p}_1(\psi, \vartheta) \quad (2.6.38)$$

harmonics of the potential V are always nondiagonal in n :

$$V_{\mathbf{m}, \mathbf{m}'}(1, 2) \neq \delta_{n, n'} V_{\mathbf{m}_1, \mathbf{m}'_1, n}(1, 2). \quad (2.6.39)$$

In this case the equilibrium Eq. (2.6.32) can be satisfied only by the most general distribution:

$$f \propto \exp\left(-\frac{\mathcal{H}_0}{T}\right) = \exp\left(-\frac{\omega_1 \mathbf{I}_1 + v_s \Delta p + R_0 \omega'_0 \Delta p^2 / 2}{T}\right) \quad (2.6.40)$$

(i.e., by the completely thermalized Boltzmann's distribution). As can be seen from Eq. (2.6.40) this process is accompanied by the deceleration of the beam as a whole. Since Eq. (2.6.40) cannot describe any reasonable beam distribution, one may state that in strong focusing rings IBS will always heat the beam:

$$\Lambda = (d \ln \Delta \Gamma / dt) > 0. \quad (2.6.41)$$

The reason for the self-heating of the beam due to IBS in a strong focusing lattice is obvious. If the spectrum of the potential V contains enough off-diagonal harmonics ($n \neq n'$), the particles move in the field

$$V \propto \exp(iq\psi_1 + il\vartheta). \quad (2.6.42)$$

Then the equation $\omega(1) = \omega(2)$ coincides with the sum resonance condition, which causes the energy exchange between the orbital and thermal motions of the particles.

Assuming a Gaussian distribution in the beam and the dominance of fast collisions, one can calculate Λ in more detail. (See also Ref. 22.) Using these assumptions we can rewrite Eq. (2.6.30) in the form

$$\Lambda = \frac{N}{2} \int d(1) d(2) f(1) f(2) J_\alpha J_\beta \frac{\partial V}{\partial b_\alpha} \int_{-\infty}^{\infty} d\tau \left(\frac{\partial V}{\partial b_\beta} \right)_\tau, \quad (2.6.43)$$

where

$$J_\alpha = \frac{\partial \ln f(1)}{\partial p_{1\alpha}} - \frac{\partial \ln f(2)}{\partial p_{2\alpha}}, \quad (2.6.44)$$

$\mathbf{b} = \mathbf{r}_1 - \mathbf{r}_2$. If ϵ_x and ϵ_z are respectively the horizontal and vertical emittances of the beam, for the most general Gaussian distribution, the expression

$$-\ln f = \frac{1}{2R_0} \left(\frac{a_x^2}{\epsilon_x} + \frac{a_z^2}{\epsilon_z} \right) + \frac{\Delta p^2}{2\sigma_{\parallel}^2} \quad (2.6.45)$$

is a quadratic form of $\mathbf{r}$ and $\mathbf{p}$, which one can write as

$$-\ln f = \frac{1}{2} A_{\alpha\beta} p_\alpha p_\beta + \frac{1}{2} B_{\alpha\beta} r_\alpha p_\beta + \frac{1}{2} C_{\alpha\beta} r_\alpha r_\beta, \quad (2.6.46)$$

where matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ can be calculated using Eqs. (1.6.22) and (2.6.45). We shall need only the matrix

$$p_s \hat{A} = \begin{pmatrix} \frac{R_0 f_x^2}{\epsilon_x} & -\frac{\Phi f_x}{\epsilon_x} & 0 \\ -\frac{\Phi f_x}{\epsilon_x} & \frac{1}{R_0 \epsilon_x} \left(\frac{\eta^2}{f_x^2} + \Phi^2 \right) + \left(\frac{p_s}{\sigma_{\parallel}} \right)^2 & 0 \\ 0 & 0 & \frac{R_0 f_z^2}{\epsilon_z} \end{pmatrix}. \quad (2.6.47)$$

The calculation of the integrals from Eq. (2.6.43) is simplified when it is performed in the reference system of the center of mass of the colliding particles. Selecting in Eq. (2.6.43) coordinates of the center of mass and relative motion

$$\mathbf{p}_1 = \mathbf{P} + \mathbf{q}/2, \quad \mathbf{p}_2 = \mathbf{P} - \mathbf{q}/2, \quad (2.6.48)$$

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{b}/2, \quad \mathbf{r}_2 = \mathbf{R} - \mathbf{b}/2, \quad (2.6.49)$$

one can rewrite Eq. (2.6.43) in the form

$$\Lambda = \omega_s \frac{4\pi^2 N r_p^2 L_C}{(\gamma\beta)^4 \epsilon_x \epsilon_z (\sigma_{\parallel}/p_s)} \int_0^{2\pi} \frac{d\vartheta_s}{2\pi} F, \quad (2.6.50)$$

where $\mathbf{q} = p_s \boldsymbol{\theta}$, $r_p = e^2/mc^2$,

$$F = \int \frac{d^3\theta}{4\pi} \frac{\delta_{\alpha\beta} \theta^2 - \theta_\alpha \theta_\beta}{\theta^3} \frac{\partial \Sigma}{\partial \theta_\alpha} \frac{\partial \Sigma}{\partial \theta_\beta} \exp(-\Sigma), \quad (2.6.51)$$

$$\Sigma(\boldsymbol{\theta}) = (A'_{\alpha\beta} \theta_\alpha \theta_\beta)/4, \quad (2.6.52)$$

and the matrix A' is obtained by the transformation of the matrix $p_s^2 \hat{A}$ to the center of mass system:

$$A' = \begin{pmatrix} \frac{R_0 f_x^2}{\epsilon_x} & -\gamma \frac{\Phi f_x}{\epsilon_x} & 0 \\ -\gamma \frac{\Phi f_x}{\epsilon_x} & \gamma^2 \left\{ \frac{1}{R_0 \epsilon_x} \left(\frac{\eta^2}{f_x^2} + \Phi^2 \right) + \left(\frac{p_s}{\sigma_{\parallel}} \right)^2 \right\} & 0 \\ 0 & 0 & \frac{R_0 f_z^2}{\epsilon_z} \end{pmatrix}. \quad (2.6.53)$$

After the integration by parts in Eq. (2.6.51) and then over $|\theta|$ the function F can be transformed into

$$F = \int \frac{d\Omega_{\theta}}{4\pi} \left(\text{Sp} \frac{\hat{A}'}{A'_{\alpha\beta} n_{\alpha} n_{\beta}} - 3 \right), \quad (2.6.54)$$

where $n_{\alpha} = \theta_{\alpha}/|\theta|$, $d\Omega_{\theta} = \sin\alpha d\alpha d\varphi$ is the element of the solid angle; α and φ are polar angles of the vector $\mathbf{n}$. The integration over the angles can be carried out in the coordinate system which has unit vectors directed along the main axes of the matrix A' . If z is chosen as the polar axis, F can be rewritten in the form

$$F = \int_0^{2\pi} \frac{d\varphi}{4\pi} \int_{-1}^1 d(\cos\alpha) \left[\frac{\text{Sp} \hat{A}'}{\lambda_z \cos^2\alpha + \sin^2\alpha (\lambda_1 \sin^2\varphi + \lambda_2 \cos^2\varphi)} - 3 \right], \quad (2.6.55)$$

where λ_z, λ_1 , and λ_2 are eigenvalues of the matrix A' which generally depend on the particle azimuth

$$\lambda_z = \beta_z / \epsilon_z, \quad \beta(\vartheta) = R_0 f^2(\vartheta), \quad (2.6.56)$$

$$\begin{aligned} \lambda_{1,2} = & \frac{\beta_x}{2\epsilon_x} \left\{ 1 + \gamma^2 \left[\frac{\eta^2}{\beta_x^2} + \frac{\Phi^2}{R_0 \beta_x} + \frac{\epsilon_x}{\beta_x} \left(\frac{p_s}{\sigma_{\parallel}} \right)^2 \right] \right. \\ & \left. \pm \sqrt{\left[1 - \gamma^2 \left(\left[\frac{\eta^2}{\beta_x^2} + \frac{\Phi^2}{R_0 \beta_x} - \frac{\epsilon_x}{\beta_x} \left(\frac{p_s}{\sigma_{\parallel}} \right)^2 \right] \right)^2 + 4\gamma^2 \frac{\Phi^2}{R_0 \beta_x} \right]} \right\}. \end{aligned} \quad (2.6.57)$$

The integration over φ can be done easily and yields

$$F = \int_0^1 du \left[\frac{\text{Sp} \hat{A}'}{\{ [\lambda_1 + (\lambda_z - \lambda_1)u^2] [\lambda_2 + (\lambda_z - \lambda_2)u^2]^{1/2} \} - 3} \right]. \quad (2.6.58)$$

With the arbitrary ratio between λ_z, λ_1 , and λ_2 the function $F(\lambda_z, \lambda_1, \lambda_2)$ is expressed in terms of elliptic integrals.¹⁴ But in some special cases this can yield the result in elementary functions. For instance, in the smoothed focusing approximation ($\Phi \simeq 0$) we have

$$\lambda_z = \beta_z / \varepsilon_z, \quad \lambda_1 = \beta_x / \varepsilon_x, \quad \lambda_2 = \gamma^2 \left[\frac{\eta^2}{\beta_x \varepsilon_x} + \frac{1}{\delta^2} \right], \quad \delta = \sigma_{\parallel} / p_s \quad (2.6.59)$$

and provided

$$\lambda_2 > \lambda_1, \quad \lambda_2 > \lambda_z, \quad \frac{\beta_x}{\varepsilon_x} = \frac{\beta_z}{\varepsilon_z} = \frac{\beta}{\varepsilon}, \quad (2.6.60)$$

Eq. (2.6.58) yields

$$F = \frac{\text{Sp} \hat{A}'}{\sqrt{\lambda_1(\lambda_2 - \lambda_z)}} \arcsin \sqrt{1 - \frac{\lambda_z}{\lambda_2}} - 3. \quad (2.6.61)$$

Substituting this expression into Eq. (2.6.50), we can write

$$\Lambda = \omega_s \frac{2\pi^3 N r_p^2 L_C}{\gamma^5 (v/c)^4} \int_0^{2\pi} \frac{d\vartheta_s}{2\pi} \frac{2\frac{\beta}{\varepsilon} + \gamma^2 \left(\frac{\eta^2}{\beta \varepsilon} + \frac{1}{\delta^2} \right)}{\varepsilon \sqrt{\beta \varepsilon + (\eta \delta)^2}}. \quad (2.6.62)$$

The function F has simple properties. It is a homogeneous function of λ_1 , λ_2 , and λ_z :

$$F(\alpha \lambda_1, \alpha \lambda_2, \alpha \lambda_z) = F(\lambda_1, \lambda_2, \lambda_z) \quad (2.6.63)$$

and thus depends only on two parameters. The function F is positive and, as can be seen from Eq. (2.6.58),

$$F(\lambda_1 = \lambda_2 = \lambda_z) = 0. \quad (2.6.64)$$

From Eq. (2.6.53) one can conclude that this equation will be satisfied provided that

$$\begin{aligned} 1. \quad & \Phi = 0, \\ 2. \quad & \frac{\varepsilon_x}{\beta_x} = \frac{\varepsilon_z}{\beta_z} = \delta^2 \left[\frac{\eta^2}{\beta_x^2} - \frac{1}{\gamma^2} \right], \\ 3. \quad & \gamma < \beta_x / \eta. \end{aligned} \quad (2.6.65)$$

In fact, the last condition is equivalent to the requirement $\gamma < \gamma_{tr}$. Thus, above the transition energy the beam self-heating will take place more or less independently of the azimuthal nonsymmetry of the ring focusing. Below the transition energy of the ring the influence of Floquet modulation on the IBS relaxation becomes significant after comparison of the radial and longitudinal temperatures. The inspection of matrix elements from Eq. (2.6.53) enables one to estimate the region where the smoothed focusing approximation is valid:

$$\gamma \frac{|\Phi|}{R_0 f_x} \ll \left| \frac{\gamma^2}{v_x^2} - 1 + \frac{\varepsilon_x v_x}{R_0 \delta^2} \right| \ll \left| \frac{\gamma^2}{v_x^2} - 1 \right| \frac{T_x}{T_{\parallel}}, \quad T_x \gg T_{\parallel}. \quad (2.6.66)$$

For the non-Coulomb interaction of particles, condition (2.6.66) can be rewritten in an equivalent form, but referring to off-diagonal elements of V .

The effect of the beam self-heating due to IBS sets the lower limit for the sum of decrements, which is necessary to cool the beam to the equilibrium state. This becomes especially important in cases when achievable cooling decrements are limited by complementary reasons.

We have also to note that the amplification of the focusing strength generally limits the region of impact parameters giving the main contribution to $\Lambda(b \leq u/\omega_\alpha)$. For the Coulomb interaction this, for example, can cause a decrease of the Coulomb logarithm. One may also expect that the enlargement of the collision time ($\tau_{st}\omega_\alpha/\vartheta_0 \gg 1$, ϑ_0 is the lattice period) will smooth the betatron functions and therefore decrease the contribution of Floquet modulation to Λ .

Since for off-equilibrium states one has $(dS/dt) \geq 0$, IBS always increases the phase space volume of the beam and, thus, gives some self-heating of the beam. The possibility for unlimited self-heating due to IBS is determined by the focusing properties of the lattice. Results of this section indicate that in storage rings two mechanisms can be responsible for these effects. The first is associated with the motion of particles along closed orbits. Above the transition energy of the ring, due to the coupling between the radial and longitudinal motions, IBS yields the simultaneous growth of betatron emittances and beam momentum spread. The second mechanism is caused by the Floquet modulation of betatron functions of the ring. In this case the phase space volume grows due to sum resonances excited by the interaction. In other words, the average kinetic energies of particles for various degrees of freedom (i.e., the beam partial temperatures) are modulated along the ring and therefore cannot reach the constant value, which is necessary to reach the equilibrium state. This effect is generally independent of whether or not the orbit is closed and can cause the growth of the phase space volumes of the beam in both circular and linear machines.²³

We did the calculations for coasting beams. The generalization for bunched beams can be done very easily. Since typically the frequencies of synchrotron oscillation are small, one can expect that collisions always will be fast for this degree of freedom.

It has been mentioned in §2.5 that, in storage rings, IBS can be associated not only with the Coulomb interaction but also with the scattering of particles by the fields, which are induced in the elements of the vacuum chamber surrounding the beam. Provided coherent tune shifts from this interaction are smaller than the beam frequency spreads and the dissipation is not so important, the relaxation of the beam due to this interaction will be described by Eq. (2.5.15) with the current $j^{(2)}$. It is obvious that the relaxation due to such IBS will have all the specific properties discussed above. But in contrast with the Coulomb case, the azimuthal modulation of the interaction here can be caused by azimuthal space that is occupied by this element. This can enhance the contribution from resonant effects even in the ring with smoothed focusing.

CHAPTER 3

Theory of Linear Coherent Oscillations

The increase in beam intensity or the decrease in its frequency spread will violate condition (2.5.1). Kinetic effects in these (we can call them “cold” or intense) beams are generally affected by collective degrees of freedom of the beam. The calculation of corresponding collision integrals and related phenomena will be discussed in the seventh and subsequent chapters. Discussion of these items will be relieved if we clear up before the main properties of the collective (or coherent) oscillations of the stored beam, particularly since they are of special interest themselves.

In this chapter we describe the theory of linear coherent oscillations of the beam in a storage ring. The main focus of this theory is the calculation of the stability criteria as well as of the spectra of the coherent oscillations of a beam. Besides, these results form the useful background for the construction of collision integrals of cold beams.

The theory of coherent oscillations was developed by many people. Initially these studies focused on the inspection of collective stability of the coasting beams and on attempts to use the results of these studies for predicting the collective behavior of the bunched beams. The straightforward study of the stability of the synchrotron coherent oscillations, which are specific for the bunched beams, was started in 1967 by the pioneering paper of A. N. Lebedev.²⁵

A universal theory of coherent oscillations of the beams in a storage ring without beam cooling was developed in 1969 by Ya. S. Derbenev and N. S. Dikansky.²⁶⁻²⁸ It was based on the application of the averaging method to solution of Vlasov's equation resulting in some general integral equation describing both the beam collective spectra and the modes of the arbitrary geometry in the beam phase space. A similar approach was later developed by F. Sacherer.²⁹ In Ref. 30-35 this method was more developed and applied to numerous particular problems. Somewhat later, to support measurements carried out at the Institute of Nuclear Physics at Novosibirsk to study the method of electron cooling,³⁶ it was found out³⁷ that strong beam cooling can essentially affect the spectrum of coherent oscillations, especially in coasting beams.

3.1. COHERENT OSCILLATIONS OF AN UNCOOLED BEAM

Coherent oscillations of an uncooled beam that interacts with the surrounding electrodes (an external system) are described by the Vlasov's equation

$$\frac{\partial f}{\partial t} + [\mathcal{H}, f] = 0 \quad (3.1.1)$$

and by the field equations

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla A_0, \quad \mathbf{H} = \text{rot} \mathbf{A}, \\ \text{div} \mathbf{A} + \frac{1}{c} \frac{\partial A_0}{\partial t} &= 0, \\ \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_0(\mathbf{r}, t) &= -4\pi N e \int d^3 p f, \\ \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A}(\mathbf{r}, t) &= -4\pi N e \int d^3 p \frac{\mathbf{v}}{c} f. \end{aligned} \quad (3.1.2)$$

Here, $\mathbf{A}$ and A_0 are respectively vector and scalar potentials of the fields, which are induced by the beam, and $\mathcal{H}$ is the Hamiltonian, describing the particle motion in focusing and induced fields. As in previous chapters, we write $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_0 - L, \quad L = \frac{e}{c} \mathbf{v} \mathbf{A} - e A_0.$$

By Eq. (3.1.1) the distribution function f is conserved during its evolution. Hence, this equation has an exact solution:

$$f(\mathbf{r}, \mathbf{p}, t) = f^{(0)}[\mathbf{r}_0(\mathbf{r}, \mathbf{p}, t), \mathbf{p}_0(\mathbf{r}, \mathbf{p}, t)],$$

which expresses instantaneous distribution $f(\mathbf{r}, \mathbf{p}, t)$ in terms of initial conditions and of the particle laws of the motion:

$$\mathbf{r}_0 = \mathbf{r}_0(\mathbf{r}, \mathbf{p}, t), \quad \mathbf{p}_0 = \mathbf{p}_0(\mathbf{r}, \mathbf{p}, t).$$

Unfortunately, except for maybe very special cases these trajectories cannot be found analytically for particles moving in the fields induced by the beam itself. Therefore to do particular predictions with Eqs. (3.1.1) and (3.1.2) we need some perturbative approach. As stated in Chapter 1, due to periodicity of the particle motion described by the Hamiltonian $\mathcal{H}$, this can be based on the averaging method.

Without perturbations the particles execute betatron and maybe synchrotron oscillations around the given closed orbit. The use of action-phase variables of these oscillations generally simplifies Eq. (3.1.1) and, thus, the description of perturbed motion. In these variables the Hamiltonian of unperturbed oscillations $\mathcal{H}_0$ depends only on action variables $\mathcal{H}_0(\mathbf{I})$:

$$\dot{I}_\alpha = -\frac{\partial \mathcal{H}_0}{\partial \psi_\alpha} \equiv 0, \quad \dot{\psi}_\alpha = \frac{\partial \mathcal{H}_0}{\partial I_\alpha} = \omega_\alpha.$$

The same property has the distribution function describing the beam without coherent oscillations $f \rightarrow f_0(I)$. Below we shall call such a state of the beam a stationary one.

The excitation of collective motion in the beam is described by a nonstationary addition to f_0 :

$$f(I, \psi, t) = f_0(I) + \tilde{f}(I, \psi, t). \quad (3.1.3)$$

In this case $\mathbf{A}$ and $\mathbf{A}_0$ acquire corresponding components

$$(\mathbf{A}, \mathbf{A}_0) = (\mathbf{A}, \mathbf{A}_0)_0 + (\tilde{\mathbf{A}}, \tilde{\mathbf{A}}_0). \quad (3.1.4)$$

Provided that the characteristics of the stationary state $f_0, (\mathbf{A}, \mathbf{A}_0)_0$ are known, Eqs. (3.1.1) and (3.1.2) can be more or less easily solved if these equations can be linearized in the deviations from the stationary state. This fact separates two types of problems. The first is the development of initial excitations of the beam. The linearization of Vlasov's equation (3.1.1) in this case requires the smallness of initial excitation as well as the deviation $\tilde{f}$ during the whole time interval of interest.

The second type of problems can be associated with studying the stability of the given distribution function f_0 , which is assumed to be stationary, relative to the excitation in the beam of small collective oscillations. To this end, the infinitesimal initial deviations from f_0 are assumed beforehand, when Vlasov's equation can be linearized by the definition. The main items of the theory in this case become the calculation of eigenfrequencies and eigenmodes of the beam as well as getting the stability criteria of those modes. Provided the eigenmodes of the beam are found, they can be used to calculate the Green function of the solution, solving the time-domain problem:

$$f(\Gamma, t) = \int d\Gamma' R(\Gamma, \Gamma', t) f^{(0)}(\Gamma').$$

Since the potentials $(\mathbf{A}, \mathbf{A}_0)_0$ describe the stationary part of induced fields, they are periodical functions of phases ψ and equilibrium azimuth $\vartheta_s = \omega_s t$. In §1.5 we saw that in the case when the working point of the storage ring is far from the machine resonances, these perturbations result only in some frequency shift of unperturbed oscillations. Assuming that these frequency shifts are already eliminated in $\mathcal{H}_0$, we write

$$\mathcal{H} = \mathcal{H}_0 - \tilde{L}. \quad (3.1.5)$$

Making use of Eq. (2.5.9) we can express $\tilde{L}$ in terms of the Green function of the electrodynamic problem

$$\tilde{L} = Ne^2 \int d\Gamma_2 \int_0^\infty d\tau G(1,2,\tau) \tilde{f}(2,t-\tau). \quad (3.1.6)$$

The linearized equation for $\tilde{f}$ in action-phase variables has the form

$$\frac{\partial \tilde{f}}{\partial t} + \omega \frac{\partial \tilde{f}}{\partial \psi} + \frac{\partial \tilde{L}}{\partial \psi} \frac{\partial f_0}{\partial \mathbf{I}} = 0. \quad (3.1.7)$$

While solving this equation we must take into account that besides a nonstationary dependence of $\tilde{L}$ on time it is also a periodical function of the azimuth ϑ and of the oscillation phases. Accordingly we may expect the eigensolutions of Eq. (3.1.7) in the form

$$\tilde{f} = e^{-i\omega t} \sum_{\mathbf{m},n} f_{\mathbf{m},n}^\omega(\mathbf{I}) e^{im\psi + in\omega_s t}.$$

Their substitution in Eq. (3.1.7) leads to the following system of equations for amplitudes $f_{\mathbf{m},n}^\omega$ and for the eigenfrequencies:

$$\begin{aligned} (\omega - \omega_{mn}) f_{\mathbf{m},n}^\omega &= Ne^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \sum_{n_1, n_2} \sum_{\mathbf{m}_2} \int d\Gamma_2 f_{\mathbf{m}_2, n-n_1+n_2}^\omega \\ &\times \{ G_{n_1, n_2} [1, 2, \omega + (n_1 - n)\omega_s] \exp[in_1 \Delta\vartheta_1 - in_2 \Delta\vartheta_2] \}_{\mathbf{m}_1, \mathbf{m}_2}, \\ \omega_{mn} &= \mathbf{m}\omega + n\omega_s, \quad \Delta\vartheta(\mathbf{I}, \psi) = \vartheta - \omega_s t, \\ X_{\mathbf{m},n} &= \int_0^{2\pi} \frac{d^3\psi}{(2\pi)^3} \int_0^{2\pi} \frac{d\vartheta}{2\pi} X(\psi, \vartheta) e^{-in\vartheta - im\psi}. \end{aligned} \quad (3.1.8)$$

If the distortions of the particle trajectories due to the interaction with induced fields on time intervals of the order of oscillation periods ($\simeq 1/\omega_s$) are small, one may expect normal frequencies of Eq. (3.1.8) in the close vicinity of unperturbed ones:

$$\omega = \omega_{mn} + \Delta\omega_{mn}, \quad |\Delta\omega_{mn}| \ll \min(\omega_s, \nu_\alpha \omega_s).$$

On the other hand, the off-diagonal in m and n terms in the right-hand side of Eq. (3.1.8) effectively contain frequency denominators:

$$\omega - m_2\omega_2 - (n - n_1 + n_2)\omega_s = \Delta\omega_{mn} + (m_1 - m_2)\omega + (n_1 - n_2)\omega_s.$$

If, therefore, coherent frequency shifts ($\Delta\omega_{mn}$) are small compared to the distance in the spectrum of unperturbed oscillations $|\Delta\omega_{mn}| \ll |l\omega + q\omega_s|$, where l and q are arbitrary integers, the contribution of the off-diagonal elements in the right-hand side of Eqs. (3.1.8) can be neglected. In this case, the relative error in the calculation of the eigenfrequencies ω can be estimated as

$$\frac{|\Delta\omega_{mn}|}{|l\omega + q\omega_s|} \ll 1, \quad \Delta\omega_{mn} = \omega - m\omega - n\omega_s.$$

With this accuracy Eq. (3.1.8) can be substituted for a system of uncoupled equations

$$\Delta\omega_{mn}f_{\mathbf{m},n}^\omega = Ne^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \int d\Gamma_2 K_n(1,2,\omega) f_{\mathbf{m},n}^\omega, \quad (3.1.9)$$

where

$$K_n = \sum_{n_1} \{ G_{n_1, n_1} [1, 2, \omega + (n_1 - n)\omega_s] e^{in_1(\Delta\vartheta_1 - \Delta\vartheta_2)} \}_{\mathbf{m}, \mathbf{m}}. \quad (3.1.10)$$

The eigenfrequencies ω enter Eq. (3.1.9) in combinations $\omega - n\omega_s$. Therefore, the spectra of all these equations differ only by a shift by $n\omega_s$ and in this sense coincide. Hence, the eigenfrequencies ω can be found by solving any of these equations. Taking in Eq. (3.1.9) $n = 0$, we can finally write

$$\Delta\omega_{\mathbf{m}}f_{\mathbf{m}}^\omega = Ne^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \int d\Gamma_2 K_0(1,2,\omega) f_{\mathbf{m}}^\omega. \quad (3.1.11)$$

This translation invariance of the spectra of coherent oscillation is similar to that appearing in calculations of betatron tunes in periodical lattices. The associated modulation of solutions with the revolution frequency ω_0 indicates the local character and the periodicity of the perturbation.

It is easy to verify that Eq. (3.1.11) corresponds to the calculation of $f_{\mathbf{m}}^\omega$ in the first approximation of the averaging method. In fact, in agreement with the general idea of this method the separation in Eq. (3.1.7) of slow dependencies of functions $f_{\mathbf{m}}$ and $L_{\mathbf{m}}$ on time

$$\tilde{f} = (\overline{\tilde{f}}) + (\tilde{\tilde{f}}), \quad \tilde{L} = (\overline{\tilde{L}}) + (\tilde{\tilde{L}}),$$

where

$$\overline{X}(t) = \int_0^{2\pi} \frac{d\vartheta_s}{2\pi} X(\vartheta_s, t), \quad \tilde{X} = X - \overline{X},$$

after the substitution $(\tilde{\tilde{f}}_{\mathbf{m}})(t) = f_{\mathbf{m}}^\omega e^{-i\omega t}$ results in the integral equation

$$\Delta\omega_{\mathbf{m}}f_{\mathbf{m}}^\omega = \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) (\overline{\tilde{L}}_{\mathbf{m}})(\mathbf{I}, \omega) = Ne^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \int d\Gamma_2 K_0(1,2,\omega) f_{\mathbf{m}}^\omega(2),$$

coinciding with Eq. (3.1.11).

For a coasting beam Eq. (3.1.11) can be simplified even more. In this case one has $f_0 = f_0(\mathbf{I}_\perp, \Delta p)$, and $\psi_{\parallel} = \omega'_0 \Delta p t + \psi_0$, which after the integration over $\psi_{\parallel}$ split Eq. (3.1.9) into uncoupled equations for azimuthal harmonics

$f_{mn}e^{-in\psi_1}$. Redefining the frequency $\omega \rightarrow \omega - n\omega'_0\Delta p$, we can rewrite Eq. (3.1.11) in the form

$$[\omega - \mathbf{m}_1 \omega_1 - n\omega'_0(\Delta p)]f_{\mathbf{m},n} = Ne^2 \left(\mathbf{m}_1 \frac{\partial f_0}{\partial \mathbf{I}_1} + \frac{n}{R_0} \frac{\partial f_0}{\partial \Delta p} \right) \times \int d\Gamma_1 d\Delta p \{G_{n,n}(1,2,\omega)\}_{\mathbf{m}_1, \mathbf{m}_1} f_{\mathbf{m},n}. \quad (3.1.12)$$

Integers $\mathbf{m}$ contributing into Eq. (3.1.11) determine the multipole numbers of the coherent modes. Generally these describe the phase symmetry of the oscillation in the phase space. Since the kernel of Eq. (3.1.11) depends on multipole numbers $\mathbf{m}$ as on parameters, this equation enables one to study the stability of coherent oscillations of arbitrary multipolarity. The various collective oscillations in the beam can be classified by three multipole numbers m_α and the number of the solution of Eq. (3.1.11). For instance, the combination $\Sigma_\alpha |m_\alpha| = 1$ corresponds to dipole modes, the combination $\Sigma_\alpha |m_\alpha| = 2$ corresponds to quadrupole ones, and so on.

To calculate $f_{\mathbf{m}}(t)$ one should retain in Eq. (3.1.8) the term describing the initial conditions.

$$\left(\frac{\partial f_{\mathbf{m}}}{\partial t} \right)_\omega = \int_0^\infty dt e^{i\omega t} \frac{\partial f_{\mathbf{m}}}{\partial t} = -f_{\mathbf{m}}(I, t=0) - i\omega f_{\mathbf{m}}(I, \omega).$$

To make sense the integral over t must converge even for growing solutions. This implies at least $\text{Im}\omega > 0$.^{*} In this case Eqs. (3.1.8) and (3.1.11) become inhomogeneous integral equations:

$$\Delta\omega_{\mathbf{m}} f_{\mathbf{m}}^\omega = i f_{\mathbf{m}}(I, 0) + Ne^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \int d\Gamma_2 K_0(1,2,\omega) f_{\mathbf{m}}^\omega. \quad (3.1.13)$$

If solutions of this equation are known, the amplitudes $f_{\mathbf{m}}(I, t)$ can be calculated using the transformation

$$f_{\mathbf{m}}(I, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f_{\mathbf{m}}(I, \omega), \quad \text{Im}\omega > 0.$$

Since we discuss solutions calculated in the first-order approximation, these equations will give reliable results, provided $|f_{\mathbf{m}}| \ll f_0$, or taking into account that

^{*}If $f_{\mathbf{m}} \propto \exp(\delta t)$, $\delta > 0$, obviously $\text{Im}\omega > \delta$.

$$|f_m| \propto \left| \frac{L_m}{\Delta\omega_m} \left(m \frac{\partial f_0}{\partial I} \right) \right| \propto \left| \Delta I_{m_1} \frac{\partial f_0}{\partial I} \right|,$$

$$\left| \Delta I_{m_1} \frac{\partial \ln f_0}{\partial I} \right| \ll 1.$$

For distribution functions close to the statistical equilibrium one, we may estimate

$$\left| \frac{\partial \ln f_0}{\partial I_\alpha} \right| \propto \frac{1}{p_s \epsilon_\alpha}.$$

The last inequality can be rewritten as the requirement of smallness for the amplitudes of coherent oscillations compared to the beam size $|\Delta I_m| \ll p_s \epsilon_\alpha$. As was mentioned in the beginning of the section, this means that Eq. (3.1.11) describes the development of oscillations in time domain only for stable solutions, when $f(t \rightarrow \infty) \rightarrow f_0$. For unstable solutions Eq. (3.1.11) apart from the calculation of increments and threshold parameters of the beam, describes coherent oscillations until $|\tilde{f}_m(t)| \ll f_0$ (i.e., on the limited time intervals). Besides, the formal similarity of equations Eqs. (3.1.7) and (2.4.28) makes it possible to apply the developed theory to the study the dynamics of the beam coherent fluctuations δF and to calculations of corresponding correlators as well as collision integrals.

Among other problems below we shall discuss the interaction of beams with low Q -value environments, when induced fields may decay faster than the revolution period. In this case the sum over n in K_0 [see Eq. (3.1.10)] weakly depends on the exact value of frequency ω , which, thus, can be replaced in the right-hand side of Eq. (3.1.11) by the unperturbed value $m\omega$:

$$G(1,2,\omega+n\omega_s) \rightarrow G(1,2,m\omega+n\omega_s).$$

The beam interaction with the surrounding electrodes is described by Green functions, admitting Taylor-series expansion of fields in powers of transverse deviations of particles from the equilibrium orbit $\mathbf{r}_\perp$. In this case, calculation of Fourier harmonics of G_{mn} in betatron phases can be simplified. In fact, for any function of the form

$$X(x) = \sum_{l=0}^{\infty} \frac{x^l}{l!} \frac{\partial^l X}{\partial x^l}, \quad x = \sqrt{J_x \beta_x} \cos[\psi_x + \chi_x(\vartheta)]$$

one can write

$$X_m = \int_0^{2\pi} \frac{d\psi_x}{2\pi} e^{-im_x\psi_x} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\partial^l X}{\partial x^l} \left(\frac{J_x \beta_x}{2} \right)^{l/2} \\ \times \sum_{k=0}^l \binom{l}{k} \exp\{i(l-2k)[\psi_x + \chi_x(\vartheta)]\},$$

where $\binom{l}{k}$ are binomial coefficients. Taking into account that

$$\int_0^{2\pi} \frac{d\psi}{2\pi} \exp(i[l-2k-m]\psi) = \delta_{l,m+2k},$$

X_m can be rewritten in the form

$$X_m = \left(\frac{\sqrt{J_x \beta_x}}{2} e^{i\chi_x(\vartheta)} \right)^{m_x} \left\{ \frac{1}{m_x!} \frac{\partial^{m_x} X}{\partial x^{m_x}} \right. \\ \left. + \sum_{k=0}^{\infty} \binom{m_x+2k}{k} \frac{1}{(m_x+2k)!} \left(\frac{\sqrt{J_x \beta_x}}{2} e^{i\chi_x} \right)^{2k} \frac{\partial^{m_x+2k} X}{\partial x^{m_x+2k}} \right\}.$$

If the dependence of X on transverse coordinates is scaled with parameter l_1^* and $a = \sqrt{J\beta} \ll l_1$, then with an accuracy of up to (a^2/l_1^2) one has

$$X_m = \left(\frac{\sqrt{J_x \beta_x}}{2} e^{i\chi_x(\vartheta)} \right)^{m_x} \frac{1}{m_x!} \frac{\partial^{m_x} X}{\partial x^{m_x}}. \quad (3.1.14)$$

For a coasting beam interacting with electrodes this yields

$$G_{mn}(1, 2, \omega_{mn}) \simeq (I_x I'_x)^{m_x/2} (I_z I'_z)^{m_z/2} g_{mn}(\omega_{mn}), \quad (3.1.15)$$

where g_{mn} is expressed through the derivatives of G_{mn} on the closed orbit. Integral equation (3.1.12) with the kernel in Eq. (3.1.15) can be solved directly. After the substitution in Eq. (3.1.12) of the functions[†]

$$f_{mn} = C_{mn} \frac{I_x^{m_x/2} I_z^{m_z/2}}{\omega - \omega_{mn}} \left(\mathbf{m}_1 \frac{\partial f_0}{\partial \mathbf{I}_1} \right), \quad m_1^2 \neq 0, \quad (3.1.16)$$

where C_{mn} is the normalization constant, it reads

$$C_{mn} \left\{ 1 - \frac{\Omega_{mn}}{\mathcal{A}_m} \int d^2 I_1 d\Delta p \frac{I_x^{|m_x|} I_z^{|m_z|} \left(\mathbf{m}_1 \frac{\partial f_0}{\partial \mathbf{I}_1} \right)}{\omega - \omega_{mn}(\mathbf{I}_1, \Delta p)} \right\} = 0.$$

For nontrivial solutions $C_{mn} \neq 0$ this yields dispersion equation of the problem defining the eigenfrequencies of modes:

*Typically l_1 is the vacuum pipe radius.

†The second term in round brackets in Eq. (3.1.12) is obviously of the second order in $(a/l_1)^2$.

$$\epsilon_{mn}(\omega) = 1 - \frac{\Omega_{mn}}{\mathcal{A}_m} \int d^2 I_{\perp} d\Delta p \frac{I_z^{|m_z|} \left(\mathbf{m}_{\perp} \frac{\partial f_0}{\partial I_{\perp}} \right)}{\omega - \omega_{mn}(I_{\perp}, \Delta p) I_x^{|m_x|}} = 0 \quad (3.1.17)$$

where we introduced

$$\mathcal{A}_m = \int d^2 I_{\perp} d\Delta p I_x^{|m_x|} I_z^{|m_z|} \left(\mathbf{m}_{\perp} \frac{\partial f_0}{\partial I_{\perp}} \right) \quad (3.1.18)$$

and

$$\Omega_{mn} = Ne^2 \mathcal{A}_m g_{mn}(\omega_{mn}). \quad (3.1.19)$$

The physical sense of Ω_{mn} can be clarified if we solve Eq. (3.1.17) for the beam without frequency spread (monochromatic beam). When ω_{mn} does not depend on both $I_{\perp}$ and Δp , the frequency denominator in Eq. (3.1.17) can be factored out of the integral sign, while the remaining integral coincides with $\mathcal{A}_m$. This shows that the introduced parameter Ω_{mn} gives the deviation of the frequency of coherent mode from unperturbed value ω_{mn} (the so-called coherent frequency shift):

$$\omega = \mathbf{m}_{\perp} \omega_{\perp} + n\omega_0 + \Omega_{mn}.$$

Eqs. (3.1.18) and (3.1.19) indicate that for dipole coherent oscillations ($m_{\perp}^2 = 1$) the value of Ω_{mn} does not depend on the transverse sizes of the beam:

$$\Omega_{mn} = Ne^2 g_{mn}(\omega_{mn}), \quad \mathcal{A}_{\pm 1} = -1,$$

while for multipole modes ($m_{\perp}^2 \neq 1$) it decays exponentially

$$\Omega_{mn} \propto (a/l_{\perp})^{2|m|-2}.$$

Such a dependence of Ω_{mn} on the beam size is essentially due to the interaction of the particles with image charges and currents distributed outside the beam. There can be, however, another broad class of problems related to the space charge phenomena, when induced fields can essentially vary on the transverse size of the beam. In these cases no one term from Taylor expansion can be selected as the leading one and, thus, the decay of Ω_{mn} , when m increases, can occur essentially slower than exponential.

In a monochromatic beam the condition $\text{Im}\Omega_{mn} > 0$ corresponds to unstable oscillations. Since $g(\omega)$ is the Fourier image of the real function, the imaginary part of $g(\omega)$ on the real axis is an odd function ω :

$$g^*(\omega) = g(-\omega).$$

Besides, for passive systems one has $\omega \text{Im}g(\omega) > 0$. Combining these with Eq. (3.1.19), we can write the stability condition for coherent oscillations of a monochromatic, coasting beam in the form

$$(m_{\perp} \omega_{\perp} + n\omega_s) \mathcal{A}_m < 0. \quad (3.1.20)$$

For one-dimensional betatron modes (say, $m_x=0$) this criterion gets the simplest form²⁷:

$$m_z(m_z v_z + n) > 0. \quad (3.1.21)$$

The second term in round brackets in Eqs. (3.1.20) and (3.1.21) appears due to excitation of fields by longitudinal motion of particles. It changes the sign at $n = -m_z v_z$. Therefore, the use of passive systems cannot provide, for example, a simultaneous damping of all the modes of the coasting beam.²⁷

The physical sense of the function $\epsilon_{mn}(\omega)$ introduced in Eq. (3.1.17) can be clarified by studying the development in the coasting beam of initial excitation, when harmonics f_m obey Eq. (3.1.13). For a coasting beam it has the solution

$$f_{mn} = \frac{if_{mn}^{(0)}}{\omega - \omega_{mn}} + C_{mn} \frac{I_x^{m_x/2} I_z^{m_z/2}}{\omega - \omega_{mn}} \left(m_{\perp} \frac{\partial f_0}{\partial I_{\perp}} \right),$$

which after its substitution in Eq. (3.1.13) gives

$$C_{mn} = \frac{iNe^2 g_{mn}}{\epsilon_{mn}(\omega)} \int d\Gamma d\Delta p \frac{I_x^{m_x/2} I_z^{m_z/2} f_{mn}^{(0)}}{\omega - \omega_{mn}}.$$

Using this solution we can write

$$\begin{aligned} \overline{L_{mn}(\omega)} &= Ne^2 \int d\Gamma G_{mn} f_{mn} \\ &= Ne^2 \int d\Gamma G_{mn} \left[\frac{if_{mn}^{(0)}}{\omega - \omega_{mn}} + C_{mn} \frac{I_x^{m_x/2} I_z^{m_z/2}}{\omega - \omega_{mn}} \left(m_{\perp} \frac{\partial f_0}{\partial I_{\perp}} \right) \right] \\ &= \overline{L_{mn}^{(0)}(\omega)} + \overline{L_{mn}^{(0)}(\omega)} \frac{1 - \epsilon_{mn}}{\epsilon_{mn}} = \frac{\overline{L_{mn}^{(0)}(\omega)}}{\epsilon_{mn}(\omega)}, \end{aligned}$$

where

$$\overline{L_{mn}^{(0)}(\omega)} = iNe^2 \int d\Gamma G_{mn} \frac{f_{mn}^{(0)}}{\omega - \omega_{mn}}$$

are harmonics of the Lagrangian calculated neglecting the interaction between beam particles. The comparison of these two equations shows that the value ϵ_{mn} describes the compensation (or amplification) of induced fields due to collective reaction of the beam and in this sense coincides with its dielectric constant.

As pointed out by C. Pellegrini³⁸ and M. Sands,³⁹ the stability of bunched beams can be strongly affected by the lattice chromaticity. Therefore, while calculating harmonics G_{mn} over the phases of synchrotron oscillation one should

take into account the dependence of betatron frequencies on the particle momentum $\omega_{\perp} = \omega_{\perp}(p)$. For instance, for vertical oscillations this yields

$$\frac{d\psi_z}{dt} = \omega_z(p_s) + \left(\frac{\partial \omega_z}{\partial p} \right)_s \Delta p = \omega_{sz} + \frac{d\varphi}{dt} \left(\frac{\partial \omega_z}{\partial \omega_0} \right)_s,$$

or

$$\psi_z = \psi_z^s + \frac{d\omega_z}{d\omega_0} \varphi,$$

and, hence, the phase harmonics must be calculated from the expressions

$$\exp \left[i \left(n + \mathbf{m}_{\perp} \frac{d\omega_{\perp}}{d\omega_0} \right) \varphi \right].$$

For small synchrotron oscillations $\varphi = \varphi_c \cos \psi_c$ this yields

$$\int_0^{2\pi} \frac{d\psi_c}{2\pi} \exp(-im_c \psi_c + in_1 \varphi) = J_{m_c}(n_1 \varphi_c),$$

$$n_1 = n + \mathbf{m}_{\perp} \frac{d\omega_{\perp}}{d\omega_0},$$

where $J_m(x)$ is the Bessel function.¹⁴ Together with Eq. (3.1.15) this gives for G_{mn} the expression

$$G_{mn} \simeq (I_x I'_x)^{m_x/2} (I_z I'_z)^{m_z/2} g_{mn}(\omega_{mn}) J_{m_c}(n_1 \varphi_c) J_{m_c}(n_1 \varphi'_c).$$

It indicates that the eigensolutions of Eq. (3.1.11) might be expected in the form

$$f_m = \frac{I_x^{m_x/2} I_z^{m_z/2}}{\omega - \mathbf{m}\omega} \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \chi_m(\varphi_c). \quad (3.1.22)$$

Assuming now for the stationary distribution function

$$f_0(I_{\perp}, I_c) = F_0(I_{\perp}) \rho(\varphi_c),$$

where $\rho(\varphi_c)$ is the distribution function over the amplitudes of synchrotron oscillations, we find from Eq. (3.1.11) that χ_m obeys the integral equation

$$\begin{aligned} \chi_m = \chi^0(\varphi_c) & \sum_{n=-\infty}^{\infty} \frac{\Omega_{mn}}{\mathcal{A}_m} \int d\Gamma_{\perp} \frac{I_x^{|m_x|} I_z^{|m_z|} \left(\mathbf{m}_{\perp} \frac{\partial F_0}{\partial \mathbf{I}_{\perp}} \right)}{\omega - \mathbf{m}\omega(\mathbf{I}_{\perp}, \varphi_c)} \\ & \times J_{m_c}(n_1 \varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c), \\ \chi^0(\varphi_c) & = \begin{cases} \rho(\varphi_c), & m_1^2 \neq 0, \\ \partial \rho(\varphi_c) / \partial \varphi_c, & m_1^2 = 0. \end{cases} \end{aligned} \quad (3.1.23)$$

Since usually the kernel of this equation cannot be factorized in φ and φ' , its solution is embarrassed very much by the nonlinearity of synchrotron oscillations and associated frequency spread. If those can be neglected, Eq. (3.1.23) splits into an equation defining eigenmodes,

$$\Omega_m \chi_m(\varphi_c) = \chi^0(\varphi_c) \sum_{n=-\infty}^{\infty} \Omega_{mn} J_{m_c}(n_1 \varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c), \quad (3.1.24)$$

and a dispersion equation giving eigenfrequencies as well as stability criteria of oscillations,

$$\epsilon_m(\omega) = 1 - \frac{\Omega_m}{\mathcal{A}_m} \int d\Gamma_{\perp} \frac{I_x^{|m_x|} I_z^{|m_z|} \left(\mathbf{m}_{\perp} \frac{\partial F}{\partial \mathbf{I}_{\perp}} \right)}{\omega - \mathbf{m}_{\perp} \omega_{\perp}(\mathbf{I}_{\perp})} = 0. \quad (3.1.25)$$

The values $\mathcal{A}_m$ and Ω_{mn} in these expressions are given by Eqs. (3.1.18) and (3.1.19) respectively.

To conclude this section let us discuss the possible classification of collective effects in the bunched stored beams, which uses as the base the ratio of the decay time of induced fields to the revolution period of particles in the ring. We note that generally the summation over harmonic numbers n in Eq. (3.1.24) can be done using the summation formula

$$\sum_{n=-\infty}^{\infty} b_n = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dn b(n) \exp(2\pi i l n). \quad (3.1.26)$$

The substitution of Eq. (3.1.26) in Eq. (3.1.24) yields terms of two kinds. The first contains only the summand with $l=0$. The corresponding contribution in the kernel of Eq. (3.1.24)

$$\int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1 \varphi_c) J_{m_c}(n_1 \varphi'_c) \quad (3.1.27)$$

coincides with the kernel, calculated for the case when the bunch passes the system of electrodes only once. It is reasonable to call collective effects caused by this contribution single-turn (or single-pass) ones. For these effects, weak dependence of both decrements and coherent frequency shifts on the frequencies of particle oscillations is specific.

Let us now estimate the contributions into kernels of Eq. (3.1.24) from the terms with $l \neq 0$:

$$K_l = \sum_{l \neq 0} \int_{-\infty}^{\infty} dn \exp(2\pi i l n) \Omega_{m,n} J_{m_c}(n_1 \varphi_c) J_{m_c}(n_1 \varphi'_c). \quad (3.1.28)$$

Below we denote as ϑ_0 the longitudinal size of the electrodes interacting with the beam, and assume that the dominant contribution into solutions gives the amplitudes of synchrotron oscillations from the region

$$2\vartheta_0 + \varphi_c + \varphi'_c < 2\pi.$$

The dependence of $\Omega_{m,n}$ on n has a two-fold origin. It is caused, first, by the dependence of the Green function on $\omega_{m,n}$, and, second, by the localization of induced fields within the region $|\vartheta| \leq 2\vartheta_0$. Due to causality condition

$$G(\mathbf{r}, \mathbf{r}', t) = 0, \quad t < 0,$$

$G_{m,n}(\omega_{mn})$ is an analytic function in the upper half-plane of the complex variable ω_{mn} . This means that in Eq. (3.1.28) the terms with $l > 0$ yield the zeroth contributions into K_l . In the lower half-plane ω_{mn} the Green function $G_{m,n}(\omega_{mn})$ may have the singularities. Let, for the sake of simplicity, these singularities be the simple poles

$$(\omega_{mn})_k = \omega_k - i\lambda_k.$$

Calculating the integrals in Eq. (3.1.28) using the residues theory, we find that the relative importance of terms with particular $l < 0$ is determined by the factors

$$\exp[2\pi l \operatorname{Im}(n_k)] = \exp(-|l| T_s \lambda_k),$$

which obviously describe the suppression of an induced field after l turns. The collective effects due to contributions of the kernel K_l are called as the multi-turn ones. The strengths of these effects are determined by the memory in the system of electrodes. If, for instance, $\lambda_k T_s \gg 1$ (i.e., the induced fields decay much faster than the revolution period of particles), the kernels K_l will be exponentially small:

$$K_l \propto \exp(-T_s \lambda_k).$$

With this accuracy Eq. (3.1.24) can be replaced by

$$\Delta\omega_m \chi_m = \chi^0(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1 \varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c), \quad (3.1.29)$$

which describes the single-turn interaction of a bunch with its induced field.

On the contrary, in the region $\lambda_k T_s \ll 1$, the contributions in the right-hand side of Eq. (3.1.28) of the terms with $l \neq 0$ will have roughly the same values as the term with $l = 0$. This case corresponds to resonant interaction of the beam with its environment, when once induced the field perturbs the motion of particles

during many turns. For collective effects of this kind it is specific strong dependence of decrements and coherent frequency shifts on the particle frequencies.

The intermediate case, $\lambda_k T_s \simeq 1$, corresponds to the interference of single-turn and multi-turn effects.

Finally, we note that using simple substitutions the integral equation (3.1.24) can be transformed into an integral equation with a symmetric kernel. Let us illustrate this very important feature using as an example the integral equation for synchrotron oscillations, when $\chi^0 = \rho(\varphi_c)$. Substituting in Eq. (3.1.24) $\chi_m = w(\varphi_c) \sqrt{\rho/\varphi_c}$ we rewrite this equation in the form

$$\Omega_m w(\varphi_c) = \int_0^\infty d\varphi'_c K(\varphi_c, \varphi'_c) w(\varphi'_c), \quad (3.1.30)$$

$$K(\varphi_c, \varphi'_c) = \sqrt{\varphi_c \varphi'_c \rho(\varphi_c) \rho(\varphi'_c)} \sum_{n=-\infty}^{\infty} \Omega_{m,n} J_{m_c}(n_1 \varphi_c) J_{m_c}(n_1 \varphi'_c).$$

Due to the symmetry of the kernel $K(\varphi_c, \varphi'_c) = K(\varphi'_c, \varphi_c)$, the adjoint to Eq. (3.1.30) integral equation:

$$\Omega'_m w^*(\varphi_c) = \int_0^\infty d\varphi'_c K^*(\varphi_c, \varphi'_c) w^*(\varphi'_c) \quad (3.1.31)$$

has the same form as Eq. (3.1.30), while its eigennumbers Ω'_m are conjugated to Ω_m :

$$\Omega'_m = \Omega_m^* \quad (3.1.32)$$

Indeed, multiplying both parts in Eq. (3.1.30) by w^* and integrating this equation over φ_c we obtain

$$\Omega_m \langle |w|^2 \rangle = \int_0^\infty d\varphi_c d\varphi'_c w^*(\varphi_c) K(\varphi_c, \varphi'_c) w(\varphi'_c),$$

$$\langle |w|^2 \rangle = \int_0^\infty d\varphi_c |w(\varphi_c)|^2.$$

Due to symmetry of K we can also rewrite this equation in the form

$$\Omega_m \langle |w|^2 \rangle = \int_0^\infty d\varphi_c d\varphi'_c w^*(\varphi'_c) K(\varphi_c, \varphi'_c) w(\varphi_c),$$

or, which is the same,

$$\Omega_m \langle |w|^2 \rangle = \int_0^\infty d\varphi_c d\varphi'_c \operatorname{Re} \{w^*(\varphi_c) w(\varphi'_c)\} K(\varphi_c, \varphi'_c). \quad (3.1.33)$$

Similarly, from Eq. (3.1.31) we obtain

$$\Omega'_m \langle |w|^2 \rangle = \int_0^\infty d\varphi_c d\varphi'_c \operatorname{Re} \{ w^*(\varphi_c) w(\varphi'_c) \} K^*(\varphi_c \varphi'_c). \quad (3.1.34)$$

The summation of Eqs. (3.1.33) and (3.1.34) results in $\operatorname{Im}(\Omega_m + \Omega'_m) = 0$, while their subtraction yields $\operatorname{Re}(\Omega_m - \Omega'_m) = 0$, which proves Eq. (3.1.32).

If α marks the solutions of Eq. (3.1.30), and if we choose $\langle |w_\alpha|^2 \rangle = 1$, then Eqs. (3.1.32), (3.1.33) and (3.1.34) give the following expression for decrements $[\delta_{m,\alpha} = -\operatorname{Im}\Omega_{m,\alpha}]$ of eigenmodes:

$$\delta_{m,\alpha} = - \sum_{n=-\infty}^{\infty} \operatorname{Im} \Omega_{m,n} \left| \int_0^\infty d\varphi_c \sqrt{\varphi_c \rho(\varphi_c)} J_{m_c}(n_1 \varphi_c) w_\alpha(\varphi_c) \right|^2. \quad (3.1.35)$$

Since

$$\sum_{\alpha} w_{\alpha}^*(\varphi'_c) w_{\alpha}(\varphi_c) = \delta(\varphi_c - \varphi'_c),$$

the sum of decrements $[\sum_{\alpha} \delta_{m,\alpha}]$ can be understood as the imaginary part of the trace of the kernel K :

$$\sum_{\alpha} \delta_{m,\alpha} = - \sum_{n=-\infty}^{\infty} \operatorname{Im} \Omega_{m,n} \int_0^\infty d\varphi_c \varphi_c \rho(\varphi_c) J_{m_c}^2(n_1 \varphi_c).$$

3.2. COUPLING IMPEDANCES AND WAKES

The calculation of Green functions entering in the kernels of general integral equations (discussed in §3.1) for the cases of practical interest usually presents a very complicated problem. In certain conditions one can avoid that using the phenomenological description of the fields induced by a beam in its environment in terms of so-called coupling impedances.

First, let us discuss the spectra of longitudinal coherent oscillations in a coasting beam. For this type of oscillations, $m_x = m_y = 0$, Eq. (3.1.12) reads

$$\Delta\omega_n f_n = Ne^2 \frac{n}{R_0} \frac{\partial f_0}{\partial \Delta p} \int d\Gamma'_1 d\Delta p' G_{n,n}(1,2,\omega) f_n(\Delta p' \omega), \quad (3.2.1)$$

$$\Delta\omega_n = \omega - n\omega_0(p).$$

In this equation, if the dependence of the Green function G_{nn} on the amplitudes of particle betatron oscillations is reasonably smooth, G can be substituted for its value at the equilibrium trajectory and factored outside the integral sign. This results in

$$\Delta\omega_n f_n = Ne^2 \frac{n}{R_0} G_{nn}(\omega) \frac{\partial f_0}{\partial \Delta p} \int_{-\infty}^{\infty} d\Delta p' d\Gamma'_1 f_n. \quad (3.2.2)$$

Now we note, that the coefficient in front of $N\partial f_0/\partial\Delta p$ in the right-hand side of Eq. (3.2.2) is a Fourier harmonic of the Poisson bracket $i[\bar{L}, \Delta p]$, and since Δp is the integral of unperturbed motion, this expression can be written down in the form

$$-i[\mathcal{H}, \Delta p]_{n,\omega} = -\frac{i}{v_s} \left(\frac{\partial \mathcal{H}}{\partial t} \right)_{n,\omega} = -ie \overline{E_{\delta}[\mathbf{r}_0(t), t]_{mn}}. \quad (3.2.3)$$

Using this equation we can rewrite Eq. (3.2.2) in two equivalent forms:

$$\Delta\omega_n f_n = \frac{Ne^2\omega}{v_s} G_{nn}(\omega) \frac{\partial f_0}{\partial\Delta p} \int_{-\infty}^{\infty} d\Delta p' d\Gamma'_1 f_n \quad (3.2.4)$$

and

$$\Delta\omega_n f_{n\omega} = -iNeE_{\delta}(n,\omega) \frac{\partial f_0}{\partial\Delta p}. \quad (3.2.5)$$

Here,

$$f_{n\omega}(\Delta p) = \int d\Gamma_1 f_{n\omega}(\Delta p, \Gamma_1)$$

is the momentum distribution function of the beam. In analogy with Eq. (1.6.7) we shall write Fourier harmonics of the induced field $E_{\delta}(n,\omega)$ in the form

$$E_{\delta}(n,\omega) = -\frac{U_{n\omega}}{\Pi},$$

where $U_{n\omega}$ has the sense of the voltage induced by the beam in the surrounding electrodes. Phenomenologically for the last value one may adopt the relation with harmonics of beam current

$$I_{n\omega} = e\omega_s \int_{-\infty}^{\infty} d\Delta p f_{n\omega}(\Delta p)$$

from the circuits theory:

$$U_{n\omega} = Z_n(\omega) I_{n\omega} \equiv Z_n(\omega) e\omega_s \int_{-\infty}^{\infty} d\Delta p f_{n\omega}(\Delta p).$$

Here, Z_n is so-called coupling impedance of the beam with the surrounding electrodes. Substitution of these expressions into Eq. (3.2.5) yields

$$\Delta\omega_n f_{n\omega} = i \frac{Ne^2\omega_s Z_n(\omega)}{\Pi} \frac{\partial f_0}{\partial\Delta p} \int d\Delta p' f_{n\omega}(\Delta p'). \quad (3.2.6)$$

For problems with initial conditions $f_n(\Delta p, t=0) = f_n^{(0)}(\Delta p)$, Eq. (3.2.6) must be modified:

$$\Delta\omega_n f_{n\omega} = i f_n^{(0)} + \frac{iNe^2\omega_s Z_n(\omega)}{\Pi} \frac{\partial f_0}{\partial \Delta p} \int d\Delta p' f_{n\omega}(\Delta p'). \quad (3.2.7)$$

The comparison of Eqs. (3.2.1) and (3.2.6) [or Eq. (3.2.4)] yields the relation between $G_{nn}(\omega)$ and $Z_n(\omega)$,

$$\frac{iZ_n(\omega)}{n} = \frac{2\pi}{\omega_s} G_{nn}(\omega), \quad (3.2.8)$$

which is the same as

$$\frac{iZ_n(\omega)}{\omega} = \frac{2\pi}{\omega_s^2} G_{nn}(\omega). \quad (3.2.9)$$

The conceptual benefit from introducing the theory of coupling impedances can be associated with the possibility of the calculation of induced fields using general results of the circuits theory. Sometimes this can also simplify the concepts of the measurements of $G(\omega)$ or $Z(\omega)$ for particular elements of the vacuum chamber before mounting them in the storage ring.

The solution of Eq. (3.2.7) is

$$f_{n\omega}(\Delta p) = \frac{i f_n^{(0)}(\Delta p)}{\omega - n\omega_0} - \frac{iNeE_{\mathfrak{g}}(n, \omega)}{\omega - n\omega_0} \frac{\partial f_0}{\partial \Delta p},$$

$$E_{\mathfrak{g}}(n, \omega) = \frac{E_{n, \omega}^{(0)}}{\epsilon_n(\omega)} = -\frac{ie\omega_s Z_n}{\Pi \epsilon_n(\omega)} \int_{-\infty}^{\infty} \frac{d\Delta p f_n(\Delta p, 0)}{\omega - n\omega_0}. \quad (3.2.10)$$

Here,

$$\epsilon_n(\omega) = 1 + \frac{\Omega_n^2}{n\omega_0'} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\omega - n\omega_0(p)} \quad (3.2.11)$$

is the dielectric constant of a coasting beam, and we introduced

$$\Omega_n^2 = n^2 \frac{Ne^2\omega_s\omega_0'}{\Pi} \left(\frac{-iZ_n}{n} \right) = U_n + iV_n. \quad (3.2.12)$$

The spectrum of coherent oscillations is determined by the dispersion equation

$$\epsilon_n(\omega) = \epsilon'(\omega) + i\epsilon''(\omega) = 0,$$

or

$$1 + \frac{\Omega_n^2}{n\omega_0'} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\omega - n\omega_0(p)} = 0. \quad (3.2.13)$$

Solution of this equation is especially easy for the case of a monochromatic beam $f_0(\Delta p) = \delta(\Delta p)$, when it reads

$$1 - \frac{\Omega_n^2}{\Delta\omega_n^2} = 0, \quad \Delta\omega_n = \omega - n\omega_s.$$

If the impedance entering in Ω_n^2 is weakly dependent on the exact value of ω , then substituting $\omega \rightarrow n\omega_s$ in argument $Z_n(\omega)$ we obtain eigenfrequencies

$$\omega_n = n\omega_s \pm \sqrt{\Omega_n^2}.$$

This relation clarifies the physical sense of the parameter Ω_n^2 as the square of the coherent frequency shift, corresponding to the impedance Z_n . In the general case the impedance Z_n is a complex value $Z_n = Z_n' + iZ_n''$. Therefore, one of the roots describes the increase in oscillations with an increment

$$\delta = -\text{Im}\omega = |n|\omega_s \sqrt{\frac{Ne^2\omega_s|\alpha_p|}{2\pi p v}} \begin{cases} \sqrt{(|Z_n| - Z_n'')/2n}, & \alpha_p > 0, \\ \sqrt{(|Z_n| + Z_n'')/2n}, & \alpha_p < 0. \end{cases} \quad (3.2.14)$$

In particular, this expression shows that without dissipation ($Z_n' = 0$) oscillations will be unstable provided

$$\alpha_p \frac{Z_n''}{n} < 0.$$

Since α_p changes the sign, when the particle energy crosses the transition energy of the ring, the interaction with capacitive and inductive elements of the vacuum chamber will cause the instability by different sides of the transition energy.

A beam with a finite frequency spread can be stabilized by Landau damping. These issues will be discussed in more detail in §3.4. Here we simply mention that the roots of Eq. (3.1.11) having $\text{Im}\omega \rightarrow +0$ correspond to the threshold of instability, when the real part of Eqs. (3.2.13):

$$\epsilon' = 1 + U_n \mathcal{P}(\omega) + V_n J(\omega) = 0 \quad (3.2.15)$$

can be solved only at a definite particle number $N = N_{th}$. The second equation,

$$\epsilon'' = V_n \mathcal{P}(\omega) - U_n J(\omega) = 0, \quad (3.2.16)$$

defines the value of the coherent frequency shift at the threshold of instability $\Delta\omega_n = \omega - n\omega_s$. Here

$$\mathcal{P}(\omega) = \frac{1}{n\omega'_0} \mathcal{P} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\Delta\omega_n - n\omega'_0 \Delta p}, \quad (3.2.17)$$

$$J(\omega) = \frac{\pi}{n\omega'_0} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0}{\partial \Delta p} \delta(\Delta\omega_n - n\omega'_0 \Delta p). \quad (3.2.18)$$

The sign $\mathcal{P}$ in front of the integral here and below denotes the calculation of the principal value of the corresponding integral:

$$\mathcal{P}(x) \equiv \lim_{\eta \rightarrow 0} \left[\int_{-\infty}^{x-\eta} + \int_{x+\eta}^{\infty} \right] \frac{dx'}{x-x'} \{ \dots \}.$$

For given beam parameters Eqs. (3.2.15) and (3.2.16) determine the boundary of the stability region in the plane of complex variable $U_n + iV_n$:

$$U_n = \frac{\mathcal{P}(\omega)}{\mathcal{P}^2(\omega) + J^2(\omega)}, \quad (3.2.19)$$

$$V_n = \frac{J(\omega)}{\mathcal{P}^2(\omega) + J^2(\omega)}. \quad (3.2.20)$$

The shape of this curve depends on the distribution function. If momentum distribution in the beam is an even function, the function $\mathcal{P}(\omega)$ is even, while $J(\omega)$ is an odd function of $\Delta\omega_n$. (See Fig. 3.1.) The only point where $J(\omega) = 0$ is $\Delta\omega_n = 0$. In this case $U_n = 1/\mathcal{P}(0) \simeq -(n\Delta\omega)^2$, and hence the boundary of the stability region lies to the right of the line $U_n = -(n\Delta\omega)^2$. In the region $|\Delta\omega_n| \gg n\Delta\omega$ the value of $J(\omega)$ decreases like the derivative of the distribution function, while $\mathcal{P}(\omega) \sim 1/\Delta\omega_n^2$. Therefore, here we have $U_n > 0$ and $U_n \gg |V_n|$. In the intermediate region $\Delta\omega_n \sim n\Delta\omega$ we can estimate $|J(\omega)| \simeq (n\Delta\omega)^{-2} \gg |\mathcal{P}(\omega)|$, and correspondingly $|V_n| \simeq (n\Delta\omega)^2$, $U_n \simeq 0$. For $U_n < 0$ these properties of the stability region can be united in a so-called “Z/n criterion.”⁴⁰

$$\left| \frac{Z_n}{n} \right| < \frac{p_s \nu_s \alpha_p}{eI} \left(\frac{\Delta p}{p_s} \right)^2, \quad (3.2.21)$$

where $I = Ne\omega_s/2\pi$ is the beam current. For $U_n > (n\Delta\omega)^2$ although the stability region narrows in V_n (see Fig. 3.2), the violation of criterion (3.2.21) does not mean yet that the oscillations will be unstable.

The location of the stability region with respect to the boundary curve can be found by calculating the oscillation spectrum near-threshold of the instability ($|N - N_{th}| \ll N_{th}$). If we define $\Delta\omega_n = \Delta\omega_n^{th} + \Delta_n$ and take into account that at the threshold $\partial\epsilon_n/\partial N = -1/N_{th}$, after a Taylor expansion of $\epsilon_n(\omega)$ in powers of $\Delta N = N_{th} - N$ and Δ_n we obtain

$$\epsilon_n(\omega) = -\left(\frac{\partial \epsilon_n}{\partial N}\right)_{th} \Delta N + \left(\frac{\partial \epsilon_n}{\partial \omega}\right)_{th} \Delta_n = \frac{\Delta N}{N_{th}} + \Delta_n \left(\frac{\partial \epsilon_n}{\partial \omega}\right)_{th} = 0,$$

from which

$$\Delta_n = \frac{\Delta N}{N_{th}} \frac{1}{(\partial \epsilon_n / \partial \omega)_{th}}. \quad (3.2.22)$$

Hence, the decrements of oscillations ($\delta_n = -\text{Im} \omega$) near the threshold of instability read

$$\delta_n = -\frac{\Delta N}{N_{th}} \frac{(\partial \epsilon'' / \partial \omega)_{th}}{|\partial \epsilon_n / \partial \omega|_{th}^2}. \quad (3.2.23)$$

The oscillations will decay ($\delta_n > 0$), when

$$\Delta N (\partial \epsilon'' / \partial \omega)_{th} < 0. \quad (3.2.24)$$

Let us illustrate the use of the above formulas for the case of the so-called negative mass instability.⁴¹ It is caused by the interaction of a beam with ideally conductive walls of the vacuum chamber. In this case $V_n = 0$ and, as was already mentioned, the condition of coherent oscillation stability depends on the relative

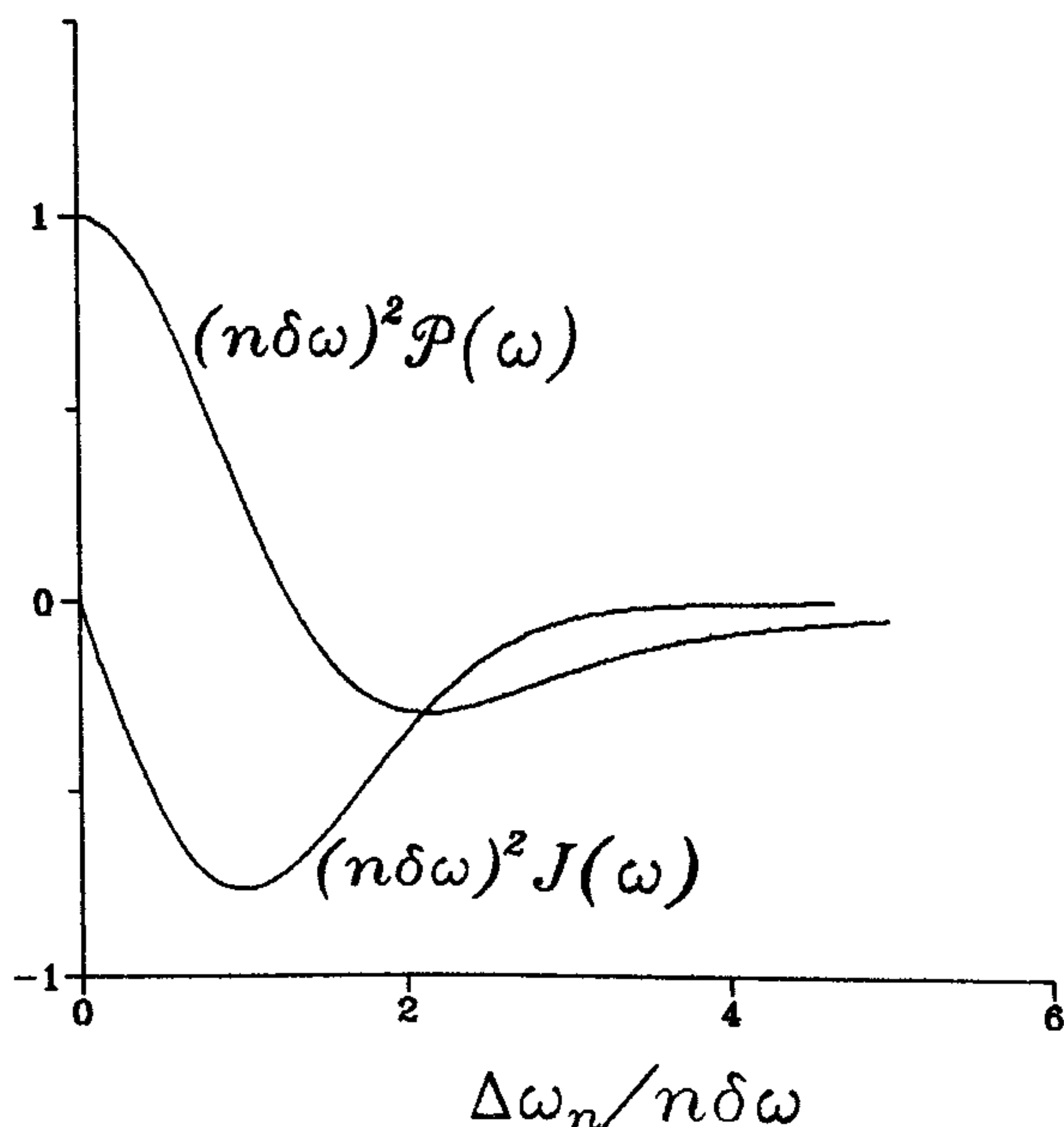


FIG. 3.1. The functions $P(\omega)$ and $J(\omega)$ from Eqs. (3.2.17) and (3.2.18) for the Gaussian momentum distribution.

position of the energy of particles and the transition energy of the storage ring.

For the sake of simplicity, we shall calculate Z_n for low-frequency oscillations

$$|n| \ll \gamma R_0 / l_\perp, \quad (3.2.25)$$

and assuming that both the vacuum chamber and the beam have the round cross sections with radii $l_\perp$ and a , respectively. In this case E_ϕ can be found using the Faraday law:

$$\oint d\mathbf{l} \mathbf{E}_\omega = \frac{i\omega}{c} \int d\Sigma \mathbf{H}_\omega, \quad \omega = n\omega_s, \quad (3.2.26)$$

the integration contour l is shown in Fig. 3.3, and Σ is the surface encircled by l . In a cylindrical system of coordinates, with the z -axis directed along that of the vacuum chamber, Eq. (3.2.26) takes the form

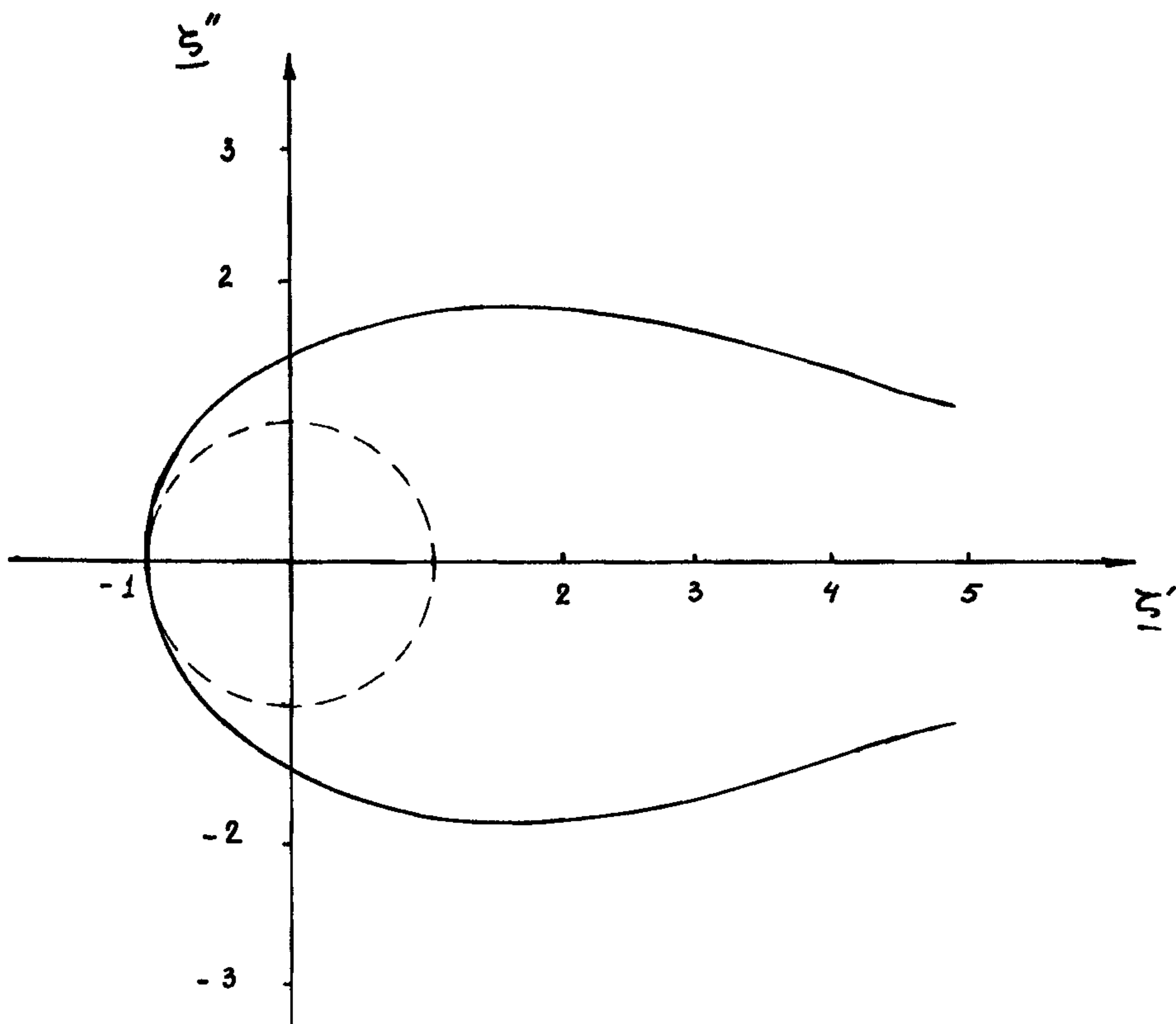


FIG. 3.2. The stability diagram for the Gaussian momentum distribution; dashed circle limits Z/n -region.

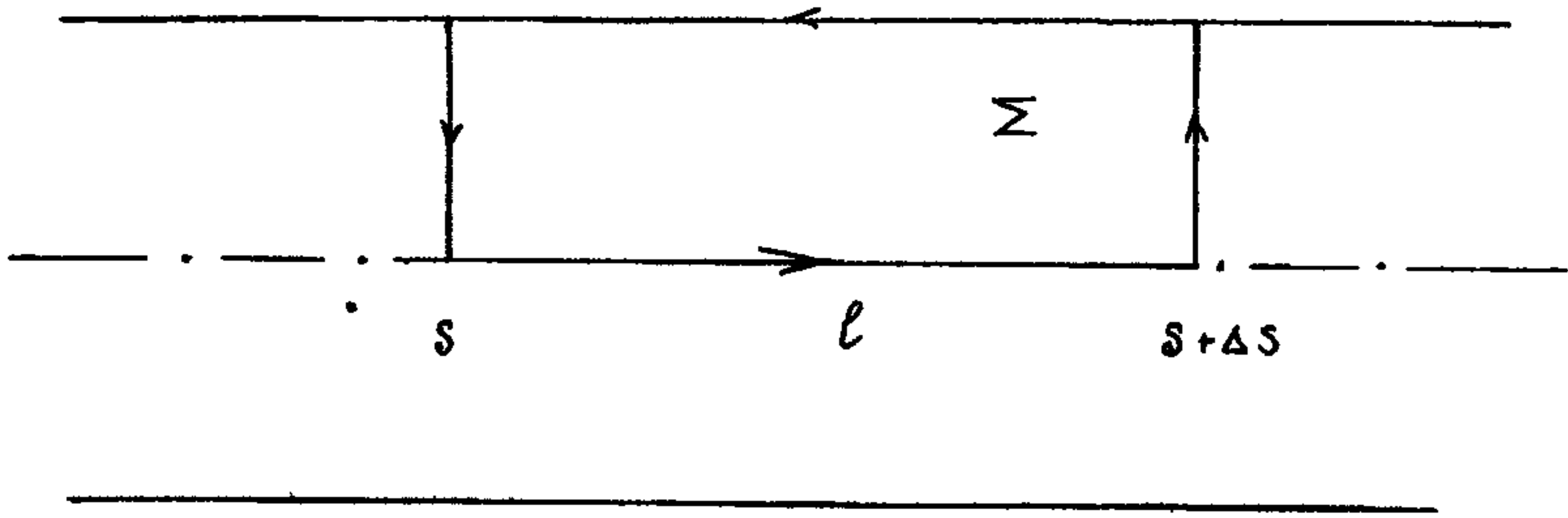


FIG. 3.3. The integration contour in Eq. (3.2.26).

$$\Delta s(E_{\vartheta,\omega} + E_{\vartheta,\omega}^w) + \int_0^{l_1} dr [E_r(r, s + \Delta s) - E_r(r, s)] = \frac{in\omega_0}{c} \Delta s \int_0^{l_1} dr H_\phi(r, s),$$

or after reducing it to azimuthal harmonics

$$E_{n,\omega} = -E_{n,\omega}^w - \frac{in}{R_0} \int_0^{l_1} dr E_{r,n} + \frac{inv_s}{R_0 c} \int_0^{l_1} dr H_{\phi,n}(r). \quad (3.2.27)$$

On the surface of the ideally conductive chamber one has $E_{n\omega}^w = 0$, while the components E_r and H_ϕ for low-frequency oscillations are calculated using the Gauss and the Amner laws. Assuming the uniform distribution of particles across the beam inside its radius:

$$f(r_1, \Delta p) = \frac{f(\Delta p)}{\pi a^2} \begin{cases} 1, & r \leq a, \\ 0, & r > a, \end{cases}$$

we obtain

$$H_\phi(n, \omega) = \frac{v_s}{c} E_r(n, \omega),$$

$$E_r(n, \omega) = \frac{e\rho_{n\omega}}{a^2 \Pi} \begin{cases} r, & r \leq a, \\ a^2/r, & r > a. \end{cases}$$

Substitution of these expressions into Eq. (3.2.27) yields

$$E_\vartheta(n, \omega) = \frac{ine}{\Pi R_0 \gamma^2} [2 \ln(l_1/a) + 1] \rho_{n,\omega} = -\frac{e\omega_s Z_n}{\Pi} \int_{-\infty}^{\infty} d\Delta p f_{n\omega}(\Delta p), \quad (3.2.28)$$

and hence, the impedance reads*

$$\frac{-iZ_n}{n} = \frac{1}{v_s \gamma^2} [2 \ln(l_1/a) + 1]. \quad (3.2.29)$$

Using Eq. (3.2.29), we can rewrite Ω_n^2 in the form

$$\Omega_n^2 = n^2 \omega_s^2 \frac{N e^2 \alpha_p}{\Pi p_s v_s \gamma^2} [2 \ln(l_1/a) + 1].$$

It is clear that $\Omega_n^2 > 0$ at $\alpha_p > 0$.

For the negative mass instability and for a beam with a Gaussian momentum distribution

$$f_0 = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\Delta p^2}{2\sigma^2}\right),$$

the calculation of roots of Eq. (3.2.13) corresponding to unstable oscillations can be simplified. Indeed, from the equation

$$\text{Im} \epsilon_n(\omega = \Delta \omega'_n + i\delta_n) \propto \int_{-\infty}^{\infty} d\Delta p \frac{\Delta p \exp(-\Delta p^2/2\sigma^2)}{(\Delta \omega'_n - n\omega'_0 \Delta p)^2 + \delta^2} = 0$$

it follows that $\Delta \omega'_n = 0$, while the value of the increment δ_n is determined by the equation

* Harmonics E_{nw}^w do not vanish on the surface of the vacuum chamber with the finite conductivity of the walls, where⁹

$$E_{nw}^w = \sqrt{\frac{-i\omega}{4\pi\sigma}} H_\phi,$$

and σ is the conductivity of the wall material. Substituting here

$$H_\phi(r=l_1) = \frac{v e \rho_{n,\omega}}{c \Pi l_1},$$

we obtain the addition to Z_n due to finite wall conductivity:

$$Z_n^\sigma = \frac{1 - i \text{sgn}(n)}{2} \frac{\delta(\omega_s)}{l_1} \frac{v}{c^2} \sqrt{|n|}, \quad \delta(\omega_s) = \frac{c}{\sqrt{2\pi\sigma\omega_s}}.$$

Typically, Z_n^σ/n is small. But for some initial n it can exceed the value defined by Eq. (3.2.29) if

$$\delta(\omega_s) \gg l_1 \frac{\ln(l_1/a)}{\gamma^2 (v/c)^2}.$$

Taking $\omega_s \sim 2 \times 10^7$ (1/s), $\sigma \sim 10^{17}$ (1/s), and $l_1/a \simeq 5$, we can estimate that this will take place for $n \simeq 1$ if $\gamma > 25$.

$$y \exp(y^2/2) [1 - \operatorname{erf}(y/\sqrt{2})] = \sqrt{2/\pi} \left(\frac{n^2 \Delta \omega^2}{|\Omega_n^2|} - 1 \right), \quad \Omega_n^2 < 0. \quad (3.2.30)$$

Here, $y = \delta_n / n \Delta \omega$, $\Delta \omega = |\omega'_0| \sigma$ is the revolution frequencies spread of the beam, and $\operatorname{erf}(x)$ is the probability integral.¹⁴ Unstable solutions occur only in the region of $|\Omega_n^2| > n^2 \Delta \omega^2$, where decrements have asymptotes

$$\delta_n \approx \begin{cases} \sqrt{|\Omega_n^2|}, & |\Omega_n^2| > (n \Delta \omega)^2, \\ (2/\pi)^{1/2} |n \Delta \omega| \left(\frac{n^2 \Delta \omega^2}{|\Omega_n^2|} - 1 \right), & |\Omega_n^2| - (n \Delta \omega)^2 \ll (n \Delta \omega)^2. \end{cases} \quad (3.2.31)$$

This equation shows that there is no unstable solution in the region $|\Omega_n^2| < (n \Delta \omega)^2$. The threshold current of instability is determined by the equation

$$|\Omega_n^2| = (n \Delta \omega)^2. \quad (3.2.32)$$

For the case of interest ($|n| \ll \gamma R_0 / l_1$) it does not depend on mode number n :

$$N_{th} \equiv N_c = \frac{\Pi}{e^2 \omega_s |\omega'_0|} \frac{v \gamma^2}{L} \Delta \omega^2, \quad (3.2.33)$$

$$L = 2 \ln(l_1 / a) + 1.$$

Near the threshold of instability we can write

$$\left(\frac{\partial \epsilon'_n}{\partial \omega} \right)_{th} = 0, \quad \left(\frac{\partial \epsilon''_n}{\partial \omega} \right)_{th} = - \sqrt{\frac{\pi}{2}} \frac{N}{N_c} \frac{1}{|n| \Delta \omega} < 0.$$

Then Eq. (3.2.23) results in

$$\delta_n = \sqrt{\frac{2}{\pi}} |n| \Delta \omega \left(\frac{N_c}{N} - 1 \right). \quad (3.2.34)$$

In agreement with Eq. (3.2.31) this predicts $\delta_n > 0$ when $N < N_c$.

Description of induced fields in terms of coupling impedances certainly can be used for studying synchrotron coherent oscillations of bunched beams as well. A relevant integral equation can be easily obtained by the substitution of Eq. (3.2.9) into general integral equation (3.1.24) ($\Delta \omega_m = \omega - m_c \omega_c$):

$$\Delta \omega_m f_m = N e^2 m_c \frac{\partial f_0}{\partial I_c} \sum_n J_{m_c}(n \varphi_c) G_n(\omega + n \omega_s) \int_0^\infty d\varphi'_c \varphi'_c J_{m_c}(n \varphi'_c) f_m. \quad (3.2.35)$$

Using Eq. (3.2.9) we can rewrite this equation in terms of the coupling impedance²⁵

$$\Delta\omega_m f_m = \frac{Ne^2\omega_s^2}{2\pi} m_c \frac{\partial f_0}{\partial I_c} \sum_n J_{m_c}(n\varphi_c) \times \frac{iZ_m(\omega + n\omega_s)}{\omega + n\omega_s} \int_0^\infty d\varphi'_c \varphi'_c J_{m_c}(n\varphi'_c) f_m. \quad (3.2.36)$$

For the single-turn interaction of the beam with its environment, when $Z_n(\omega)$ is a smooth function of ω , we can replace in Eqs. (3.2.35) and (3.2.36) the summation over n by the integration and ω in the argument of G by its unperturbed value $m_c\omega_c$. This results in

$$\Delta\omega_m f_m = \frac{Ne^2\omega_s}{2\pi} m_c \frac{\partial f_0}{\partial I_c} \int_0^\infty d\varphi'_c \varphi'_c g_{m_c}(\varphi_c, \varphi'_c) f_m, \quad (3.2.37)$$

where we introduced

$$g_{m_c}(\varphi_c, \varphi'_c) = \int_0^{2\pi} \frac{d\psi_c d\psi'_c}{(2\pi)^2} g(\varphi - \varphi') e^{im_c(\psi_c - \psi'_c)},$$

$$g(\varphi - \varphi') = \int_{-\infty}^\infty dn \frac{iZ_n(m_c\nu_c + n)}{m_c\nu_c + n} e^{in(\varphi - \varphi')}. \quad (3.2.38)$$

If, moreover, the azimuthal length of the device interacting with the beam is rather small and we can neglect in $Z_n(\omega)$ its dependence on n , the function $g(\varphi)$ in Eq. (3.2.38) can be written as

$$g(\varphi - \varphi') = e^{-im_c\nu_c(\varphi - \varphi')} w(\varphi - \varphi')$$

$$w(\varphi - \varphi') = \int_{-\infty}^\infty dn \frac{iZ(n)}{n} e^{in(\varphi - \varphi')}. \quad (3.2.39)$$

Due to the causality property, the functions $G(n\omega_s)$ as well as $iZ(n)/n$ have no singularities in the upper half-plane of the complex variable n . Therefore, $w(\varphi - \varphi') = 0$, when $\varphi > \varphi'$, and hence, this function defines the longitudinal fields, which the particle, having the position φ' in the beam, leave in the device. Usually, the function $w(\varphi)$ is referred to as the (longitudinal) wake function. Note that by the definitions both the coupling impedance and wake function are determined for the particle, whose trajectory in the interaction region is the straight line parallel to the pipe axis.

To conclude this section let us briefly discuss the use of the concept of the coupling impedances for the description of transverse coherent oscillations. We consider dipole vertical coherent oscillations of a coasting beam, when $f_{m,n}$ obeys Eq. (3.1.12) ($m_x=0$, $m_z=\pm 1$):

$$(\omega - \omega_{m,n}) f_{m,n} = Ne^2 m_z \frac{\partial f_0}{\partial I_z} \int d\Gamma' G_{m,n}(1,2,\omega + n\omega_s) f_{m,n} \quad (3.2.40)$$

Generally, the value $G_{m,n}$ contains two kinds of terms. The combination

$$G_{\parallel} = \frac{v^2}{c^2} G_{\vartheta,\vartheta} - G_{0,0} \quad (3.2.41)$$

describes the excitation of the fields by the longitudinal motion of the beam. The remaining terms in G describe the excitation of induced fields by the transverse current ($v_{\perp} \rho$) of the beam and the reaction of the beam on those fields. If we can neglect these contributions that physically correspond to the case when induced fields are mainly determined by the retarded dipole momentum of the beam, we can write harmonics $G_{m,n}$ in the form

$$G_{m,n} = \frac{R_0}{2p v_z} \sqrt{I_z I'_z} \left(\frac{\partial^2}{\partial z \partial z'} G_{\parallel}(1,2,\omega) \Big|_{r_1=0} \right).$$

The substitution of this expression into Eq.(3.2.40) yields

$$(\omega - \omega_{m,n}) f_{m,n} = \frac{Ne^2 R_0}{2p v_z} m_z \frac{\partial f_0}{\partial I_z} \sqrt{I_z} \left(\frac{\partial^2}{\partial z \partial z'} G_{\parallel}(1,2,\omega) \Big|_{r_1=0} \right) \int d\Gamma' \sqrt{I'_z} f_{m,n} \quad (3.2.42)$$

On the other hand, since without the coupling of oscillations one has

$$\dot{I}_z = \frac{\partial z}{\partial \psi_z} \mathcal{F}_z,$$

therefore

$$(\omega - \omega_{m,n}) f_{m,n} = -m_z \frac{\partial f_0}{\partial I_z}(z)_m \overline{(\mathcal{F})_n(\omega)}. \quad (3.2.43)$$

Defining now

$$\overline{(\mathcal{F})_n(\omega)} = -\frac{Ne^2 \omega_s Z_n^{\perp}(\omega + n\omega_s)}{\Pi} \frac{R_0}{2p v_z} \sqrt{I_z} \int d\Gamma' \sqrt{I'_z} f_{m,n} \quad (3.2.44)$$

we can rewrite Eq. (3.2.43) as

$$(\omega - \omega_{m,n}) f_{m,n} = m_z \frac{\partial f_0}{\partial I_z} \frac{Ne^2 R_0}{2p v_z} \sqrt{I_z} \frac{\omega_s Z_n^{\perp}(\omega + n\omega_s)}{\Pi} \int d\Gamma' \sqrt{I'_z} f_{m,n} \quad (3.2.45)$$

The comparison of Eqs. (3.2.42) and (3.2.45) yields the relation of the transverse coupling impedance $Z_n^{\perp}(\omega)$ introduced here to the derivatives of the Green function on the closed orbit:

$$Z_n^\perp(\omega) = \frac{2\pi R_0}{\omega_s} \left(\frac{\partial^2}{\partial z \partial z'} G_{\parallel}(1, 2, \omega) \Big|_{r_\perp=0} \right). \quad (3.2.46)$$

Using Eq. (3.2.9) we can rewrite Eq. (3.2.46) in the form

$$Z_n^\perp(\omega) = \frac{iv}{\omega} \left(\frac{\partial^2}{\partial z \partial z'} Z_n^\parallel(1, 2, \omega) \Big|_{r_\perp=0} \right). \quad (3.2.47)$$

Estimating the derivatives by the order of the value, we can also write

$$Z_n^\perp(\omega) \simeq \frac{v}{l_\perp^2} \frac{i Z_n^\parallel(\omega)}{\omega}. \quad (3.2.48)$$

3.3. GREEN FUNCTIONS

Let us discuss in more detail some general properties of solutions of Eq. (2.1.25). As is known, introducing the vector $\mathbf{A}(\mathbf{r}, t)$ and the scalar $A_0(\mathbf{r}, t)$ potentials

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla A_0, \quad \mathbf{H} = \text{rot} \mathbf{A}(\mathbf{r}, t), \quad (3.3.1)$$

these equations are reduced to

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi e}{c} \mathbf{j} + \nabla \left(\text{div} \mathbf{A} + \frac{1}{c} \frac{\partial A_0}{\partial t} \right), \quad (3.3.2)$$

$$\Delta A_0 + \frac{1}{c} \frac{\partial}{\partial t} \text{div} \mathbf{A} = -4\pi e \rho(\mathbf{r}, t). \quad (3.3.3)$$

Besides, due to Eqs. (3.3.1) the potentials $\mathbf{A}$ and A_0 may obey some gauge condition. For example, Eqs. (3.1.2) are written assuming the Lorentz gauge condition for potentials

$$\text{div} \mathbf{A} + \frac{1}{c} \frac{\partial A_0}{\partial t} = 0, \quad (3.3.4)$$

whereby Eqs. (3.3.2) and (3.3.3) are transformed into the equations

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi e}{c} \mathbf{j}(\mathbf{r}, t), \quad (3.3.5)$$

$$\Delta A_0 - \frac{1}{c^2} \frac{\partial^2 A_0}{\partial t^2} = -4\pi e \rho(\mathbf{r}, t). \quad (3.3.6)$$

In Lorentz gauge the potentials $\mathbf{A}$ and A_0 are described symmetrically.

With respect to the problems related to the effect of radiation processes on beam stability, the use of the Coulomb gauge condition

$$\operatorname{div} \mathbf{A} = 0 \quad (3.3.7)$$

typically turns out to be more convenient than using the Lorentz gauge condition. Condition (3.3.7) breaks the symmetry in the description of $\mathbf{A}$ and A_0 :

$$\Delta A_0(\mathbf{r}, t) = -4\pi e \rho(\mathbf{r}, t), \quad (3.3.8)$$

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi e}{c} \mathbf{j} + \frac{1}{c} \frac{\partial}{\partial t} \nabla A_0. \quad (3.3.9)$$

In this form it is obvious that Eq. (3.3.8) describes quasistatic induced fields, that is, the fields that depend on time following that of $\rho(\mathbf{r}, t)$. Radiation processes with such a gauge condition are described by Eq. (3.3.9).

As was already mentioned, potentials Eqs. (3.3.5), (3.3.6), (3.3.8), and (3.3.9) can be solved using the Green functions. In the Lorentz gauge these solutions are determined by Eq. (2.5.3). Their substitution into Eqs. (3.3.5) and (3.3.6) yields the equation for the Green functions:

$$\Delta G_{\alpha\beta} - \frac{1}{c^2} \frac{\partial^2 G_{\alpha\beta}}{\partial t^2} = -4\pi \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (3.3.10)$$

$$\Delta G_{00} - \frac{1}{c^2} \frac{\partial^2 G_{00}}{\partial t^2} = -4\pi \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (3.3.11)$$

Besides, the Green functions should satisfy the causality condition

$$G_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t - t') = 0, \quad t < t'. \quad (3.3.12)$$

Transforming Eqs. (3.3.10) and (3.3.11) to Fourier harmonics

$$G(1, 2, t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(1, 2, \omega), \quad (3.3.13)$$

let us rewrite these equations in the form

$$\Delta G_{\alpha\beta}(1, 2, \omega) + \frac{\omega^2}{c^2} G_{\alpha\beta}(1, 2, \omega) = -4\pi \delta_{\alpha\beta} \delta(1 - 2), \quad (3.3.14)$$

$$\Delta G_{00}(1, 2, \omega) + \frac{\omega^2}{c^2} G_{00}(1, 2, \omega) = -4\pi \delta(1 - 2). \quad (3.3.15)$$

From these equations it follows that the Green function $G(1, 2, \omega)$ is a symmetrical function of coordinates

$$G(1, 2, \omega) = G(2, 1, \omega). \quad (3.3.16)$$

Functions $G(1, 2, \omega)$ are complex ones:

$$G(1, 2, \omega) = G'(1, 2, \omega) + iG''(1, 2, \omega). \quad (3.3.17)$$

As can be seen from Eqs. (3.3.14) and (3.3.15) at high frequencies $\omega \rightarrow \infty$ the function $G(\omega)$ decreased like $G(\omega) \propto \omega^{-2}$. As follows from the causality principle and Eq. (3.3.13), $G(1, 2, \omega)$ is an analytical function in the upper half-plane of a complex variable ω . Representing it as the Cauchy integral

$$G(1, 2, \omega) = \oint_C \frac{d\omega'}{2\pi i} \frac{G(1, 2, \omega')}{\omega' - \omega},$$

where the contour C goes above the real axis ω and then is closed in the upper half-plane in the circumference of a large radius, one can easily find the so-called dispersion relations between the real and imaginary parts of $G(1, 2, \omega)$:

$$\begin{aligned} G'(1, 2, \omega) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} G''(1, 2, \omega'), \\ G''(1, 2, \omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} G'(1, 2, \omega'). \end{aligned} \quad (3.3.18)$$

Using the identity

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} = 0,$$

we can also rewrite Eqs. (3.3.18) in the form

$$\begin{aligned} G'(1, 2, \omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{G''(1, 2, \omega') - G''(1, 2, \omega)}{\omega' - \omega}, \\ G''(1, 2, \omega) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{G'(1, 2, \omega') - G'(1, 2, \omega)}{\omega' - \omega}, \end{aligned} \quad (3.3.19)$$

where the integrand has now no singularity at $\omega = \omega'$.

Since $G(1, 2, \omega)$ is the Fourier image of a real function, Eq. (3.3.13) results in symmetry condition

$$G^*(1, 2, \omega) = G(1, 2, -\omega^*)$$

or on the real axis ω ,

$$G'(1, 2, \omega) = G'(1, 2, -\omega), \quad G''(1, 2, \omega) = -G''(1, 2, -\omega). \quad (3.3.20)$$

If the potentials satisfy the gauge condition (3.3.4), not all the components of $G_{\alpha\beta}$ and G_{00} are independent. In fact, substituting in Eq. (3.3.4) solutions in the form of Eq. (2.5.3), we can write

$$\int d(2) \frac{\partial}{\partial x_{\alpha}^{(1)}} G_{\alpha\beta}(1,2,\omega) j_{\beta}(2,\omega) = i\omega \int d(2) G_{00}(1,2,\omega) \rho(2,\omega). \quad (3.3.21)$$

Using here the continuity equation

$$i\omega \rho(\mathbf{r},\omega) = \text{div} \mathbf{j}(\mathbf{r},\omega)$$

and integrating by parts in the right-hand side of Eq. (3.3.21)

$$\int d(2) G_{00}(1,2,\omega) \nabla_{\beta} j_{\beta} = - \int d(2) j_{\beta}(2,\omega) \frac{\partial}{\partial x_{\beta}^{(2)}} G_{00}(1,2,\omega),$$

instead of Eq. (3.3.21) we obtain

$$\int d(2) j_{\beta}(2,\omega) \left[\frac{\partial}{\partial x_{\alpha}^{(1)}} G_{\alpha\beta}(1,2,\omega) + \frac{\partial}{\partial x_{\beta}^{(2)}} G_{00}(1,2,\omega) \right] = 0.$$

Since this condition should be satisfied with an arbitrary $j(2,\omega)$, the required gauge condition for the components of the Green function is found:

$$\frac{\partial}{\partial x_{\alpha}^{(1)}} G_{\alpha\beta}(1,2,\omega) = - \frac{\partial}{\partial x_{\beta}^{(2)}} G_{00}(1,2,\omega). \quad (3.3.22)$$

Calculating the divergence of both the parts over the coordinates of the second particle and taking into account Eq. (3.3.15), we can find another useful relation:

$$G_{00}(1,2,\omega) = - \frac{c^2}{\omega^2} [4\pi\delta(1-2) - \nabla_{\alpha}^{(1)} \nabla_{\beta}^{(2)} G_{\alpha\beta}(1,2,\omega)]. \quad (3.3.23)$$

With the Coulomb gauge condition, solutions of Eqs. (3.3.8) and (3.3.9) read

$$A_0(\mathbf{r},t) = e \int d(2) G(1,2) \rho(2,t), \quad (3.3.24)$$

$$A_{\alpha}(\mathbf{r},t) = \frac{e}{c} \int d(2) \int_{-\infty}^t dt' G_{\alpha\beta}(1,2,t-t') J_{\beta}(2,t'), \quad (3.3.25)$$

where

$$J_{\alpha}(\mathbf{r},t) = j_{\alpha}(\mathbf{r},t) - \frac{1}{4\pi} \frac{\partial}{\partial t} \nabla_{\alpha} A_0(\mathbf{r},t). \quad (3.3.26)$$

In this case, $A_0(\mathbf{r},t)$ is determined by the Green function of the electrostatic problem:

$$\Delta G(1,2) = -4\pi\delta(1-2), \quad (3.3.27)$$

while all the nonstationary processes are described by $A_\alpha(\mathbf{r}, t)$ or the Green function:

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G_{\alpha\beta}(1, 2, \tau) = -4\pi\delta_{\alpha\beta}\delta(1-2)\delta(\tau). \quad (3.3.28)$$

It is easy to show that the Fourier image of $G_{\alpha\beta}(1, 2, \omega)$ satisfies both Eq. (3.3.14) and gauge condition

$$\frac{\partial}{\partial x_\alpha^{(1)}} G_{\alpha\beta}(1, 2, \omega) = 0. \quad (3.3.29)$$

It is obvious that being a function of frequency $G_{\alpha\beta}(1, 2, \omega)$ has all specific features discussed above.

Substituting Eq. (3.3.24) into Eq. (3.3.26), we can write

$$\begin{aligned} A_\alpha(\mathbf{r}, t) = & \frac{e}{c} \int d(2) \int_{-\infty}^t dt' G_{\alpha\beta}(1, 2, t-t') \\ & \times \left\{ j_\beta(2, t') - \nabla_\beta^{(2)} \int d(3) G(2, 3) \frac{\partial}{\partial t'} \rho(3, t') \right\}. \end{aligned} \quad (3.3.30)$$

Now, using the continuity equation and then integrating over the variables of the charges (3) by parts:

$$\begin{aligned} \int d(3) G(2, 3) \frac{\partial \rho}{\partial t'} &= - \int d(3) G(2, 3) \operatorname{div} \mathbf{j}(3, t') \\ &= \int d(3) j_\beta(3, t') \nabla_\beta^{(3)} G(2, 3), \end{aligned}$$

we can rewrite Eq. (3.3.30) as

$$A_\alpha(\mathbf{r}, t) = \frac{e}{c} \int d(2) \int_{-\infty}^t dt' G_{\alpha\beta}^t(1, 2, t-t') j_\beta(2, t'), \quad (3.3.31)$$

where

$$G_{\alpha\beta}^t(1, 2, \tau) = \int d(3) G_{\alpha\lambda}(1, 3, \tau) \{ \delta_{\lambda\beta} \delta(2-3) - \nabla_\lambda^{(3)} \nabla_\beta^{(2)} G(3, 2) \} \quad (3.3.32)$$

is the transverse part of the Green function $G_{\alpha\beta}(1, 2, \tau)$. In this form the gauge condition (3.3.29) is obviously satisfied.

Let us present one more form of Eq. (3.3.30) that also seems useful. We start with the identity

$$\int d(2) G_{\alpha\beta}(1,2,\tau) \nabla_{\beta}^{(2)} \frac{\partial A_0}{\partial t'} = \int d(2) \left\{ \frac{\partial}{\partial x_{\beta}^{(2)}} \left[G_{\alpha\beta}(1,2,t-t') \frac{\partial A_0}{\partial t'} \right] - \frac{\partial A_0}{\partial t'} \frac{\partial}{\partial x_{\beta}^{(2)}} G_{\alpha\beta}(1,2,t-t') \right\}. \quad (3.3.33)$$

Due to Eq. (3.3.29) the second addend in this expression becomes zero, while the first one can be transformed into a surface integral

$$\int d\Sigma_2 \frac{\partial A_0}{\partial t'} G_{\alpha\beta}(1,2,t-t') \mathbf{n}_{\beta}(2), \quad (3.3.34)$$

where the integration is performed over the surface of electrodes surrounding the beam and $\mathbf{n}$ is a unit vector of the normal to this surface. Substitution of Eq. (3.3.34) into Eq. (3.3.30) yields

$$A_{\alpha}(\mathbf{r},t) = A_{\alpha}^j(\mathbf{r},t) + A_{\alpha}^{\Sigma}(\mathbf{r},t), \quad (3.3.35)$$

where

$$A_{\alpha}^j(\mathbf{r},t) = \frac{e}{c} \int d(2) \int_{-\infty}^t dt' G_{\alpha\beta}(1,2,t-t') j_{\beta}(2,t'), \quad (3.3.36)$$

$$A_{\alpha}^{\Sigma}(\mathbf{r},t) = \frac{e}{c} \int d\Sigma_2 \int_{-\infty}^t dt' G_{\alpha\beta}(1,2,t-t') \mathbf{n}_{\beta}(2) \frac{\partial A_0}{\partial t'}. \quad (3.3.37)$$

In particular, in a pipe with the ideally conductive walls the boundary condition for $A_0(\mathbf{r},t)$ can be chosen as follows:

$$A_0(\mathbf{r},t) |_{\Sigma} = 0.$$

This yields $A^{\Sigma} = 0$, while $A = A^j$. However, if the vacuum chamber with ideally conductive walls has gaps or holes, A^{Σ} does not vanish, which describes radiation at these obstacles.

With Eq. (3.3.31) the Lagrangian describing the interaction of a particle with induced fields can be written in the form

$$L(1,t) = Ne^2 \int d\Gamma_2 \int_0^{\infty} d\tau G'(1,2,\tau) f(2,t-\tau) - eA_0(\mathbf{r},t). \quad (3.3.38)$$

Here, the second term describes the interaction of particles with quasistatic induced field. Therefore, it can cause significant perturbation of trajectories only in the case when particle tunes approach machine resonances. In the opposite case the second term in Eq. (3.3.38) can be neglected and thus, Eq. (3.3.38) gets the form coinciding with that of Eq. (11.3). The function G' in Eq. (3.3.38) is defined by the transverse Green function from Eq. (3.3.32) only:

$$G'(1,2,\tau) = \frac{v_{\alpha}^{(1)} v_{\beta}^{(2)}}{c^2} G'_{\alpha,\beta}(1,2,\tau).$$

One of the most propagated is the case when the beam interacts with low-dissipative, resonant elements. It can be, for instance, the accelerating, or some other cavity, that forms the vacuum chamber for any technical purpose. If the dissipation in this element is sufficiently low, it becomes reasonable to describe the induced field in terms of so-called field oscillators. Assuming the Coulomb gauge condition for the vector potential, we consider only the equation describing nonstationary processes:

$$\Delta A_{\alpha} - \frac{1}{c^2} \frac{\partial^2 A_{\alpha}}{\partial t^2} = -\frac{4\pi}{c} J_{\alpha}(\mathbf{r}, t). \quad (3.3.39)$$

Here, the current $J_{\alpha}(\mathbf{r}, t)$ is determined by Eq. (3.3.26). Suppose we know the eigenfunctions of the homogeneous electrodynamic problem $\mathbf{A}_k$. Neglecting the dissipation in the cavity, they satisfy the equation

$$\Delta \mathbf{A}_k + \frac{\omega_k^2}{c^2} \mathbf{A}_k = 0, \quad (3.3.40)$$

with real eigenfrequencies ω_k , the normalization condition

$$\int d^3r \mathbf{A}_k(\mathbf{r}) \mathbf{A}_{k'}^*(\mathbf{r}) = 4\pi \delta_{kk'}, \quad (3.3.41)$$

the gauge condition $\text{div} \mathbf{A}_k = 0$, and corresponding boundary conditions on electrodes. Then solution of Eq. (3.3.39) can be found expanding $\mathbf{A}(\mathbf{r}, t)$ into a series of eigenfunctions $\mathbf{A}_k$

$$\mathbf{A}(\mathbf{r}, t) = c \sum_k \mathbf{A}_k(\mathbf{r}) Q_k(t). \quad (3.3.42)$$

Substituting this expression in Eq. (3.3.39) we obtain

$$\sum_k \mathbf{A}_k(\mathbf{r}) (\ddot{Q}_k + \omega_k^2 Q_k) = 4\pi \mathbf{J}(\mathbf{r}, t).$$

Integrating this equation over the volume, we obtain

$$\ddot{Q}_k + \omega_k^2 Q_k = e J_k(t) = e \int d^3r \mathbf{J}(\mathbf{r}, t) \mathbf{A}_k^*(\mathbf{r}). \quad (3.3.43)$$

Eq. (3.3.43) describes the excitation of undamped oscillations of Q_k by a driving force. The damping of these oscillations due to finite conductivity of the cavity walls, or the radiation from its open ends, generally is taken into account through the boundary conditions. If this dissipation is small, it mainly results in

the modification of eigenfrequencies ω_k , which now gets small damping decrement

$$\omega_k \rightarrow \omega_k - i\lambda_k, \quad \lambda_k \ll \omega_k.$$

For the phenomenological description of this damping we can simply add in the left-hand side of Eq. (3.3.43) the term $2\lambda_k \dot{Q}_k$. Hence instead of Eq. (3.3.43) we shall write

$$\ddot{Q}_k + 2\lambda_k \dot{Q}_k + \omega_k^2 Q_k = eJ_k(t). \quad (3.3.44)$$

The driven solution of this equation reads

$$Q_k(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} Q_k(\omega),$$

$$Q_k(\omega) = \frac{eJ_k(\omega)}{\omega_k^2 - 2i\omega\lambda_k - \omega^2}. \quad (3.3.45)$$

The function $|Q_k(\omega)|$ has a maximum when $\omega = \omega_k$ with a width $\Delta\omega \simeq \lambda_k$. It proves that representation of a field in terms of a set of oscillators is reliable, at least for the case when $\omega_k \gg \lambda_k$. Using the condition $\omega_k \gg \lambda_k$, we can also rewrite Eqs. (3.3.45) and (3.3.42) as

$$Q_k(\omega) = \frac{eJ_k(\omega)}{\omega_k^2 - (\omega + i\lambda_k)^2}, \quad (3.3.46)$$

$$\mathbf{A}(\mathbf{r}, t) = c \sum_k \mathbf{A}_k(\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{eJ_k(\omega) e^{-i\omega t}}{\omega_k^2 - (\omega + i\lambda_k)^2}. \quad (3.3.47)$$

The part of the Lagrangian responsible for the interaction of a particle with these nonstationary induced fields reads

$$L = Ne^2 \sum_k \mathbf{v} \mathbf{A}_k(\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega_k^2 - (\omega + i\lambda_k)^2} \int d^3r' \mathbf{J}(\mathbf{r}', \omega) \mathbf{A}_k^*(\mathbf{r}'). \quad (3.3.48)$$

By the comparison of Eqs. (3.3.47) and (3.3.30) we can find the expansion $G_{\alpha\beta}(1, 2, \omega)$ in terms of field oscillators:

$$G_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega) = c \sum_k \frac{A_k^\alpha(\mathbf{r}) A_k^{\beta*}(\mathbf{r}')}{\omega_k^2 - (\omega + i\lambda_k)^2}. \quad (3.3.49)$$

Substituting this expression into Eq. (2.5.9), and then making use of Eq. (3.3.7) we get an expression defining the longitudinal coupling impedance of a cavity,

$$\frac{iZ_n(\omega)}{\omega} = 2\pi R_0^2 \sum_k \frac{|A_k^\vartheta(\mathbf{r}_0)_n|^2}{\omega_k^2 - (\omega + i\lambda_k)^2}. \quad (3.3.50)$$

The last expression can be rewritten as

$$Z_n(\omega) = 2\pi R_0^2 \sum_k \frac{Q_k}{\omega_k} \frac{|A_{kn}^\vartheta|^2}{1 + iQ_k \left(\frac{\omega_k}{\omega} - \frac{\omega}{\omega_k} \right)},$$

where $Q_k = \omega_k / 2\lambda_k$ is the quality factor of the cavity. Comparing this expression with the impedance of the resonant circuit

$$Z_n(\omega) = \frac{Z_k}{1 + iQ_k \left(\frac{\omega_k}{\omega} - \frac{\omega}{\omega_k} \right)},$$

with Z_k so-called shunt impedance of the resonator, we obtain the formula connecting $|A_k^\vartheta|^2$ and Z_k

$$|A_{kn}^\vartheta|^2 = \left| \int_{-\infty}^{\infty} \frac{ds}{\Pi} \tau A_k[\mathbf{r}_0(s), s] \exp(-ins/R_0) \right|^2 = \frac{\omega_k}{2\pi Q_k} Z_k. \quad (3.3.51)$$

In the general case calculation of eigensolutions of A_k or Green functions for the elements interacting with the beam, runs into considerable difficulties. As a rule, one has to make some additional simplifying assumptions. As they are usually of a special character, it is reasonable that we discuss them with a particular calculation as an example.

3.4. THE SUM OF DECREMENTS FOR COLLECTIVE BEAM MODES

As was stated in §3.1, the calculation of eigenfrequencies of beam coherent oscillations requires the calculation of a spectrum of eigenvalues of a rather complicated integral equation of the form of Eq. (3.1.11). In many cases of practical interest this presents serious mathematical difficulties even for numerical calculation. Under these conditions it is useful to analyze the values that show the general properties of the spectrum, but do not require the direct solution of the integral equation. The simplest value of this kind is the sum of decrements of coherent oscillations in the absence of a frequency spread.

We remind you that for a system of linear coupled oscillators the Jacobi theorem is valid. If the oscillators are described by the system of equations

$$\frac{d}{dt} \hat{X} = \hat{M} \hat{X}, \quad \hat{X} = \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \\ \vdots \end{pmatrix},$$

then the change in the phase space (or Wronskian) of the system in time is determined by the trace of matrix $\hat{M}$,

$$\Delta \Gamma \propto \exp \left[- \int dt \text{Sp} \hat{M} \right]$$

(i.e., by the sum of decrements of those oscillators). Since the trace of a matrix is invariant with respect to the orthogonal transformations, the sum of decrements does not depend on the oscillator coupling. A statement of this kind is usually called a theorem on the sum of decrements.

Let us consider from this point of view a system consisting of field oscillators and modes of collective beam oscillations. Let the field oscillators have some natural friction λ_k . Strictly speaking, the theorem on the sum of decrements can be stated only for such a closed system. In this case, it is obvious that the sum of decrements of the total system is equal to the sum of decrements of the field oscillators.

In practice, however, the main interest presents not the total sum of decrements, but the sum of decrements of those modes that contain the main part of energy of the collective motion of the beam. A general statement on the sum of these decrements can be found out for a nonresonant interaction of the beam with its environment.³⁰

Let us proceed from Eq. (3.1.11) written for a low- Q -value element ($\lambda_k \gg \omega_s$). Since the sum of characteristic numbers of the integral equation is equal to the trace of its kernel,⁴² the sum of coherent frequency shifts $\Delta \omega_\Sigma$ reads

$$\Delta \omega_\Sigma = N e^2 \sum_{m,n} \int d\Gamma \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) G_{mn}(1, 1, \omega_{mn}). \quad (3.4.1)$$

The real part of this expression $G(\omega) = G'(\omega) + iG''(\omega)$ obviously vanishes ($\Delta \omega_m = -\Delta \omega_{-m}$), while the imaginary one (with an opposite sign) determines the sum of decrements of the collective modes of the beam

$$\begin{aligned} \delta_\Sigma &= -\text{Im} \Delta \omega_\Sigma = -N e^2 \sum_{m,n} \int d\Gamma \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) G''_{mn}(1, 1, \omega_{mn}) \\ &= N e^2 \sum_{m,n} \int d\Gamma f_0 \mathbf{m} \frac{\partial}{\partial \mathbf{I}} G''_{mn}(1, 1, \omega_{mn}). \end{aligned} \quad (3.4.2)$$

This expression obviously gives the sum of decrements for both bunched and coasting beams. Though we started with an equation for uncoupled modes, Eq. (3.4.2) obviously gives the sum of decrements of coupled modes. In practice this can be important for the case of synchrotron and synchrobetatron coherent oscillations, when due to typically small values of the synchrotron frequency modes become coupled at reasonably high beam currents.

For known distribution functions and G''_{mn} the right-hand part in Eq. (3.4.2) can be obtained by direct calculation. But in our case we are more interested in the interpretation of the expression obtained.

Note that Eq. (3.4.2) for δ_Σ/N coincides with $-dS^{(1)}/dt$ in Eq. (2.5.36). But the value $-dS^{(1)}/dt$ gives the local sums of decrements of the particles interacting with the same system independently from one another. In view of

$$\frac{dS^{(1)}}{dt} = \langle \partial \dot{I}_\alpha / \partial I_\alpha \rangle,$$

we come to the statement formulated in Ref. 30: the sum of decrements of normal coherent oscillations of a beam, interacting with a low-Q system, is equal to the sum of decrements of individual particles interacting with the same system independently from each other.

$$\delta_\Sigma = N \langle -\partial \dot{I}_\alpha / \partial I_\alpha \rangle = - \sum_{a=1}^N \langle \partial \dot{I}_\alpha^a / \partial I_\alpha^a \rangle, \quad (3.4.3)$$

In fact, both Eq. (3.4.1) and Eq. (3.4.3) can be used for the direct calculations of the sum of decrements δ_Σ .

One might find the following circumstance to be unusual. The left-hand side of Eq. (3.4.3) contains the value related to the processes suffering no changes in the beam phase space volume. On the contrary, its right hand side is exactly the logarithmic rate of the phase space volume of the beam, interacting with a dissipative system. As a matter of fact, it is not contradictory. In §2.4 we saw that an equation of the form of Eq. (3.1.1) describes both coherent oscillations and coherent fluctuations of a beam (i.e., coherent oscillations generated by the particle thermal motion). Due to interaction the particles and these coherent fluctuations exchange their energies. This effect is reflected by Eq. (3.4.3) which thus indicates the mutual relations between collective and kinetic processes in a beam.

Presented calculations do not take into account Landau damping of coherent modes. Chapter 8 will show that the formulation of a similar statement for a beam with a nonzero frequency spread requires that the calculations should be made beyond the self-consistent field approximation.

In practice the question of finding the total sum of decrements δ_Σ arises in problems of damping coherent oscillations by external systems. For applying the theorem under consideration to these problems it can be formulated in a more narrow and practical sense. For collective interaction, the only area of practical

interest is the region of low frequencies determined by characteristic frequencies of particle motion in the machine and by geometric dimensions of the vacuum chamber elements and those of a beam. In this case the higher modes of excitations $|m_x| \gg 1, |m_z| \gg 1$ are damped mainly due to frequency spread and (probably) due to the beam cooling. Therefore, of interest in our case is the sum of decrements not of all the modes, but only of those that are not damped by the frequency spread (or cooling). The sums of decrements of these modes can be calculated using Eqs. (3.4.2), or (3.4.3), if we left in the right-hand sides of these equations only those field oscillators which efficiently interact with these low-frequency modes.

An important feature of the sum of decrements of collective modes is the fact that for a beam interacting with a low- Q value system $\lambda_k \gg \omega_s$, the sum of decrements weakly depends on the arrangement of normal particle oscillations with respect to the equilibrium trajectory. In particular, one can state that the sum of decrements of dipole modes is independent of radial-longitudinal coupling.

We can illustrate this fact by the following simple example. Let us consider the single-turn interaction of the bunched beam with the low- Q system, where the energy loss of a particle has the radial gradient. For the sake of simplicity we adopt here the description of the longitudinal induced fields in terms of the coupling impedances. In the case of interest instead of Eq. (3.2.28) we have to write the relation between $E_\delta(n, \omega)$ and $f_{n, \omega}$ taking into account the radial modulation of the coupling impedance:

$$E_\delta(n, \omega) = -\frac{e\omega_s}{\Pi} \int d\Gamma Z_n(x, \omega) f_{n, \omega}. \quad (3.4.4)$$

Now, using Eqs. (3.2.9) and (3.1.29), we can write the general integral equation for synchrotron coherent oscillations $m_x = m_z = 0$ in the form

$$\begin{aligned} \Delta\omega_m f_m = & \frac{Ne^2\omega_s}{2\pi} m_c \frac{\partial f_0}{\partial I_c} \int_{-\infty}^{\infty} dn J_{m_c}(n\varphi_c) \\ & \times \int_0^{\infty} d\varphi'_c \varphi'_c \left\{ \frac{iZ_n(x, n\omega_s + m_c\omega_c)}{n + m_c\nu_c} \exp(in\varphi_c \cos\psi_c) \right\}_{m_c} f_m. \end{aligned} \quad (3.4.5)$$

For more simplification we assume also that $Z_n(\omega)$ is pure resistive impedance, which linearly depends on x :

$$Z_n(x, \omega) = \left(\frac{\partial Z'}{\partial x} \right)_s x = \left(\frac{\partial Z'}{\partial x} \right)_s \eta \frac{\Delta p}{p},$$

and that for typical frequencies

$$\left| \frac{\partial Z'}{\partial \omega} \right| \ll \left| \frac{Z'}{\omega} \right|.$$

Then, harmonics over synchrotron phases in Eq. (3.4.5) can be easily calculated. Taking into account that

$$I_{m_c} = [(\Delta p/p) \exp(in\varphi_c \cos \psi_c)]_{m_c} = \int_0^{2\pi} \frac{d\psi_c}{2\pi} \frac{\Delta p}{p} \exp(in\varphi_c \cos \psi_c - im\psi_c),$$

$$I_{m_c} = - \int_0^{2\pi} \frac{d\psi_c}{2\pi} \frac{v_c}{\alpha_p} \varphi_c \sin \psi_c \exp(in\varphi_c \cos \psi_c - im\psi_c)$$

$$= \frac{m_c v_c}{\alpha_p n} J_{m_c}(n\varphi_c),$$

and neglecting in Eq. (3.4.5) the higher powers of $m_c v_c$, one can easily obtain, instead of Eq. (3.4.5),

$$\begin{aligned} \Delta \omega_m f_m = & i \frac{Ne^2 \omega_s}{2\pi} \frac{m_c^2 v_c}{\alpha_p} \frac{\partial f_0}{\partial I_c} \eta \int_{-\infty}^{\infty} \frac{dn}{n^2} \left(\frac{\partial Z'_n}{\partial x} \right)_s \\ & \times J_{m_c}(n\varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n\varphi'_c) f_m, \quad |m_c v_c| \ll 1. \end{aligned} \quad (3.4.6)$$

The substitutions $u = \sqrt{\varphi^2/2}$,

$$\Delta \omega_m = -i\Lambda \frac{Ne^2 \omega_s}{2\pi p} \frac{m_c^2}{R_0}, \quad (3.4.7)$$

and

$$f_m = \sqrt{-\frac{\partial f_0}{\partial u}} \chi_m$$

transform this equation into the following integral equation with symmetric kernel:

$$\begin{aligned} \Lambda \chi_m = & \eta \int_{-\infty}^{\infty} \frac{dn}{n^2} \left(\frac{\partial Z'_n}{\partial x} \right)_s \int_0^{\infty} du' K_m(u, u') \chi_m(u'), \\ K_m(u, u') = & \sqrt{\frac{\partial f_0}{\partial u} \frac{\partial f_0}{\partial u'}} J_{m_c}(n\sqrt{u}) J_{m_c}(n\sqrt{u'}). \end{aligned} \quad (3.4.8)$$

Its kernel will be also the positively defined one^{*} and therefore all its eigenvalues Λ will be real, positive numbers if

$$\int_{-\infty}^{\infty} dn \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s > 0. \quad (3.4.9)$$

Due to Eq. (3.4.7) the last condition yields the criterion when synchrotron coherent oscillations will decay. It has obvious physical sense. The interaction with electrodes will damp coherent synchrotron oscillations provided the coherent energy losses will grow, when the amplitude of oscillations increases. For our case this will take place if condition (3.4.8) is satisfied.

The sum of decrements of modes near given m_c obviously reads

$$\delta_{\Sigma}^{(m)} = -\omega_s \frac{Ne^2 \omega_s}{2\pi p v} m_c^2 \int_0^{\infty} du \frac{\partial f_0}{\partial u} \int_{-\infty}^{\infty} \frac{dn}{n^2} \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s J_{m_c}^2(n \sqrt{u}). \quad (3.4.10)$$

Performing here the summation over m_c , we finally obtain[†]

$$\delta_{\Sigma}^{(c)} = \sum_{m_c=-\infty}^{\infty} \delta_{\Sigma}^{(m)} = \omega_s \frac{Ne^2 \omega_s}{2\pi p v} \int_{-\infty}^{\infty} dn \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s. \quad (3.4.11)$$

Now, to prove the statement we have to calculate the sum of decrements of dipole horizontal coherent oscillations ($m_x = \pm 1$). Since we know the longitudinal electric field in the form of Eq. (3.4.4), it is more convenient to write the general integral equation, using the identity

^{*}The kernel $K(x, x')$ is called positively defined (see, for instance, Ref. 42) if for any smooth function $w(x)$

$$\int_0^{\infty} dx dx' w(x) K(x, x') w(x') > 0.$$

[†]The sum over m_c in Eq. (3.4.11)

$$S_2 = \sum_{m=-\infty}^{\infty} m^2 J_m^2(x),$$

as well as similar expressions, can be easily calculated using the expansion

$$\exp(ix \cos \psi) = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\psi}.$$

Direct calculation yields

$$S_2 = \int_0^{2\pi} \frac{d\psi}{2\pi} x^2 \sin^2 \psi = \frac{x^2}{2}.$$

$$\dot{I}_a = \frac{\partial I_a}{\partial \mathbf{p}} \vec{\mathcal{F}},$$

which yields

$$(\omega - m_x \omega_x - m_c \omega_c) f_m = -i \overline{\left(\frac{\partial I_x}{\partial \Delta p} e E_\vartheta \right)_{m_x, m_c}} \frac{\partial f_0}{\partial I_x}. \quad (3.4.12)$$

For the sake of simplicity we shall do the calculations assuming the smoothed focusing approximation when

$$I_x = \frac{p v_x a_x^2}{2R_0}, \quad \frac{\partial I_x}{\partial \Delta p} = -\frac{v_x \eta}{R_0} a_x \cos \psi_x. \quad (3.4.13)$$

The resulting integral equation reads ($\Delta \omega_m = \omega - m_x \omega_x - m_c \omega_c$):

$$\begin{aligned} \Delta \omega_m f_m = & -i \frac{\partial f_0}{\partial I_x} \frac{N e^2 \omega_s}{2 \Pi p} \sqrt{I_x} \int_{-\infty}^{\infty} dn \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s J_{m_c}(n_1 \varphi_c) \\ & \times \int d\Gamma' \sqrt{I'_x} J_{m_c}(n_1 \varphi'_c) f_m \end{aligned}$$

or, using $f_0 = F_0(I_x) \rho(\varphi_c)$ and $f_m = \chi_m(\varphi_c) \sqrt{I_x} \partial F_0 / \partial I_x$,

$$\begin{aligned} \Delta \omega_m \chi_m = & i \rho(\varphi_c) \frac{N e^2 \omega_s}{2 \Pi p} \int_{-\infty}^{\infty} dn \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s J_{m_c}(n_1 \varphi_c) \\ & \times \int_0^\infty d\varphi'_c \varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c). \end{aligned} \quad (3.4.14)$$

Direct calculation of the trace of the kernel in Eq. (3.4.14) yields the total sum of decrements ($\delta = -\text{Im} \omega$) of horizontal betatron and synchrobetatron modes:

$$\delta_\Sigma^{(x)} = \sum_{m_c=-\infty}^{\infty} (\delta_{1,m_c} + \delta_{-1,m_c}) = -\omega_s \frac{N e^2 \omega_s}{2 \pi p v} \int_{-\infty}^{\infty} dn \left(\eta \frac{\partial Z'_n}{\partial x} \right)_s. \quad (3.4.15)$$

Now, the comparison of Eqs. (3.4.11) and (3.4.15) gives

$$\delta_\Sigma = \delta_\Sigma^{(c)} + \delta_\Sigma^{(x)} = 0. \quad (3.4.16)$$

The independence of the total sum of decrements on the coupling of horizontal and longitudinal oscillations becomes especially clear if δ_Σ is expressed in terms of the particle radiation reaction force in the external system

$$\vec{\mathcal{F}}(\mathbf{p}, \mathbf{r}, t) = e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right). \quad (3.4.17)$$

Making use of Eqs. (3.4.3) and (2.5.41), we obtain

$$\delta_{\Sigma} = -N \langle \text{div}_p \vec{\mathcal{F}} \rangle = -N \left[\frac{\partial \mathcal{F}_x}{\partial p_x} + \frac{\partial \mathcal{F}_z}{\partial p_z} + \frac{\partial \mathcal{F}_{\parallel}}{\partial \Delta p} \right]. \quad (3.4.18)$$

In the second equality the derivatives should be calculated on the equilibrium trajectory.

Note that in line with (3.4.18) the system of the cooling of the beam should provide (at least "in average") the stability of the coherent oscillations of the beam. It is also noteworthy that in the general case partial decrements as well as their sum Eq. (3.4.2) are not proportional to the total power of the particle energy losses due to radiation into the beam environment

$$P_i = \langle \mathbf{v} \cdot \vec{\mathcal{F}} \rangle,$$

as occurs, for instance, in the case of synchrotron radiation. In the latter case the main contribution to the decrements is made by high frequencies $\omega \sim \gamma^3 \omega_s$; therefore, the radiation reaction force is determined only by the particle trajectory and has the opposite sign of the velocity $\vec{\mathcal{F}} = -\mathbf{v}(P_i/c^2)$.

The main contribution into decrements of collective motion gives rather low-frequency oscillations of induced fields. Typically these present an example of the so-called transient radiation, when the fields are induced due to variations of electrodynamic properties of the vacuum chamber along the orbit. In this case, the force of radiation reaction will be determined to a great extent by the geometry, dimensions, and other similar characteristics of the device interacting with the beam. Therefore, one can hardly assume it to be directed along the particle velocity. This fact can specifically be used for facilitating the damping coherent oscillations in particular degrees of freedom.

In cases when the signs of decrements of all modes near the given $\mathbf{m}$ coincide, the calculation of partial sums of decrements (as well as the value of coherent tune shifts) may be useful as an upper limit of the maximum decrements (increments) of coherent oscillations. Since the quasistatic field of the beam makes no contribution to sums of decrements, these sums converge. If α marks the modes near $\mathbf{m}$, this means that the decrements $\delta_{\mathbf{m},\alpha}$ as well as coherent frequency shifts decrease at least like $1/\alpha^2$, when α increases. Even if one assumes that this decrease occurs by the law $\delta(\alpha) \propto \delta(1)/\alpha$, which can take place with some simplifying assumptions about the geometry of induced fields, the difference between the maximum decrement δ_1 and δ_{Σ} will be determined only by a logarithmic factor. The value of this factor is usually easy to estimate taking into account physical considerations.

3.5. EVALUATION OF COLLECTIVE SPECTRA FROM THE MICROWAVE REGION

Calculation of a spectrum of coherent beam oscillations often requires one to solve an integral equation of the form

$$\Omega_m \chi_m = \rho(\varphi_c) \int_0^\infty dn \Phi_{mn} J_{m_c}(n\varphi_c) \int_0^\infty d\varphi'_c \varphi'_c J_{m_c}(n\varphi'_c) \chi_m(\varphi'_c). \quad (3.5.1)$$

This equation, for example, determines the spectrum of synchrotron oscillations of a bunched beam due to its single-turn interaction. Solution of other problems can also be reduced to solving a similar equation. Except for rare cases, the solution of Eq. (3.5.1) cannot be found in a general form. Therefore, of special importance here are approximation methods of calculating its spectrum.

For definiteness, let us consider synchrotron oscillations. Then $\rho(\varphi_c)$ is the distribution in amplitudes of synchrotron oscillations, $2R_0\varphi_b$ is the beam length, while Φ_{mn} as a function of n is determined on the azimuthal length θ_0 of the region of beam particle interaction through an external system. If φ_b and θ_0 differ considerably from one another and $\rho(\varphi_c)$ is a monotonous function concentrated near zero, the spectrum of Eq. (3.5.1) can be obtained by extrapolation out of the short-wave region $|n| \gg |m_c|/\theta_0$.³²

Indeed, substituting

$$C(x) = \int_0^\infty dy y J_{m_c}(xy) \chi_m(y), \quad x = n\varphi_b, \quad y = \varphi_c/\varphi_b, \quad (3.5.2)$$

let us transform Eq. (3.5.1) into the equation for momenta $C(x)$:

$$\Omega_m C(x) = \int_0^\infty dx' \rho(x, x') \Phi_m(x' \vartheta_0 / \varphi_b) C(x'). \quad (3.5.3)$$

Here,

$$\rho(x, x') = \int_0^\infty dy y \rho(y) J_{m_c}(xy) J_{m_c}(x'y), \quad (3.5.4)$$

and $\rho(y)$, by definition, starts to decrease from $y \simeq 1$. In the short-wave region $x, x' \gg m_c$ the product $J_{m_c}(xy) J_{m_c}(x'y)$, beginning from $y \propto \min\{m_c/x, m_c/x'\}$ is a fast-oscillating function. Therefore, factoring $\rho(y)$ at $y = 0$ outside the integral sign in Eq. (3.5.4) yields

$$\rho(x, x') \simeq \rho(0) \frac{\delta(x - x')}{x}, \quad x, x' \gg m_c. \quad (3.5.5)$$

In reality the “width” of the δ -function in Eq. (3.5.5) is of the order of $\Delta x \propto 1$. Taking this into account the spectrum Ω_m can be estimated in two param-

eter regions. The first is $\theta_0 \ll \varphi_b$. The function $\Phi(x)$ is a slowly varying function of x on the interval $\Delta x \propto 1$. Then substitution of Eq. (3.5.5) into (3.5.3) yields

$$\Omega_m(x) \simeq \rho(0) \frac{\Phi_m(x)}{x}, \quad x \gg m_c. \quad (3.5.6)$$

The second parameter region is $\theta_0 \gg \varphi_b$. The function $\Phi(x)$ is a fast oscillating function in the range $\Delta x \propto 1$. In this case the integral over x' can be estimated by substituting Φ_m within the interval $\Delta x \propto 1$ by

$$\overline{\Phi_m(x)} = \int_{\Delta x \simeq 1} d(\Delta x) \Phi_m(x + \Delta x),$$

then

$$\Omega_m(x) \simeq \rho(0) \frac{\overline{\Phi_m(x)}}{x}. \quad (3.5.7)$$

Combining Eqs. (3.5.6) and (3.5.7) we can write

$$\Omega_m(x) \simeq \frac{\rho(0)}{x} \begin{cases} \Phi_m(x), & \varphi_b \gg \vartheta_0, \\ \overline{\Phi_m(x)}, & \varphi_b \ll \vartheta_0. \end{cases} \quad (3.5.8)$$

The sum of eigenvalues of Eq. (3.5.3) is determined by the expression

$$\begin{aligned} \Sigma \Omega_m &= \int_0^\infty dx \rho(x, x) \Phi_m(x) \\ &= \int_0^\infty dy y \rho(y) \int_0^\infty dx J_{m_c}^2(xy) \Phi_m(x) \\ &= \int_{xy < m_c} dx dy y \Phi_m(x) \rho(y) \frac{(xy)^{2m_c}}{(m_c!)^2} + \rho(0) \int_{x \simeq m_c}^\infty dx \frac{\Phi_m(x)}{x}. \end{aligned} \quad (3.5.9)$$

It shows that the spectrum of short-wave excitations $x > m_c$ is adjacent to the spectrum of long-wave ones $x < m_c$. Therefore, eigenvalues Ω_m for long-wave oscillations can be obtained by the extrapolation of Eq. (3.5.8) in the range of $x \simeq m_c$:

$$\Omega_m \simeq \frac{\rho(0)}{m_c} \begin{cases} \Phi_m(m_c), & \varphi_b \gg \vartheta_0, \\ \overline{\Phi_m(m_c)}, & \varphi_b \ll \vartheta_0. \end{cases} \quad (3.5.10)$$

In some cases Eq. (3.5.3) can be transformed into a differential one. In particular, if $\rho(y)$ is the Lorentz distribution,

$$\rho(y) = 1/(1+y^2),$$

Eq. (3.5.3) gets the form

$$\begin{aligned} \Omega_m C_m(x) = & K_{m_c}(x) \int_0^x dx' \Phi_m(x) C_m(x') I_{m_c}(x') \\ & + I_{m_c}(x) \int_x^\infty dx' \Phi_m(x') C_m(x') K_{m_c}(x'), \end{aligned} \quad (3.5.11)$$

and $K_m(x)$ and $I_m(x)$ are the Bessel function of the 3rd kind and the Bessel function of imaginary argument, respectively. Double differentiation of Eq. (3.5.10), as well as the use of equations for $K_m(x)$ and $I_m(x)$,¹⁴ yields the differential equation

$$\left\{ \frac{1}{x} \frac{d}{dx} \left(x \frac{d}{dx} \right) + \frac{\Phi_m(x)}{\Omega_m x} - 1 - \frac{m_c^2}{x^2} \right\} C_m(x) = 0, \quad (3.5.12)$$

which is equivalent to Eq. (3.5.3). Substituting in Eq. (3.5.12)

$$C_m(x) = \frac{w(x)}{x^{1/2}}$$

it can be transformed to the normal form

$$w'' + k^2(x)w = 0, \quad k^2(x) = \frac{\Phi_m(x)}{\Omega_m x} - 1 - \frac{m_c^2 - \frac{1}{4}}{x^2}. \quad (3.5.13)$$

Although the equation obtained is just as complicated as Eq. (3.5.3), in some cases it provides for the oscillation spectra calculation.

As an illustration, we can use Eq. (3.5.13) to find the increments of synchrobetatron modes discussed in the previous section in Eqs. (3.4.14). In this case

$$\Phi_m = 2i \frac{Ne^2 \omega_s}{\Pi p} \left(\eta \frac{\partial Z'_n}{\partial r} \right)_s. \quad (3.5.14)$$

Neglecting here the dependence of Z' on n , which corresponds to the interaction with a rather short device, and substituting in Eq. (3.5.13) $z = x/2$, we can transform this equation into

$$\frac{d^2 w}{dz^2} + \left\{ -\frac{1}{4} + \frac{\lambda}{z} - \frac{m_c^2 - \frac{1}{4}}{z^2} \right\} w = 0, \quad \lambda = \frac{\Phi_m}{2\Omega_m}. \quad (3.5.15)$$

Its solution reads

$$w = \left(\frac{x}{2} \right)^{|m_c| + 1/2} e^{-x/4} F(|m_c| + \lambda + 1/2, 2|m_c| + 1, x/2), \quad (3.5.16)$$

where $F(\alpha, \gamma, x)$ is the confluent hypergeometric function.¹⁴ Eigenfunctions $w_{m_c, l}$ will not grow, when $x \rightarrow \infty$, if

$$|m_c| + \lambda + 1/2 = -l, \quad l = 0, 1, \dots$$

This equation yields the increments of modes in the form

$$\delta_{m_c, l} = -\omega_s \frac{Ne^2 \omega_s}{2\pi p v} \left(\eta \frac{\partial Z'}{\partial r} \right) \frac{1}{2(|m_c| + l) + 1}. \quad (3.5.17)$$

More generally, if the real and imaginary parts of $\Phi(x)$ significantly differ by the value, the coefficient $k^2(x)$ is a real value. In the region of $x \gg |m_c|$ eigenvalues Ω_m of the ground state can be estimated by the condition

$$k^2(x, \Omega_m) = 0$$

or

$$\Omega_m(x) \simeq \frac{\Phi_m(x)}{x}.$$

3.6. LANDAU DAMPING

Without the beam cooling, the frequency spread of the beam can stabilize its coherent oscillations due to Landau damping. The required frequency spread can be accounted for by several reasons. Here, for the sake of simplicity, we consider the cases when it is due to nonlinearity of betatron oscillations. In this section we also assume that the beam interacts with a nonresonant environment. The opposite case is somewhat different and will be considered in the next chapter.

For the oscillations of an arbitrary multipole number the dispersion equation [see Eq. (3.1.25)] has the form

$$1 = \frac{\Omega_m}{\mathcal{A}_m} \int d^2 \mathbf{I}_1 \frac{I_x^{|m_x|} I_z^{|m_z|} (\mathbf{m}_1 \partial F_0 / \partial \mathbf{I}_1)}{\omega - \mathbf{m}_1 \omega_1(\mathbf{I}_1)}. \quad (3.6.1)$$

Here, we use the same notations as in §3.1. Let us first transform this equation into some canonical form. Taking in Eq. (3.6.1) as a new variable

$$w = \mathbf{m}[\omega(\mathbf{I}_1) - \omega(0)] = \mathbf{m}_1 \Delta\omega_1(\mathbf{I}_1), \quad (3.6.2)$$

and defining an effective frequency distribution in a beam

$$g_m(w) = \int \frac{d^2\mathbf{I}_1}{\mathcal{A}_m} I_x^{|\mathbf{m}_x|} I_z^{|\mathbf{m}_z|} \left(\mathbf{m}_1 \frac{\partial F_0}{\partial I_1} \right) \delta[w - \mathbf{m}_1 \Delta\omega_1(\mathbf{I}_1)] \quad (3.6.3)$$

we can rewrite Eq. (3.6.1) as

$$\epsilon_m(\Delta\omega_m) = 1 - \Omega_m \int_{-\infty}^{\infty} dw \frac{g_m(w)}{\Delta\omega_m - w}, \quad \text{Im}\omega > 0, \quad (3.6.4)$$

where $\Delta\omega_m = \omega - \mathbf{m}_1 \omega_1(0)$.

As is known (see, for instance, Ref. 43), a universal solution of Eq. (3.6.4) can be found only provided the distribution $g_m(w)$ is close to the Gaussian one and has a width (i.e., the frequency spread in a beam), considerably exceeding the value of the coherent frequency shift Ω_m . In this case we may expect that, due to strong interference, induced fields will decay faster than Ω_m and the roots of Eq. (3.6.4) will occur in the lower half-plane of ω (i.e., the oscillations will decay). However, whereas Eq. (3.6.4) is written in the upper half-plane of ω , we can use it to find the frequencies of stable oscillations only after the analytic continuation of the function $\epsilon_m(\Delta\omega_m)$ to the lower half-plane of $\Delta\omega_m$. That can be done by the deformation of the integration contour in Eq. (3.6.4) in such a way that it always goes below the singularity of the integrand at $w = \Delta\omega_m$ (see Fig. 3.4; the so-called Landau contour). Calculation of the residue of the integrand in this simple pole results in

$$\epsilon_m(\Delta\omega_m) = 1 - \Omega_m \int_{-\infty}^{\infty} dw \frac{g_m(w)}{\Delta\omega_m - w} + 2\pi i \Omega_m g_m(\Delta\omega_m) = 0. \quad (3.6.5)$$

In some cases this analytic continuation is obvious. If, for example, $g_m(w)$ is the Lorentz distribution $g_m = (\Delta_m/\pi)/(w^2 + \Delta_m^2)$, the calculation of the integral in Eq. (3.6.4) yields

$$\epsilon_m = 1 - \frac{\Omega_m}{\Delta\omega_m + i\Delta_m}.$$

On the other hand, substitution of this $g_m(w)$ in Eq. (3.6.5) yields exactly the same result:

$$1 = \frac{\Omega_m}{\Delta\omega_m - i\Delta_m} - 2i\Omega_m \frac{\Delta_m}{\Delta\omega_m^2 + \Delta_m^2}$$

$$= \frac{\Omega_m}{\Delta\omega_m + i\Delta_m}.$$

For a Gaussian frequency distribution:

$$g_m(\omega) = \frac{1}{\sqrt{2\pi}|m\Delta\omega|} \exp(-\omega^2/2|m\Delta\omega|^2),$$

neglecting the values of the order of $|\Omega_m|/|m\Delta\omega|$, we rewrite Eq. (3.6.5) in the form

$$1 \simeq -i\sqrt{2\pi}\xi \exp\left(-\frac{y^2 - 2ixy - x^2}{2}\right),$$

$$\xi = \frac{\Omega_m}{|m\Delta\omega|} = \xi' + i\xi'', \quad y - ix = \frac{\Delta\omega'_m - i\delta_m}{|m\Delta\omega|}.$$

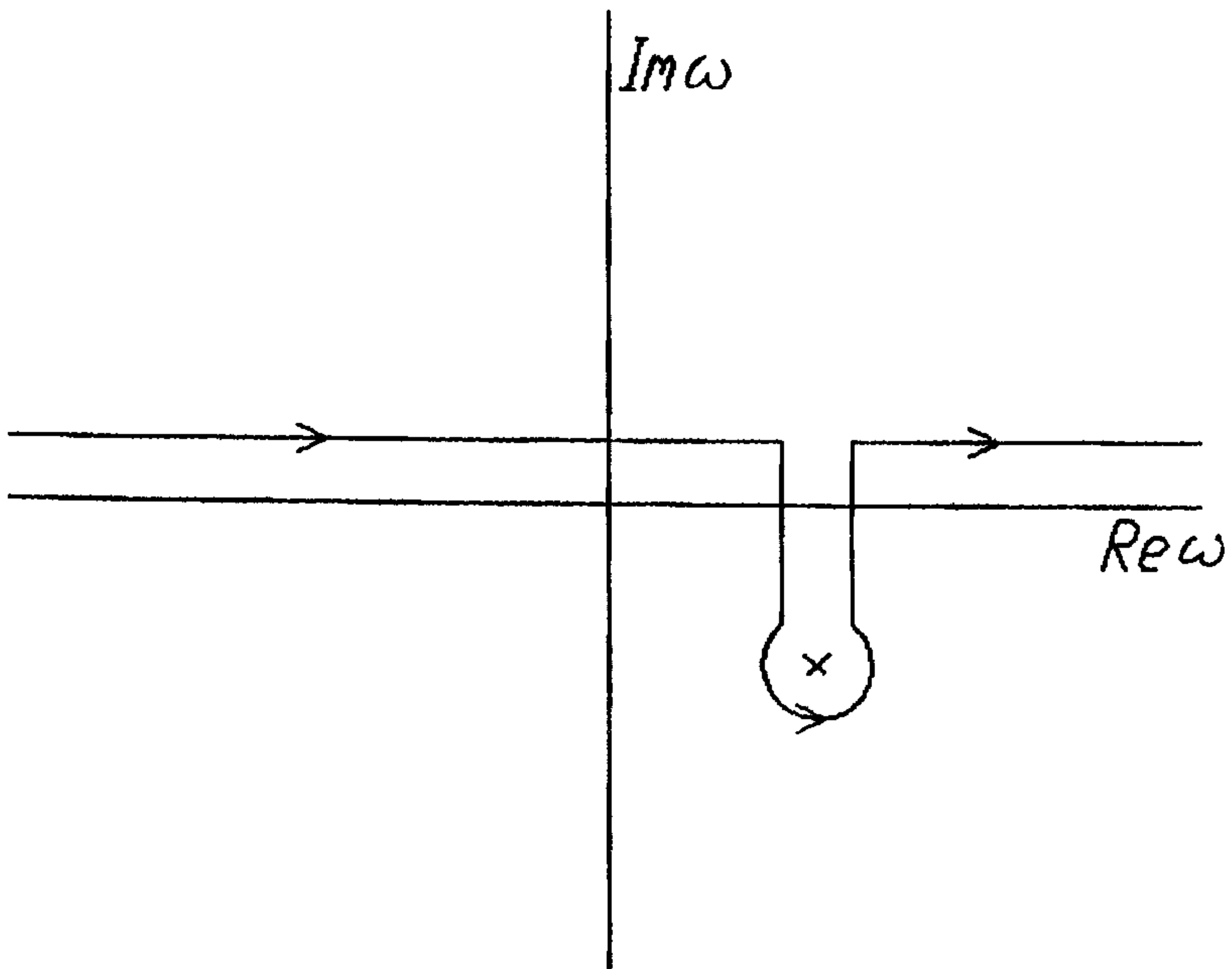


FIG. 3.4. The integration contour in Eq. (3.6.4) after the sifting of its roots in the lower half-plane of ω .

Selecting in the dispersion equation the real and imaginary parts, we can rewrite it as

$$\tan(2yx) = -\frac{\xi'}{\xi''}, \quad x^2 - y^2 = -2\Lambda, \quad \Lambda = \ln 2\pi. \quad (3.6.6)$$

By the order of value the smallest x obeying these equations can be estimated* by $x_{\min} \sim 1$, where

$$\Delta\omega_m \sim -i|m\Delta\omega|.$$

If the value of the coherent tune shift $|\Omega_m|$ is not small compared to $|m\Delta\omega|$ the oscillations may include unstable ones. In this case instead of calculation of the roots of Eq. (3.6.4), which typically requires numerical methods, it is more important to find out conditions when this equation has no solutions in the upper half-plane of ω and therefore, the oscillations are stable. With the given frequency distribution $g_m(\omega)$ this problem can be solved in a very general way. Since the function $\epsilon_m(\omega)$ has no singularities in the upper half-plane of ω , and $\epsilon(\omega) \rightarrow 1$, when $|\Delta\omega_m| \rightarrow \infty$, by the Cauchy theorem⁵ ϵ_m will not have these roots if it will have the zero phase advance when $\Delta\omega_m$ varies slightly above the real axis. In other words, the contour $-\infty < \text{Re}\Delta\omega_m < \infty$ after its transformation in the plane of the complex variable $\epsilon_m(\omega)$ [†] should not encircle the origin. This statement is usually called the Nyquist criterion. Apart from straightforward numerical calculations, one can use the Nyquist criterion to find out the more simplified criteria (such as the Z/n criterion discussed in §3.2) relevant for engineering estimations. We note, for instance, that the Nyquist criterion will be valid if for any point $\text{Im}\epsilon_m(\omega_\alpha) = 0$, where α marks solutions, one will have $\text{Re}\epsilon_m(\omega_\alpha) < 0$. This condition will be fulfilled beforehand, provided^{32,35}

$$1 - \pi \frac{|\Omega_m|^2}{\text{Im}\Omega_m} g_{\max} > 0, \quad (3.6.7)$$

where $g_{\max}$ is the maximal value of $g(\omega)$ over the whole interval of ω variation. Condition (3.6.7) means that the stability of coherent motion is obviously satis-

*Equation (3.6.6) can be solved exactly. (See, for instance, Ref. 43.) Namely, from the first equation we have

$$y = \frac{\phi_k}{2x}, \quad \phi_k = [k\pi - \arctan(\xi'/\xi'')], \quad k = 1, 2, \dots$$

Then, from the second equation,

$$x_k = \sqrt{\Lambda^2 + \frac{\phi_k}{4} - \Lambda}.$$

[†]Such a curve is called the hodograph of the function $\epsilon_m(\omega)$.

fixed when parameters of the external system are chosen such that the complex coherent tune shift Ω_m might get inside the circle

$$\left| \Omega_m - \frac{i}{2\pi g_{\max}} \right| < \frac{1}{2\pi g_{\max}}. \quad (3.6.8)$$

Violation of conditions (3.6.7) and (3.6.8) does not necessarily mean that oscillations become unstable. The conditions of coherent oscillation stability in the presence of frequency spread can be stipulated more precisely by studying the solution of Eq. (3.6.4) at the threshold of instability $\Delta\omega_m \rightarrow \Delta\omega_{th} + i\Delta$, $\Delta \rightarrow 0$, when it splits into the pair of equations

$$\begin{aligned} \frac{\text{Im}\Omega_m}{|\Omega_m|^2} &= \pi g_m(\Delta\omega_{th}), \\ \frac{\text{Re}\Omega_m}{|\Omega_m|^2} &= \mathcal{P} \int_{-\infty}^{\infty} dw \frac{g_m(w)}{\Delta\omega_{th} - w} = \mathcal{P}(\Delta\omega_{th}). \end{aligned} \quad (3.6.9)$$

With a given frequency distribution $[g_m(w)]$ Eqs. (3.6.9) in a parametric form give the stability diagram of coherent oscillation. By the definition, this is a curve in the plane of the complex variable Ω_m , separating the regions containing only stable and unstable solutions. The calculation of the decrements of oscillations near the threshold of instability ($\Delta N = N_{th} - N \ll N$; see in §3.2)

$$\delta_m = \frac{\Delta N}{N} \frac{(\partial \epsilon_m'' / \partial \omega)_{th}}{|(\partial \epsilon_m / \partial \omega)_{th}|^2}, \quad \epsilon_m = \epsilon_m' + i\epsilon_m'', \quad (3.6.10)$$

yields the position of the stable region relative to the boundary curve:

$$\Delta N (\partial \epsilon_m'' / \partial \omega)_{th} > 0. \quad (3.6.11)$$

Using Eqs. (3.6.9) this condition can be reduced to the form²⁶

$$\frac{\partial \mathcal{P}}{\partial \omega_{th}} \left\{ \frac{\text{Im}\Omega'_m}{|\Omega'_m|^2} - \pi g_m(\Delta\omega_{th}) \right\} > 0. \quad (3.6.12)$$

Here, Ω' is a point in the plane of the complex variable Ω_m outside the boundary curve, and $\Delta\omega_{th}$ is a corresponding point on the boundary curve.

The value of the threshold current of instability N_{th} and the coherent frequency shift at the threshold $\Delta\omega_{th}$ at given parameters of the external system and a given frequency spread are determined from the same system of equations (3.6.9), which we rewrite now in the form

$$\pi \frac{g_m(\Delta\omega_{th})}{\mathcal{P}(\Delta\omega_{th})} = \frac{\text{Im}\Omega_m}{\text{Re}\Omega_m},$$

$$N_{th} = \frac{\text{Im}\bar{\Omega}_m}{|\bar{\Omega}_m|^2} = \frac{1}{\pi g_m(\Delta\omega_{th})}, \quad (3.6.13)$$

where $\bar{\Omega}_m = \Omega_m/N$ is the coherent tune shift meant for one particle.

For further analysis it is necessary to specify the density $g_m(w)$. Let us consider, for example, the oscillation $\omega \simeq m_z \omega_z$. Let us also assume that the dependence of ω_z on I is due to the cubic nonlinearity of the guiding field. Then

$$\omega_z(I) = \omega_z(0) + \frac{\partial \omega_z}{\partial I_z} I_z + \frac{\partial \omega_z}{\partial I_x} I_x = \omega_z(0) + \alpha_{zz} I_z + \alpha_{zx} I_x.$$

The form of the frequency distribution function $g_m(w)$ essentially depends on which dimension of the beam (vertical or radial) determines the frequency spread. Let the frequency spread of vertical oscillations be determined by the vertical beam dimension

$$|\alpha_{zz}\langle I_z \rangle| \gg |\alpha_{zx}\langle I_x \rangle|.$$

In this case the dispersion equation can be written as

$$1 = \frac{\Omega_m}{m_z \alpha_{zz} \mathcal{A}_m} \int_0^\infty \frac{dw w^{|m_z|}}{t-w} \frac{\partial F_0}{\partial w}, \quad (3.6.14)$$

where $t = \Delta\omega_m/\alpha_{zz}$. In this case the density $g_m(w)$ is equal to

$$g_m(w) = \frac{w^{|m_z|}}{m_z \alpha_{zz} \mathcal{A}_m} \frac{\partial F_0}{\partial w} \theta(w),$$

$$\theta(w) = \begin{cases} 1, & w > 0, \\ 0, & w < 0. \end{cases} \quad (3.6.15)$$

In an opposite limiting case

$$|\alpha_{zz}\langle I_z \rangle| \ll |\alpha_{zx}\langle I_x \rangle|$$

the spread is determined by the radial dimension. In this case

$$g_m(w) = \frac{F(w)\theta(w)}{m_z \alpha_{zx}}. \quad (3.6.16)$$

Equation (3.6.15) clearly indicates that for the given mode $\{m\}$ the shape of the frequency distribution of the beam is determined not only by the particle distribution function over the amplitudes of oscillations, but also by the multipole number m_z . Note, in particular, that the effective width of distribution $g_m(w)$

grows approximately as $(m_z)^{1/2}$. Besides, $g_m(w)$ has at least one maximum that is located for monotonous $F_0(I)$ near $w = |m_z| \langle I_z \rangle$.

Also essential is the fact that $g_m(w)$ generally is not symmetrical with respect to positions of its maximums. Therefore, the stability boundary curve

$$\begin{aligned} \frac{\text{Im}\Omega_m}{|\Omega_m|^2} &= \frac{\theta(w)}{|m_z\alpha_{zz}|} \frac{w^{|m_z|}}{\mathcal{A}_m} \frac{\partial F_0}{\partial w}, \\ \frac{\text{Re}\Omega_m}{|\Omega_m|^2} &= \frac{\mathcal{P}(w)}{m_z\alpha_{zz}}, \end{aligned} \quad (3.6.17)$$

is not symmetrical with respect to the axis $\text{Im}\Omega_m$. (See Fig. 3.5.) In particular, it coincides with the axis $\text{Re}\Omega_m$ at

$$\text{Re}\Omega_m < \text{Re}\Omega_m^{tr} \frac{m_z\alpha_{zz}}{\mathcal{P}(0)}. \quad (3.6.18)$$

For these (generally, big) coherent frequency shifts the stabilization of coherent oscillations by Landau damping becomes impossible.

As is seen from Eqs. (3.6.17), with the change of the sign of α_{zz} to the opposite one the stability region is reflected with respect to the axis $\text{Im}\Omega_m$. Due to its asymmetry, after such a reflection the points from the stable region may occur closer to the boundary curve or even outside the stability diagram. The nonsymmetrical dependence of the threshold current on the lattice cubic nonlinearity has been observed, for instance, at the Cambridge Electron Accelerator facility.⁴⁴

In case of external spread the integral $\mathcal{P}(w)$ diverges logarithmically at $w \rightarrow 0$. Nevertheless, it affects only the value of the boundary coherent tune shift $\text{Re}\Omega_m^{tr}$ which cannot now be calculated using Eq. (3.6.18). However, qualitatively the results remain the same.

The value of the threshold current instability can be easily calculated for monotonous $F_0(I)$ in the case when $|\text{Im}\Omega_m| \gg |\text{Re}\Omega_m|$. In this case $g_m(\Delta\omega_{th}) \gg \mathcal{P}(\Delta\omega_{th})$ and, hence, the roots of the first equation in Eqs. (3.6.13) should be near $\bar{w}$ corresponding to the maximum of $g_m(w)$, that is, $|\omega_{th} - \bar{\omega}_m| \ll |m\Delta\omega|$. The value of the threshold current can be estimated by the formula

$$N_{th} \simeq \frac{1}{\pi g_{\max} \text{Im}\bar{\Omega}_m} \simeq \frac{|m\Delta\omega|}{\text{Im}\bar{\Omega}_m}, \quad (3.6.19)$$

where $\Delta\omega$ is the betatron oscillation frequency spread.

Substituting excitation increments into Eq. (3.6.19) one can obtain a formula for the beam threshold current interacting with surrounding elements. Most important, from a practical point of view, is the lowest value of the threshold current $N_{th}^{\min}$, as this value is detected experimentally. Therefore, in applications it is sufficient to find the maximal eigenvalues of integral equations Ω_m .

If the value of the coherent frequency shift is large ($|\operatorname{Re}\Omega_m| \gg |\operatorname{Im}\Omega_m|$), then $\mathcal{P}(\Delta\omega_{th}) \gg g_m(\Delta\omega_{th})$, which is valid only in case $|\omega_{th} - \omega_m| \gg |m\Delta\omega|$.

In this case $\mathcal{P}(\Delta\omega_{th}) \simeq 1/\Delta\omega_{th}$ and, according to Eq. (3.6.13), $\Delta\omega_{th} \simeq \operatorname{Re}\Omega_m$. Hence, we can calculate N_{th} from a transcendental equation:

$$\operatorname{Im}\bar{\Omega}_m = \pi N_{th} (\operatorname{Re}\bar{\Omega}_m)^2 g_m(\operatorname{Re}\Omega_m). \quad (3.6.20)$$

With the cubic nonlinearity of betatron oscillations the frequency distribution in a beam could (at least in one direction) be infinite. This does not hold true when the leading nonlinearity is due to the space charge forces. As we saw in Chapter 1 and shall see in the next chapter, for these forces the decay of the incoherent tune shift is specific with the increase in the particle oscillation amplitude

$$\Delta\omega_\alpha = - \left\langle \frac{\partial L}{\partial I_\alpha} \right\rangle_{\psi, \vartheta} \simeq \Delta\omega_\alpha^{(0)} \begin{cases} 1, & I \rightarrow 0, \\ I_0/I, & I \gg I_0. \end{cases} \quad (3.6.21)$$

Frequency distribution in such a beam is always limited within an interval equal to a linear tune shift $\Delta\omega_\alpha^{(0)}$. This imparts definite characteristics to the behavior of the stability region boundary and to Landau damping.

In a general case, the relation $\Delta\omega_\alpha(I)$ is determined by a rather complicated expression, whose asymptotic characteristics are reflected in Eq. (3.6.21). For

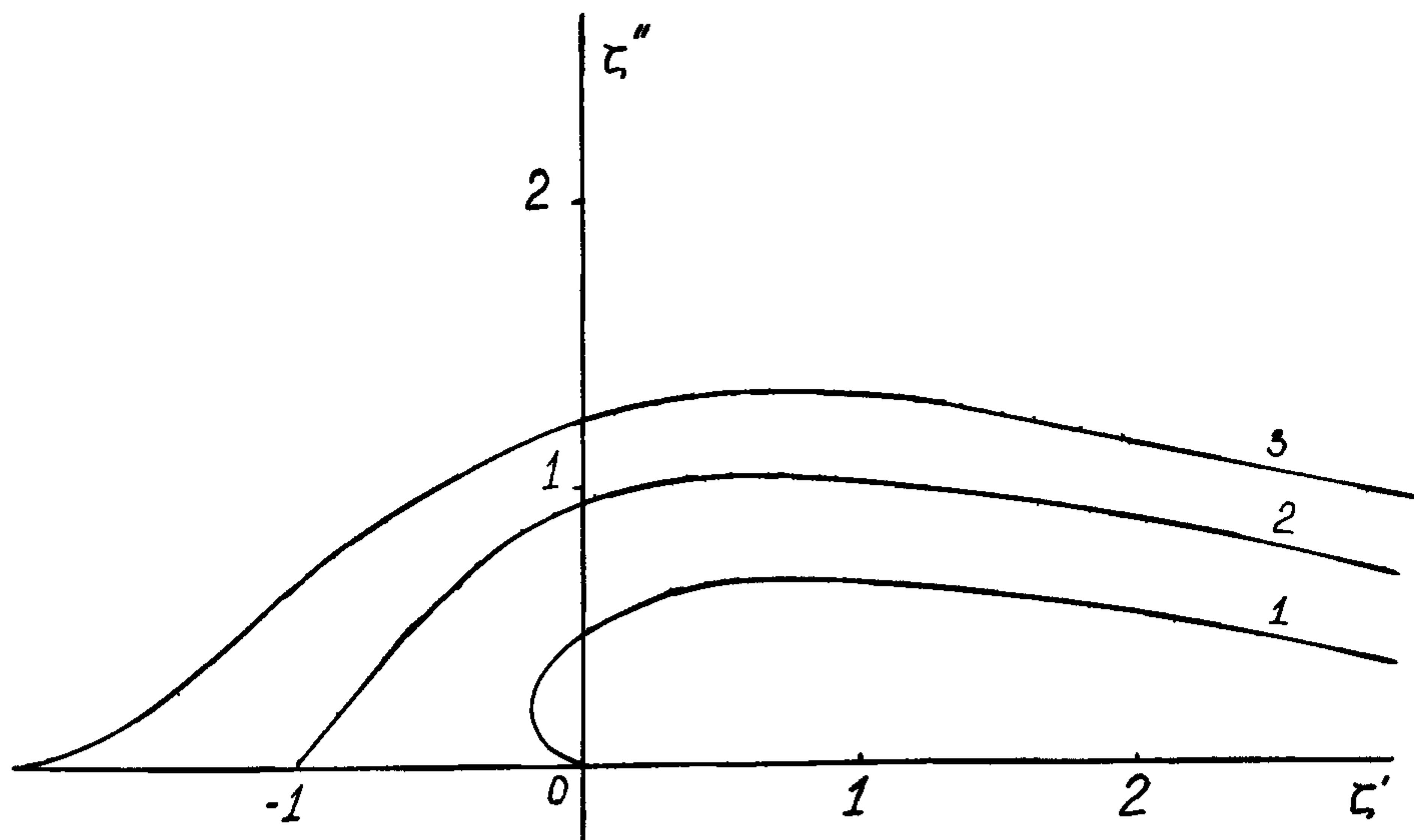


FIG. 3.5. The stability diagram for vertical coherent oscillations ($\xi = \Omega_m/m\Delta\omega$); 1.—Dipole vertical oscillations are stabilized by the frequency spread of horizontal incoherent oscillations; 2.—Dipole vertical oscillations are stabilized by the frequency spread of vertical incoherent oscillations; 3.—The same as in 2., but for quadrupole mode ($m_z=2$).

the sake of simplicity, let us consider a specific case, when the coherent oscillation is one-dimensional, $\mathbf{m} = (m_z, 0, 0)$, while $\Delta\omega_z(I)$ is described by a simple formula:

$$\Delta\omega_z(I) = \frac{\Delta\omega_z^{(0)}}{1 + I/I_0}, \quad (3.6.22)$$

and

$$F_0(I) = \exp(-I/I_0)/I_0.$$

According to these assumptions Eq. (3.6.1) will take the form

$$1 = \frac{\Omega_m}{m!} \int_0^\infty dx \frac{x^m e^{-x}}{\Delta\omega_m - \frac{m\Delta\omega_z^{(0)}}{1+x}}. \quad (3.6.23)$$

The substitution

$$w = \frac{m\Delta\omega_z^{(0)}}{1+x} \quad (3.6.24)$$

transforms Eq. (3.6.24) to a standard form:

$$1 = \frac{\Omega_m m \Delta\omega_z^{(0)}}{m!} \int_0^{m\Delta\omega_z^{(0)}} \frac{dw}{w^2} \frac{\exp\left(1 - \frac{m\Delta\omega_z^{(0)}}{w}\right)}{\Delta\omega_m - w} \left(\frac{m\Delta\omega_z^{(0)}}{w} - 1\right)^m \quad (3.6.25)$$

with frequency distribution

$$g_m(w) = \frac{m\Delta\omega_z^{(0)}}{w^2} \left(\frac{m\Delta\omega_z^{(0)}}{w} - 1\right)^m \exp\left(1 - \frac{m\Delta\omega_z^{(0)}}{w}\right), \quad (3.6.26)$$

in a limited interval $0 < w < m\Delta\omega_z^{(0)}$ ($m\Delta\omega_z^{(0)} > 0$) (Fig. 3.6).

As a result of studying the oscillation stability region with the dispersion equation (3.6.25) one can state (see also Fig. 3.7) that the stabilizing effect of the "spread" $\Delta\omega_z^{(0)}$ is in general rather small. Roughly speaking, the increments of coherent oscillations induced by the interaction should not exceed $0.15\Delta\omega_z^{(0)}$. The width of the frequency distribution at half-height is shown to be approximately $0.3\Delta\omega_z^{(0)}$, which is in reasonable agreement with the corresponding widths of the stability region.

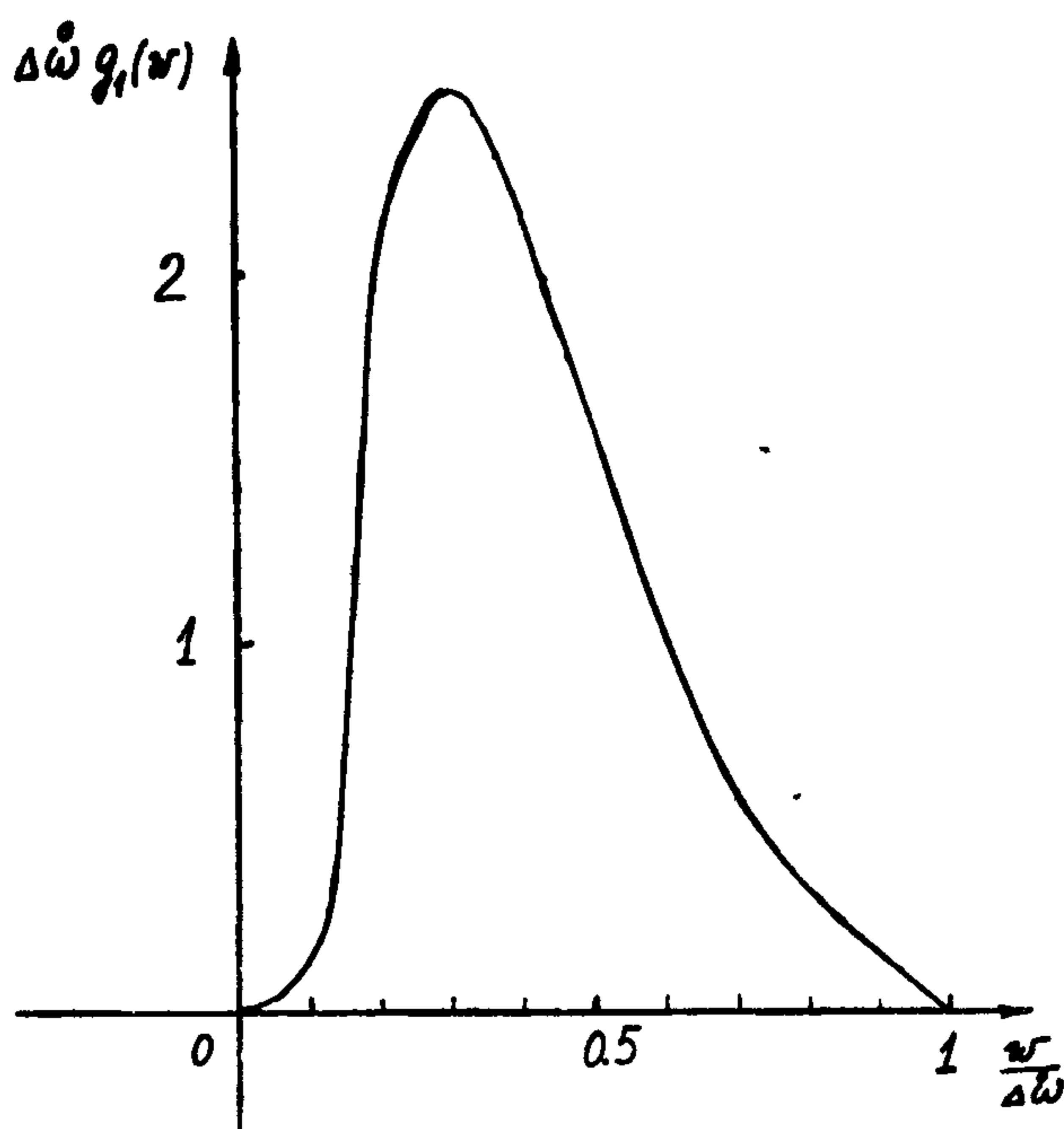


FIG. 3.6. Schematic behavior of the frequency distribution in Eq. (3.6.26)

3.7. LONGITUDINAL INSTABILITIES OF THE COOLED COASTING BEAMS

3.7.1 For storage rings with electron cooling the typical values of longitudinal cooling decrements $\lambda_{\parallel}$ significantly exceed the equilibrium beam revolution frequency spread $\Delta\omega_0$. In this region of beam parameters, coherent oscillations generally cannot be described within the formalism discussed in §3.1–3.6. To calculate the collective oscillation spectrum as well as threshold beam currents one should use a more complete kinetic equation with collision integral.³⁷ Generally, the result of these calculations can be strongly affected by the properties of the collision integral used. In this book we discuss the case when the cooling can be described by the Fokker-Planck equation. For the sake of simplicity, we assume that the momentum cooling decrement $\lambda_{\parallel}$ and the diffusion coefficient $d_{\parallel}$ are constant, which holds near the equilibrium. With these assumptions, instead of Eq. (3.2.5) the harmonics $f_{n\omega}$ obey the equation

$$-i(\omega - \omega_n)f_{n\omega} + \frac{Ne^2\omega_s Z_n}{\Pi} \frac{\partial f_0}{\partial \Delta p} \int_{-\infty}^{\infty} d\Delta p f_{n\omega} = \frac{\partial}{\partial \Delta p} \left(\lambda_{\parallel} \Delta p f_{n\omega} + \frac{d_{\parallel}}{2} \frac{\partial f_{n\omega}}{\partial \Delta p} \right). \quad (3.7.1)$$

As in Eq. (3.2.7) we assume the phenomenological ratio between the harmonics $E_{\delta}(n, \omega)$ and $f_{n\omega}$ using the longitudinal coupling impedance

$$E_{\parallel}(n, \omega) = -\frac{e\omega_s Z_n(\omega)}{\Pi} \int_{-\infty}^{\infty} d\Delta p f_{n\omega}. \quad (3.7.2)$$

The stationary distribution $f_0(\Delta p)$ is no longer arbitrary, as it should satisfy the equilibrium equation

$$\lambda_{\parallel} \Delta p f_0 + \frac{d_{\parallel}}{2} \frac{\partial f_0}{\partial \Delta p} = 0, \quad (3.7.3)$$

which with constant $\lambda_{\parallel}$ and $d_{\parallel}$ results in the Gaussian function

$$f_0(\Delta p) = \frac{1}{\sqrt{2\pi}\sigma_{\parallel}} \exp\left(-\frac{\Delta p^2}{2\sigma_{\parallel}^2}\right), \quad \sigma_{\parallel}^2 = \frac{d_{\parallel}}{2\lambda_{\parallel}}. \quad (3.7.4)$$

Here we shall assume nonresonant dependencies of Z_n on ω and thus shall replace ω in the argument of Z_n by $n\omega_s$. Using the substitutions

$$x = \frac{\Delta p}{\sigma_{\parallel}}, \quad a = -i \frac{\Delta\omega_n}{\lambda_{\parallel}}, \quad b = -\frac{in\Delta\omega_n}{\lambda_{\parallel}}, \quad \Delta\omega = \omega'_0 \nabla_{\parallel}$$

$$x = \sqrt{2}z + 2b, \quad f_{n\omega} = \exp\left(-\frac{x^2}{2} + bx\right) W(z), \quad \Delta\omega_n = \omega - n\omega_s, \quad (3.7.5)$$

one can transform Eq. (3.7.1) to a form more convenient for solution:

$$\left\{ \frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2(b^2 - a) \right\} W = \frac{i\Omega_n^2 2^{3/2}}{\pi^{1/2} \lambda_{\parallel} n \Delta\omega} (z + \sqrt{2}b) \exp(-b\sqrt{2}z + 2b)$$

$$\times \int_{-\infty}^{\infty} dz \exp(-z^2 + \sqrt{2}bz) W(z). \quad (3.7.6)$$

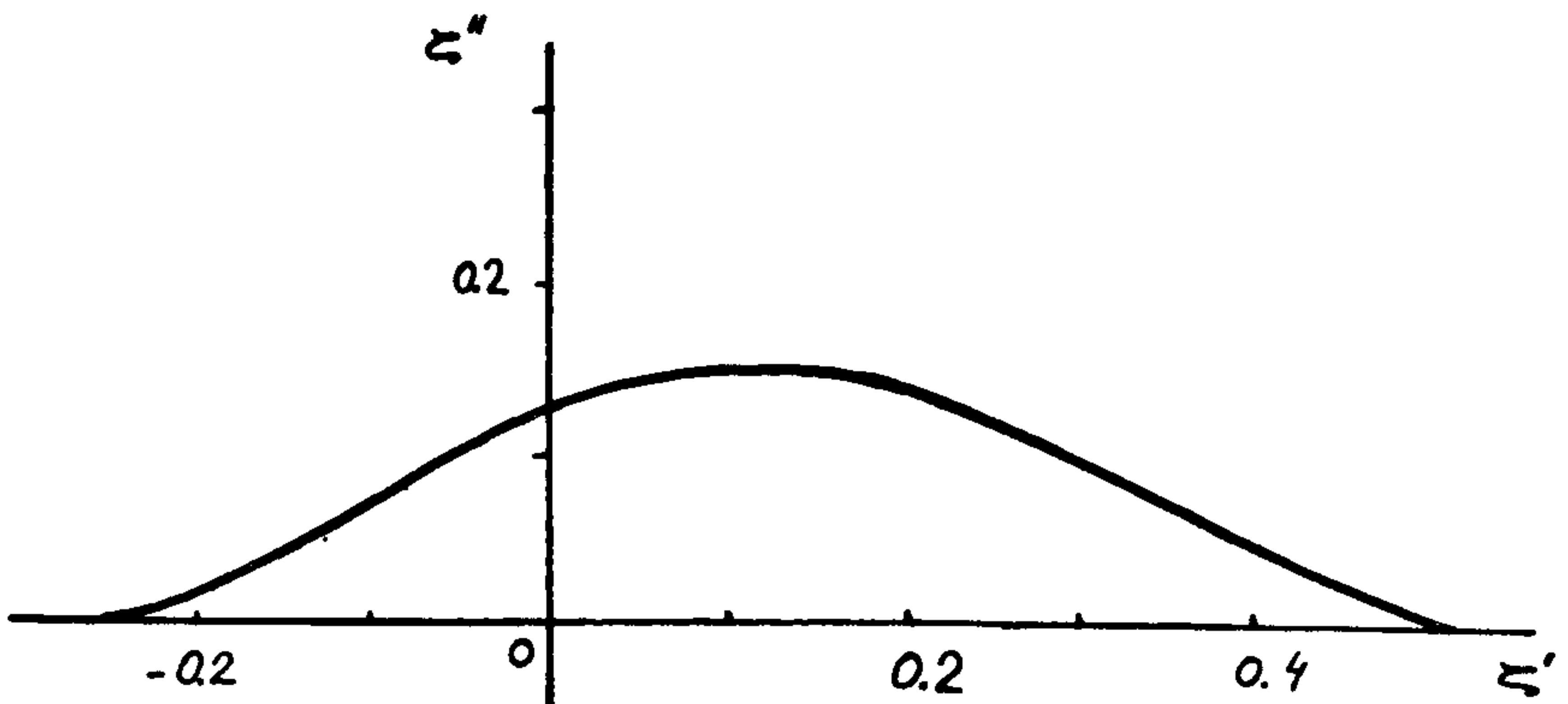


FIG. 3.7. The stability diagram, corresponding to the frequency distribution shown in Fig. 3.6.

3.7.2 Let us first discuss solutions of Eq. (3.7.6) for a low-intensity beam ($N \rightarrow 0$). In this case the oscillation spectrum and the eigenfunctions of the problem are determined by Eq. (3.7.6) without its right-hand side:

$$\left\{ \frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2(b^2 - a) \right\} W = 0.$$

As eigenfunctions must not grow, when $\Delta p \rightarrow \pm \infty$, this equation yields

$$b^2 - a = l, \quad W(z) = H_l(z), \quad l = 0, 1, 2, \dots, \quad (3.7.7)$$

where $H_l(z)$ are Hermite polynomials.¹⁴

Relaxation of an arbitrary initial distribution $f_n(\Delta p, t=0)$ can be described by expanding harmonics $f_n(\Delta p, t)$ in a row in terms of the eigenfunctions obtained:

$$f_n(x, t) = e^{-\frac{x^2}{2} + bx} \sum_{l=0}^{\infty} C_l H_l(z) e^{-\lambda_{\parallel} t - \frac{(\Delta \omega_n)^2}{\lambda_{\parallel}} t}. \quad (3.7.8)$$

Using the orthogonal properties of Hermite polynomials, we can calculate the coefficients C_l through the harmonics of the initial distribution function

$$C_l = \frac{1}{\pi^{1/2} 2^l l!} \int_{-\infty}^{\infty} dz \exp(bz/2) H_l(z) f_n(x, t=0).$$

Substitution of this expression in Eq. (3.7.8) and subsequent summation over l yields a fundamental solution of the problem

$$f_n(x, t) = \int_{-\infty}^{\infty} dx' g_n(x, x', t) f_n(x', 0), \quad (3.7.9)$$

where $g_n(x, x', 0) = \delta(x - x')$ and

$$g_n(x, x', t) = \frac{\exp[-G(x, x', t)]}{\sqrt{2\pi(1 - e^{-2\lambda_{\parallel} t})}},$$

$$G(x, x', t) = \sqrt{2}b(z - z') + \frac{(z - z'e^{-\lambda_{\parallel} t})^2}{1 - \exp(-2\lambda_{\parallel} t)} + \frac{(n\Delta\omega)^2}{\lambda_{\parallel}} t. \quad (3.7.10)$$

Integrating Eq. (3.7.8) over momenta $\Delta p = \sigma_{\parallel} x$, one can calculate the dependence of harmonics of the beam linear density on time:

$$\rho_n(t) = \int_{-\infty}^{\infty} d\Delta p f_n(\Delta p, 0) \exp \left\{ -\frac{in\omega'_0 \Delta p}{\lambda_{\parallel}} (1 - e^{-\lambda_{\parallel} t}) - \frac{(n\Delta\omega)^2}{\lambda_{\parallel}^2} \Psi(\lambda_{\parallel} t) \right\},$$

$$\Psi(x) = x + (1 - e^{-2x})/2 - 2(1 - e^{-x}). \quad (3.7.11)$$

Equation (3.7.11) shows that at small time intervals ($\lambda_{\parallel} t \ll 1$) the harmonic ρ_n as a function of time is determined by the initial frequency spread and the form of the distribution function

$$\rho_n(t) \cong \int_{-\infty}^{\infty} d\Delta p f_n(\Delta p, 0) e^{-in\omega'_0 \Delta p t}. \quad (3.7.12)$$

On the contrary, at times exceeding the cooling time ($\lambda_{\parallel} t \gg 1$) regardless of the form of the initial distribution $f_n(\Delta p, 0)$ the harmonics ρ_n are exponentially damped with decrements $(n\Delta\omega)^2 \lambda_{\parallel}$. This result corresponds to the diffusional dilution of azimuthal inhomogeneities in a cooled beam and generally is specific for systems, composed of Brownian particles with friction. The origin of this diffusion is easy to trace. Moving in the cooled beam, besides the average losses of their energy, the particles feel the random kicks with average frequency of the order of the cooling rate $\lambda_{\parallel}$. Between successive kicks a particle having the revolution frequency ω_a changes its azimuthal position in a beam by $\Delta\vartheta_a \simeq \Delta\omega_a / \lambda_{\parallel}$, where $\Delta\omega_a = \omega_a - \omega_s$. On average over kicks one has $\langle \Delta\vartheta \rangle = 0$, and $\langle \Delta\vartheta^2 \rangle = (\Delta\omega_a / \lambda_{\parallel})^2$. This corresponds to the diffusion of a particle along the beam with the diffusion coefficient $d\langle \Delta\vartheta^2 \rangle / dt = \Delta\omega^2 / \lambda_{\parallel}$. At the time intervals $t \gg 1/\lambda_{\parallel}$ the evolution of the lineal density of the beam $\rho(\vartheta, t)$ is described by the diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{\Delta\omega^2}{\lambda_{\parallel}} \frac{\partial^2 \rho}{\partial \vartheta^2}. \quad (3.7.13)$$

We can illustrate everything just mentioned concerning the behavior of $\rho(\vartheta, t)$ by an example where the initial distribution function has the form

$$f(\Delta p, \vartheta, 0) = \delta(\vartheta) \frac{\exp(-\Delta p^2 / 2\sigma_{\parallel}^2)}{(2\pi)^{1/2} \sigma_{\parallel}}.$$

Summing the harmonics from Eq. (3.7.12), we obtain for $\rho(\vartheta - \omega_s t, t)$

$$\rho(\varphi, t) = \vartheta_3 \left(\frac{\varphi}{2\pi} \middle| \langle \varphi^2(t) \rangle \right),$$

where $\varphi = \vartheta - \omega_s t$, $\vartheta_3(u|\tau)$ is the Jacobi function,¹⁴ and

$$\langle \varphi(t)^2 \rangle = \left[\frac{\Delta\omega}{\lambda_{\parallel}} (1 - e^{-\lambda_{\parallel} t}) \right]^2 + 2 \left(\frac{\Delta\omega}{\lambda_{\parallel}} \right)^2 \Psi(\lambda_{\parallel} t) = \begin{cases} (\Delta\omega t)^2, & \lambda_{\parallel} t \ll 1, \\ \frac{\Delta\omega^2}{\lambda_{\parallel}} t, & \lambda_{\parallel} t \gg 1. \end{cases}$$

This formula indicates that, for very cold beams ($\Delta\omega \ll \lambda_{\parallel}$), damping times of the beam density harmonics can essentially exceed $(n\Delta\omega)^{-1}$. This possibility should be taken into account while interpreting the measurements of debunching times in strongly cooled beams.

3.7.3 Let us return now to the calculation of the beam collective spectra. Equation (3.7.6) can be solved by expanding $W(z)$ in a series of Hermite polynomials:

$$W(z) = \sum_{l=0}^{\infty} C_l H_l(z). \quad (3.7.14)$$

Substitution of this expression in Eq. (3.7.6) yields a system of algebraic equations for coefficients C_b

$$C_l = \frac{-e^{-b^2}}{(b^2 - a - l)!} \left(\frac{\Omega_n}{n\Delta\omega} \right)^2 \left(\frac{-b}{\sqrt{2}} \right)^l \sum_{l'=0}^{\infty} (-\sqrt{2}b)^{l'} (b^2 - l') C_{l'}, \quad (3.7.15)$$

whose obvious solution results in a dispersion equation determining oscillation eigenfrequencies

$$1 = -\xi e^q \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \frac{q+l}{-\frac{i\Delta\omega_n}{\lambda_{\parallel}} + q+l}. \quad (3.7.16)$$

Here,

$$q = \left(\frac{n\Delta\omega}{\lambda_{\parallel}} \right)^2, \quad \xi = \frac{\Omega_n^2}{(n\Delta\omega)^2} = \xi' + i\xi''.$$

Correspondingly, the dielectric constant of a beam $\epsilon_n(\omega)$ is determined by the expression

$$\epsilon_n(\omega) = 1 + \xi e^q \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \frac{q+l}{-\frac{i\Delta\omega_n}{\lambda_{\parallel}} + q+l}. \quad (3.7.17)$$

The series involved in Eqs. (3.7.16) and (3.7.17) can be expressed in terms of the confluent hypergeometric function

$$e^x \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} \frac{1}{x+\alpha} = \frac{1}{\alpha} F(1, 1+\alpha, x).$$

Allowing for this relation, Eq. (3.7.17), for example, can be written

$$\epsilon_n(\omega) = 1 + \xi \Phi(\Delta\omega_n), \quad (3.7.18)$$

where

$$\Phi(\Delta\omega) = 1 + \frac{i\Delta\omega}{q\lambda_{\parallel} - i\Delta\omega} F\left(1, 1+q - \frac{i\Delta\omega}{\lambda_{\parallel}}, q\right). \quad (3.7.19)$$

In some cases the latter representation can be convenient for numerical calculations.

Let us first discuss some general properties of the dispersion equation (3.7.16), resulting from the specific features of the unperturbed spectrum and spectral characteristics of induced fields.

Due to the discreteness of the unperturbed motion spectrum (at $N \rightarrow 0$, $\Delta\omega_n \rightarrow -i\lambda_{\parallel} - i(n\Delta\omega)^2/\lambda_{\parallel}$) the right-hand side of Eq. (3.7.16) is an analytical function over the whole plane of the complex variable $\Delta\omega_n$, and the spectrum of collective motion is also discrete.

Collective modes are characterized by two numbers: the number of the azimuthal harmonic n and the solution number of Eq. (3.7.16), which tends to l at small currents. (See below.) Using Eqs. (3.7.5) one can show that the contribution of collective fields in the sum of eigenfrequencies is zero:

$$\left(\Delta\omega_n + i\lambda_{\parallel} + i \frac{(n\Delta\omega)^2}{\lambda_{\parallel}} \right)_{\Sigma} = \xi e^q \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} (q+l) = 0.$$

Therefore, the sum of decrements of collective modes is positive and coincides with the sum of decrements without the interaction:

$$\delta_{\Sigma} = \frac{(n\Delta\omega)^2}{\lambda_{\parallel}} \sum_l + \lambda_{\parallel} \sum_l l.$$

The physical reason for this is evident. Eq. (3.7.6) describes the coupling of modes with various l due to collective interaction, resulting in some redistribution of the unperturbed normal frequencies, while the sum of decrements determined by the matrix trace in Eq. (3.7.15) is independent of this coupling. Let us also note that dispersion Eq. (3.7.16) describes, strictly speaking, collective modes without Landau damping. For finite friction the stabilization of unstable modes takes place due to friction and diffusion of azimuthal position of particles.

The last statement needs some explanation. As was shown in subsection 3.7.2, the spectrum of induced fields remains the continuous one during a period of time of the order of $\lambda_{\parallel}^{-1}$, when the damping of density harmonics is caused by the frequency spread in the beam. Therefore, for the modes with a large spread $|n\Delta\omega|$ compared to the cooling decrement, the effect of interactions on the collective spectrum is not a factor, and Eq. (3.7.16) is transformed (with an accuracy up to the terms of an order of $\lambda_{\parallel}/\Delta\omega$) into a dispersion one that is valid in the absence of collisions.

The asymptote $\lambda_{\parallel} \rightarrow 0$ in Eq. (3.7.16) can be easily calculated using for the sum in Eq. (3.7.16) a representation in the form of the Melline integral:

$$\sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \frac{q+l}{y+i(q+l)} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{q^{-s} \Gamma(s) (q-s)}{y+i(q-s)},$$

$$\lambda_{\parallel} y = \Delta\omega_n, \quad 0 < c < q. \quad (3.7.20)$$

Thus, we have to calculate the expression

$$D(y) = \lim_{\lambda_{\parallel} \rightarrow 0} e^q \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{q^{-s} \Gamma(s)(q-s)}{y+i(q-s)}, \quad 0 < c < q. \quad (3.7.21)$$

Taking in Eq. (3.7.21) $s = t + q$ as a new integration variable and using the Stirling formula

$$\Gamma(t+q) \simeq \sqrt{2\pi} (t+q)^{t+q} e^{-q-t}, \quad q \gg 1,$$

one can write

$$\begin{aligned} D(y) &= \frac{i}{\sqrt{2\pi q}} \int_{c-i\infty}^{c+i\infty} \frac{dt}{y-it} e^{-t} \left(1 + \frac{t}{q}\right)^{t+q}, \quad -q < c < 0, \\ &\simeq \frac{i}{\sqrt{2\pi q}} \int_{c-i\infty}^{c+i\infty} \frac{dt}{y-it} e^{t^2/2q}. \end{aligned} \quad (3.7.22)$$

Taking as a new variable $u = it\lambda_{\parallel}$ and combining Eqs. (3.7.22) and (3.7.16) we obtain dispersion equation (3.2.13).

We also can calculate this asymptote using the substitution

$$\frac{1}{q+l-iy} = \int_0^{\infty} ds \exp(-s[q+l] + isy),$$

when $\epsilon_n(y)$ reads

$$\begin{aligned} \epsilon_n &= 1 - \xi e^q \int_0^{\infty} ds e^{isy} \frac{d}{ds} e^{-sq} \sum_{l=0}^{\infty} \frac{(-qe^{-s})^l}{l!} \\ &= 1 - \xi \int_0^{\infty} ds e^{isy} \frac{d}{ds} \exp[q(1-s-e^{-s})]. \end{aligned}$$

In the region where $q \gg 1$ we can replace

$$\exp[q(1-s-e^{-s})] \simeq \exp(-qs^2/2),$$

which after the substitution $u = \sqrt{q}s$ yields Eq. (3.2.11), written for the Gaussian momentum distribution function

$$\epsilon_n \simeq 1 - \xi \int_0^{\infty} du \exp\left(\frac{iyu}{\sqrt{q}}\right) \frac{d}{du} e^{-u^2/2}.$$

Thus, in a strongly cooled beam ($\Delta\omega \ll \lambda_{\parallel}$) stabilization of high-frequency modes $|n| > \lambda_{\parallel}/\Delta\omega$ is due to Landau damping, while that of low-frequency ones $|n| < \lambda_{\parallel}/\Delta\omega$ is caused by the beam cooling and diffusion. The first case is thoroughly studied in §3.6 so, here we shall restrict ourselves to studying the stability of low-frequency modes.

The character of oscillations is determined by the relationship between the parameters Ω_n^2 , $n\Delta\omega$ and $\lambda_{\parallel}$. There are no unstable solutions in the region of Z/n criterion [$|\Omega_n^2| < (n\Delta\omega)^2$]. In fact, since for any $\Delta\omega_n$ from its real axis we may estimate

$$\left| \xi e^q \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \frac{(q+l)^2}{\left(\frac{\Delta\omega_n}{\lambda_{\parallel}}\right)^2 + (q+l)^2} \right| < |\xi|,$$

the equation $\text{Re}\epsilon_n = 0$ definitely will have no roots if $|\xi| < 1$. In this region decrements of oscillations can be calculated by perturbation theory. In a first approximation over ξ/q , $q \ll 1$, the decrements linearly depend on the number of particles in the beam N :

$$\delta_{n,l} = \lambda_{\parallel} (q+l) \left[1 + \xi e^q \frac{(-q)^l}{l!} \right]. \quad (3.7.23)$$

As is seen from this expression, only the mode with $l = 0$ can decay substantially slower than $\lambda_{\parallel}$. Its decrement is

$$\delta_n = \frac{(n\Delta\omega)^2 + \Omega_n^2}{\lambda_{\parallel}}. \quad (3.7.24)$$

This expression is negative when

$$(n\Delta\omega)^2 < -\Omega_n^2. \quad (3.7.25)$$

Thus, the negative mass instability is not stabilized by the beam cooling. The reason is that the friction equally suppresses both the excitation of oscillations $\Omega_n^2/\lambda_{\parallel}$ and the longitudinal particle mobility, responsible for the stabilization of oscillations. This circumstance is general and follows directly from Eq. (3.7.16). The boundary of the stability region (see Fig. 3.8) is described by the equations

$$\xi' = -\frac{\Phi'(\Delta\omega)}{|\Phi(\Delta\omega)|^2}, \quad \xi'' = -\frac{\Phi''(\Delta\omega)}{|\Phi(\Delta\omega)|^2}. \quad (3.7.26)$$

Here, $\Phi(\Delta\omega)$ is determined using Eq. (3.7.19). Since $\Phi(0) = 1$, the stability diagram is situated to the right of the line $\xi' = -1$, corresponding to the stabilization of negative mass instability by the frequency spread.

For the beams with strong cooling the condition $|\xi| < 1$ indeed can correspond to rather small beam currents. For instance, for parameters of the cooling ring NAP-M⁴⁵ (see also the table in §7.2) it was satisfied at $N < 10^6$. However, as seen in Fig. 3.8, this restriction concerns mostly the negative mass instability. In the region $\xi' > 0$ (which was specific for experiments at NAP-M) friction results in an essential expansion of the stability region. The beam currents in this

ring could exceed by several orders of magnitude the limiting values with respect to the Z/n criterion.

In the case of $q \ll 1$, which is most specific for storage rings with electron cooling, the calculation of decrements is simplified. Retaining in Eq. (3.7.16) the leading terms in q , it can be written as

$$1 = \frac{\zeta q}{w(w+i)}, \quad w = \frac{\Delta\omega_n}{\lambda_{\parallel}} + q. \quad (3.7.27)$$

Whence,

$$w = -\frac{i}{2} \pm \sqrt{\frac{\Omega_n^2}{\lambda_{\parallel}^2} - \frac{1}{4}},$$

and oscillation decrements are determined via

$$\delta_n = \frac{\lambda_{\parallel}}{2} + \frac{(n\Delta\omega)^2}{\lambda_{\parallel}} \pm \sqrt{\frac{1}{2} (\sqrt{X^2 + V_n^2} - X)},$$

$$X = U_n - \frac{\lambda_{\parallel}^2}{4}. \quad (3.7.28)$$

Simple calculations result in the stability criterion

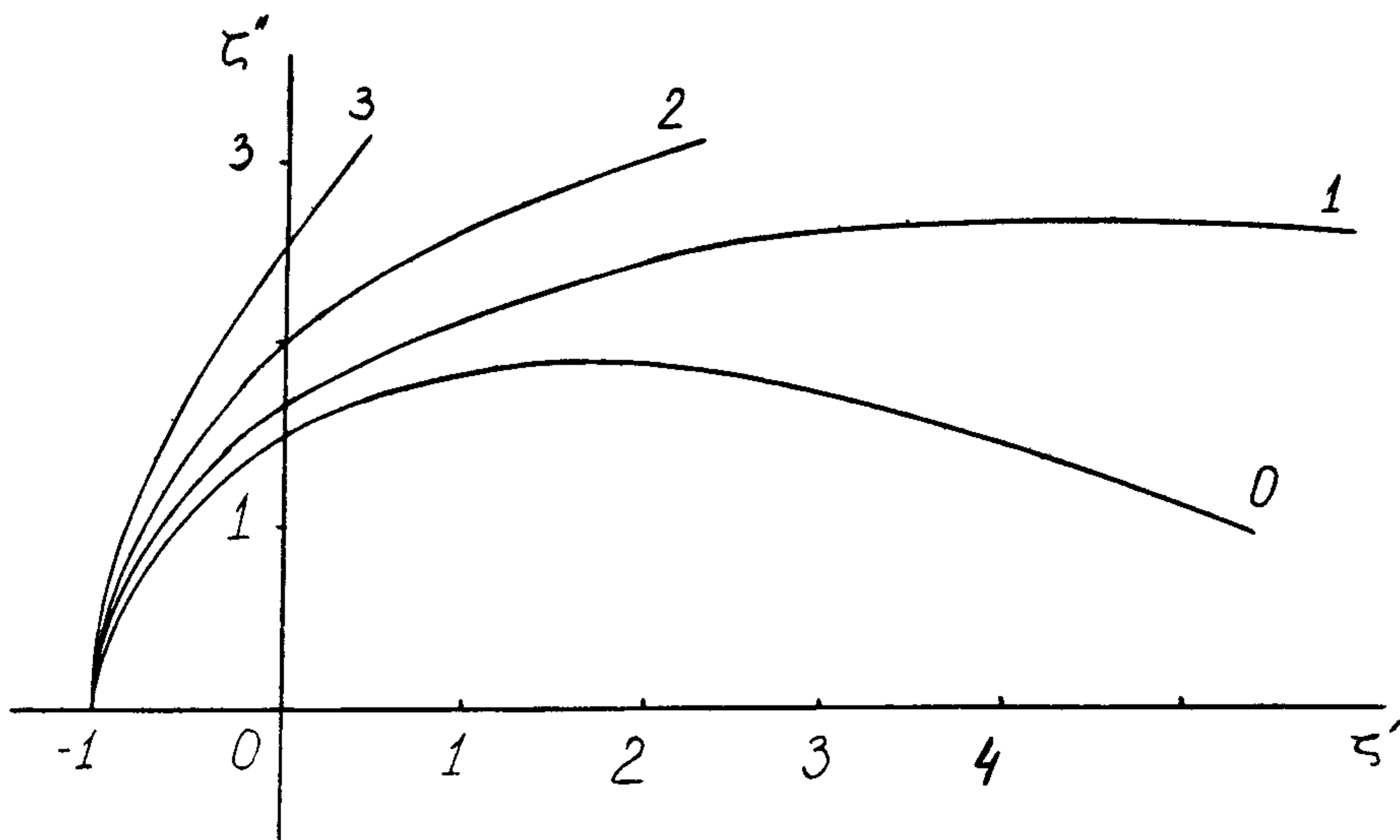


FIG. 3.8. The stability diagram for the cooled beam; 1.—No beam cooling; 2.— $q=4$; 3.— $q=1$; 4.— $q=0.25$. Due to the mirror symmetry only the upper half of the diagram is shown.

$$U_n + (n\Delta\omega)^2 > \frac{V_n^2}{\lambda_{\parallel}^2}. \quad (3.7.29)$$

The value of the instability threshold current is determined by the same relation. If we denote

$$\mu_n = nZ'_n/Z''_n,$$

then

$$\frac{N_{th}}{N_c} = \frac{\mu_n}{2} \left(\frac{\lambda_{\parallel}}{n\Delta\omega} \right)^2 \left\{ \text{sgn}(\omega'_0) + \sqrt{1 + \frac{1}{\mu_n} \left(\frac{2n\Delta\omega}{\lambda_{\parallel}} \right)^2} \right\}, \quad (3.7.30)$$

where N_c is the negative mass instability threshold. In the region $\omega'_0 > 0$ the instability threshold currents are seen to essentially exceed N_c :

$$\frac{N_{th}}{N_c} \simeq \begin{cases} \left(\frac{\lambda_{\parallel} \mu_n}{n\Delta\omega} \right)^2, & \lambda_{\parallel} \mu_n \gg |n\Delta\omega|, \\ \frac{\lambda_{\parallel}}{|n\Delta\omega|}, & \lambda_{\parallel} \mu_n \ll |n\Delta\omega|, \end{cases} \quad (3.7.31)$$

and N_{th} as a function of revolution frequency spread turns out more slow. In the region $\omega'_0 < 0$ the value of N_{th} practically coincides with that of N_c . In this energy range stabilization of coherent oscillation should be provided by using (active or passive) damping systems.

The rather complicated expression for oscillation decrements in Eq. (3.7.28) in some cases can be simplified. For example, in the region $|\Omega_n^2| < \lambda_{\parallel}^2/4$, which corresponds to experimental conditions at the storage ring NAP-M, Eq. (3.7.28) is transformed into

$$\delta_n = \frac{(n\Delta\omega)^2 + U_n}{\lambda_{\parallel}}, \quad (3.7.32)$$

which coincides with Eq. (3.7.24).

3.8. TRANSVERSE COHERENT OSCILLATIONS OF THE COOLED BEAM

As mentioned before, a strong cooling of a beam causes definite anomalies in coherent oscillation spectra. In this connection, it would be interesting to obtain dispersion equations for transverse coherent oscillations. Let us start with the calculations for the coasting beam. We assume that all the frequency spread in the beam is caused by the energy spread, while the dependence of betatron frequencies on amplitudes can be neglected. This case corresponds, for example, to the case of a heavy particle relativistic beam being cooled by an electron beam.

In the presence of a fast energy cooling Eq.(3.1.7) is substituted for by

$$\frac{\partial \tilde{f}}{\partial t} + \omega \frac{\partial \tilde{f}}{\partial \psi} + \frac{\partial \tilde{L}}{\partial \psi} \frac{\partial f_0}{\partial \mathbf{I}} = \frac{\partial}{\partial \Delta p} \left(\lambda_{\parallel} \Delta p \tilde{f} + \frac{d_{\parallel}}{2} \frac{\partial \tilde{f}}{\partial \Delta p} \right). \quad (3.8.1)$$

Performing another calculation from Eq. (3.1.7) to (3.1.22) with a substitution

$$\tilde{f}_{mn} = a_x^{|m_x|} a_z^{|m_z|} \left(\mathbf{m}_{\perp} \frac{\partial f_0}{\partial \mathbf{I}_{\perp}} \right) \chi(\Delta p) \quad (3.8.2)$$

we obtain an equation

$$i(\Delta \omega_{mn} - \xi \Delta p) \chi + i \Omega_{mn} f_0(\Delta p) \int_{-\infty}^{\infty} d\Delta p' \chi(\Delta p') = \frac{\partial}{\partial \Delta p} \left(\lambda_{\parallel} \Delta p \chi + \frac{d_{\parallel}}{2} \frac{\partial \chi}{\partial \Delta p} \right), \quad (3.8.3)$$

where Ω_{mn} is determined by Eq. (3.1.23), and

$$\xi = \left(\frac{d\omega_0}{dp} \right)_s (\mathbf{m}_{\perp} \mathbf{v}_{\perp} + n) + \left(\mathbf{m}_{\perp} \frac{\partial \mathbf{v}_{\perp}}{\partial p} \right)_s \omega_s. \quad (3.8.4)$$

Due to cooling the stationary distribution $f_0(\Delta p)$ is no longer arbitrary:

$$f_0(\Delta p) = \frac{\exp(-\Delta p^2 / 2\sigma_{\parallel}^2)}{\sqrt{2\pi}\sigma_{\parallel}}.$$

By successive substitutions

$$a = \frac{-i\Delta \omega_{mn}}{\lambda_{\parallel}}, \quad b = \frac{i\xi \sigma_{\parallel}}{\lambda_{\parallel}} = i \frac{\Delta}{\lambda_{\parallel}},$$

$$\Delta p = \sigma_{\parallel} x, \quad x = \sqrt{2}z - 2b, \quad \chi = \exp\left(-\frac{x^2}{2} - bx\right) W(z), \quad (3.8.5)$$

where Δ is the frequency spread, Eq. (3.8.3) is reduced to the form

$$\left\{ \frac{d^2}{dz^2} - 2z \frac{d}{dz} + 2(b^2 - a) \right\} W = \frac{2i\Omega_{mn}}{\pi^{1/2}\lambda_{\parallel}} \exp(\sqrt{2}bz - 2b^2)$$

$$\times \int_{-\infty}^{\infty} dz' W(z') \exp(-z'^2 + \sqrt{2}bz'). \quad (3.8.6)$$

Substituting here $W(z)$ as a series in Hermite polynomials

$$W(z) = \sum_{l=0}^{\infty} C_l H_l(z)$$

and using¹⁴

$$\int_{-\infty}^{\infty} dz \exp(-z^2 + \sqrt{2}bz) H_l(z) = \pi^{1/2} (\sqrt{2}b)^l e^{b^2/2},$$

we find the dispersion equation

$$1 = \Omega_{mn} \exp\left(\frac{\Delta^2}{\lambda_{\parallel}^2}\right) \sum_{l=0}^{\infty} \frac{(-\Delta^2/\lambda_{\parallel}^2)^l}{l! [\Delta\omega_m + i\lambda_{\parallel} (l + \Delta^2/\lambda_{\perp}^2)]}. \quad (3.8.7)$$

Similar to Eq. (3.7.16), Eq. (3.8.7) describes the suppression of Landau damping by a strong momentum cooling:

$$\Delta\omega_{mn} \simeq \Omega_{mn} - i \frac{\Delta^2}{\lambda_{\parallel}}, \quad \Delta \ll \lambda_{\parallel}. \quad (3.8.8)$$

Due to a quick damping of a particle momentum deviation from its equilibrium value, the stabilizing effect of the energy spread is reduced by the cooling by $\Delta/\lambda_{\parallel}$ times.

The contribution of the frequency spread due to the nonlinearity of betatron oscillations changes Eq. (3.8.7) as follows:

$$1 = \frac{\Omega_{mn} \exp(\Delta^2/\lambda_{\parallel}^2)}{\mathcal{A}_m} \sum_{l=0}^{\infty} \frac{(-1)^l (\Delta/\lambda_{\parallel})^{2l}}{l!} \times \int d\Gamma_{\perp} \frac{I_x^{|m_x|} I_z^{|m_z|} (\mathbf{m}_{\perp} \partial F_0 / \partial \mathbf{I}_{\perp})}{\omega - \mathbf{m}_{\perp} \omega_{\perp} - n\omega_s + i\lambda_{\parallel} (l + \Delta^2/\lambda_{\parallel}^2)}. \quad (3.8.9)$$

If $\Delta \ll \lambda_{\parallel}$, Eq. (3.8.9) is transformed in Eq. (3.1.17), written for a beam with a zero momentum spread:

$$1 = \frac{\Omega_{mn}}{\mathcal{A}_m} \int d\Gamma_{\perp} \frac{I_x^{|m_x|} I_z^{|m_z|} (\mathbf{m}_{\perp} \partial F_0 / \partial \mathbf{I}_{\perp})}{\omega - \mathbf{m}_{\perp} \omega_{\perp} (\mathbf{I}) - n\omega_s + i\Delta^2 \lambda_{\parallel}}. \quad (3.8.10)$$

For the opposite limiting case of $\Delta \gg \lambda_{\parallel}$, the repeated calculation starting from Eq. (3.8.7), similar to that given in §3.7, yields a similar result as in Eq. (3.1.17).

The influence of the cooling of the betatron oscillations on collective spectra can be described in a similar way. For the sake of simplicity, we assume that $\Delta/\lambda_{\parallel} \rightarrow 0$ calculate a dispersion equation for one-dimensional betatron oscillations ($m_z = m, m_x = 0$). The collision integral for transverse degrees of freedom of the beam has the form

$$\text{St} = \frac{\partial}{\partial I} \left(\lambda I f + D I \frac{\partial f}{\partial I} \right) + \frac{D}{4I} \frac{\partial^2 f}{\partial \psi^2}. \quad (3.8.11)$$

Its first term describes the cooling of the beam emittance toward the equilibrium,

$$\frac{d}{dt}\langle I \rangle = -\lambda I + D,$$

while the second describes the diffusion of phases of betatron oscillations due to collisions:

$$\frac{d}{dt}\langle (\Delta\psi)^2 \rangle = \overline{\left(\frac{\partial\psi}{\partial p_z}\right)^2} \frac{d}{dt}\langle (\Delta p_z)^2 \rangle = \frac{D}{4I}. \quad (3.8.12)$$

Eq. (3.8.11) yields the right-hand side of the equation, which replaces Eq. (3.1.7) in the case of strong cooling of betatron oscillations:

$$\frac{\partial \tilde{f}}{\partial t} + \omega_z \frac{\partial \tilde{f}}{\partial \psi_z} + \frac{\partial \tilde{L}}{\partial \psi_z} \frac{\partial f_0}{\partial I_z} = \text{St}, \quad (3.8.13)$$

Expanding $\tilde{f}$ in harmonics, we can rewrite this equation as

$$\begin{aligned} & -i\Delta\omega_{mn}f_{mn} + \frac{i\Omega_{mn}}{m!} \left(\frac{I}{I_0}\right)^{|m_z|/2} e^{-I/I_0} \int_0^\infty dI' I'^{|m_z|/2} f_{mn} \\ & = \frac{\partial}{\partial I} \left(\lambda I f_{mn} + D I \frac{\partial f_{mn}}{\partial I} \right) - \frac{m^2 D}{4I} f_{mn}. \end{aligned} \quad (3.8.14)$$

Here, we used the fact that the equilibrium distribution in the cooled beam is $f_0 \propto \exp(-I/I_0)$ with $I_0 = D/\lambda$.

Using in Eq. (3.8.14) the dimensionless variables

$$x = \frac{I}{I_0}, \quad a = \frac{i\Delta\omega_{mn}}{\lambda}, \quad b = -\frac{im_z\Delta\omega_z}{\lambda}, \quad (3.8.15)$$

and substituting

$$f_{mn} = x^{|m_z|/2} \exp[-(1-\sigma)x] W, \quad x = \kappa z,$$

we can rewrite it as

$$\begin{aligned} & \frac{d^2 W}{dz^2} + (|m_z| + 1 - z) \frac{dW}{dz} + \kappa \left[a - \frac{|m_z|}{2} + \sigma(|m_z| + 1) \right] W \\ & = \frac{i\Omega_{mn}}{m_z!} e^{-\sigma x} \int_0^\infty dx' x'^{|m_z|} e^{-(1-\sigma)x'} W. \end{aligned} \quad (3.8.16)$$

Here, $\sigma = (\kappa - 1)/2\kappa$, $\kappa = (1 - 4b)^{-1/2}$, and $\Delta\omega_z = I_0(\partial\omega_z/\partial I)$ is the frequency spread of betatron oscillations. Expanding $W(z)$ in a series of Laguerre polynomials¹⁴

$$W(z) = \sum_{l=0}^{\infty} C_l L_l^{|m_z|}(z)$$

and performing some transformations we obtain the dispersion equation

$$1 = \Omega_{mn} \left(\frac{2}{u_+} \right)^{2|m_z|+2} \sum_{l=0}^{\infty} \frac{(|m_z|+l)!}{|m_z|l!} \frac{[-(u_-/u_+)^2]^l}{\Delta\omega_{mn} + \frac{i\lambda[2l+|m_z|]}{2\kappa} - \frac{i u_-}{2}}, \quad (3.8.17)$$

where $u_{\pm} = (\kappa \pm 1)/\kappa$. Equation (3.8.17) becomes especially simple for a beam with a zero frequency spread $b = 0$, when it reads

$$\Delta\omega_{mn} = \Omega_{mn} - i \frac{|m_z|\lambda}{2}. \quad (3.8.18)$$

This equation indicates significant difference in the damping of coherent oscillations by the beam betatron and momentum cooling. Inside the betatron potential well the cooling directly adds to Ω_m the decrement $|m|\lambda/2$. Since without collective interaction the multipole $z^{|m_z|}$ must decay with a decrement $|m_z|\lambda/2$, the additional decrement in Eq. (3.8.18) increases $\propto |m_z|$. In formal respect, this result can be obtained if we replace in the Vlasov equation ω_α by $\omega_\alpha - i\lambda_\alpha/2$. This fact can be used to simplify the calculations related to coherent stability of the betatron and synchrotron oscillation, when the cooling decrements significantly exceed the frequency spread of the beam.

In beams with a finite, but small frequency spread of the betatron oscillations ($|m_z\Delta\omega_z| \ll \lambda$), the spectrum of coherent oscillations will differ from that given by Eq. (3.8.18). In this case, as seen in Eq. (3.8.17), the linear (in b) corrections do not seriously change the real part of the coherent frequency shift, while the corresponding corrections to the decrements are proportional to $\lambda b^2 = (m_z\Delta\omega_z)^2/\lambda$ (i.e., are determined by the diffusion of particles in the betatron phases like in the case of longitudinal motion).

The increase in the multipole number m_z at a given λ violates the condition $|b| \ll 1$ and moderates the role of beam cooling for stabilizing these oscillations. In the limit $|m_z\Delta\omega_z| \gg \lambda$ Eq. (3.8.17) has as an asymptote Eq. (3.1.17), written for a beam with an exponential distribution over I .

A dispersion equation that determines the eigenfrequencies of coherent oscillations of a bunched beam can be obtained similarly. It coincides with Eq. (3.8.17), with values Ω_{mn} to be substituted for by eigenvalues Ω_m from Eq. (3.1.22).

CHAPTER 4

Resonant Coherent Instabilities

This chapter begins the analysis of special problems. We shall start with the discussion of main properties of resonant instabilities of the beam coherent oscillations when one or some normal frequencies of field oscillation ω_k occur close to one from the combination frequencies of the beam current

$$\omega \simeq m\omega + n\omega_s = \omega_{mn}, \quad \omega_k \simeq \omega_{mn}.$$

Due to the resonant dependence of the increments of these instabilities on the frequencies these instabilities can be very strong and without special efforts resonant instabilities can be observed before others.^{28,46} On the other hand, because of strong dependence of decrements of these instabilities on frequencies, the resonant instabilities can be cured by detuning unstable modes, suppressing the resonance modes in field oscillations, or using narrow bandwidth feedback systems.

Due to the finite Q -value of the RF-cavity the resonance coupling of the beam and field oscillations redistributes decrements between coupled modes of the beam and field oscillations. This condition can be used to damp particular unstable modes of the beam.

The special interest presents the case when coherent oscillations of bunches become unstable due to the so-called beam-beam interaction. Since the interaction of colliding bunches couples their collective modes, unstable coherent oscillations occur near machine resonances

$$mv \simeq n.$$

4.1. RESONANT LONGITUDINAL INSTABILITIES OF A COASTING BEAM

The phenomenological theory of longitudinal coherent oscillations in a coasting beam has been described in §3.2, where we assumed a low Q -value surrounding the beam electrodes resulting in a rather smooth dependence of Z_n on ω . Here we shall consider a quite different case, when the coasting beam interacts with a high Q -value cavity. Let us start with the calculations for the longitudinal modes. The cavity will be described by the impedance

$$Z_n(\omega) = \frac{Z_0}{1 + iQ_k(\omega_k/\omega - \omega/\omega_k)}. \quad (4.1.1)$$

Here Z_0 is the shunt impedance of the cavity, Q_k is its Q -value, and ω_k is the frequency of the resonant mode. In off-resonant region $\omega \ll \omega_k$ (or $\omega \gg \omega_k$) Eq. (4.1.1) yields respectively the inductive or capacitive impedance. So, with the definite sign of the synchrotron mass (ω'_0) one may expect the different stability properties for coherent oscillations whether the beam is tuned below or above the resonance frequency ω_k .

Both the frequencies of normal oscillations and stability criteria for the resonance region $\omega \simeq n\omega_s$ can be calculated using the dispersion equation (3.2.13)

$$1 = \frac{Ne^2\omega_s j Z_n(\omega_s)}{\Pi} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\Delta\omega_n - n\omega'_0 \Delta p}, \quad \text{Im}\Delta\omega_n > 0, \quad (4.1.2)$$

$$\Delta\omega_n = \omega - n\omega_s.$$

Substituting here $Z_n(\omega)$ from Eq. (4.1.1) and introducing the values

$$\Omega_n^3 = n^2\omega_s^3 \frac{Ne^2\omega_s (Z_0/2Q_k)}{2\pi p v} \alpha_p \quad (4.1.3)$$

and $\Delta = \omega_k - n\omega_s$ we can rewrite Eq. (4.1.2) in the resonance region ($|\Omega_n| > |\Delta|$) as

$$1 = - \frac{\Omega_n^3}{(\Delta\omega_n - \Delta + i\lambda_k)n\omega'_0} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\Delta\omega_n - n\omega'_0 \Delta p}, \quad \text{Im}\Delta\omega_n > 0. \quad (4.1.4)$$

To avoid too complicated calculations with Eq. (4.1.4), let us consider some special cases. First, let us neglect the frequency spread in the beam ($|\Omega_n| \gg \delta\omega$, $\delta\omega = |n\omega'_0|\sigma$, σ being the beam momentum spread) and make the damping of field oscillations in the cavity ($|\Omega_n| \gg \lambda_k$, $\lambda_k = \omega_k/2Q_k$). With these assumptions Eq. (4.1.4) reads

$$1 = \Omega_n^3 D(\Delta\omega_n), \quad D(\Delta\omega_n) = 1/[\Delta\omega_n^2(\Delta\omega_n - \Delta)]. \quad (4.1.5)$$

Since Eq. (4.1.5) is a cubic equation with real coefficients, we can find the stability criteria by calculating the number of crossings between branches of the function $D(\Delta\omega_n)$ with the horizontal line $1/\Omega_n^3$. (See Fig. 4.1.) This indicates that coherent oscillations can be stable if

$$\Delta\Omega_n^3 < 0 \quad (4.1.6)$$

and

$$|\Omega_n^3| \leq \frac{4}{27} |\Delta|^3. \quad (4.1.7)$$

The substitution here of Eq. (4.1.3) yields the threshold current of the beam

$$eI_{th} = \left(\frac{Ne^2\omega_s}{2\pi} \right)_{th} = \frac{4}{27} (|\Delta|/\omega_s)^3 \frac{p_s v_s}{n^2 |\alpha_p| (Z_0/2Q_k)}. \quad (4.1.8)$$

Note that this value is proportional to the cubic power of the beam detuning from the resonance Δ . As can be seen from Eq. (4.1.5), the most simple expressions for eigenfrequencies $\Delta\omega_n$ can be obtained for the worst case of exact resonance $\Delta=0$:

$$\Delta\omega_n = \Omega_n \begin{cases} 1, \\ \frac{1 \pm i\sqrt{3}}{2} \end{cases}. \quad (4.1.9)$$

This yields for the rise time of the instability

$$1/\tau_n = \frac{\sqrt{3}}{2} \omega_s \left\{ n^2 \frac{eI_{th} Z_0/2Q_k}{p_s v_s} |\alpha_p| \right\}^{1/3}. \quad (4.1.10)$$

It is remarkable that in contrast with Eq. (14.12) this instability rate is proportional to $(n^2 N)^{1/3}$. Note also that far from the resonance ($|\Delta| \gg |\Omega_n|$) this dependence will return to the normal one

$$1/\tau_n \propto |n| N^{1/2}.$$

The physical reasons for this effect are simple. In the coasting beam the resonant interaction couples three oscillators: the induced field, the slow wave, and the fast wave.

Both the dissipation in the cavity and the frequency spread in the beam can stabilize the oscillations. The simplest expressions for the roots of the dispersion equation can be obtained for the Lorentz momentum distribution function

$$f(\Delta p) = \frac{1}{\pi} \frac{\sigma}{\Delta p^2 + \sigma^2}, \quad (4.1.11)$$

when the dispersion equation reads

$$1 = \frac{\Omega_n^3}{(\Delta\omega_n + i\delta\omega)^2 (\Delta\omega_n - \Delta + i\lambda_k)}. \quad (4.1.12)$$

The roots of this equation are⁴⁷

$$(\Delta\omega_n)_k = -i\lambda_k + \Lambda \{1 + 2\cos[(\alpha + 2\pi k)/3]\}, \quad k=0,1,2, \quad (4.1.13)$$

$$\cos\alpha = 1 - \Omega_n^3/(2\Lambda^3), \quad \Lambda = \frac{1}{3}(\Delta + i\delta\omega - i\lambda_k).$$

These expressions can be easily calculated numerically. Typical dependencies of the oscillation decrements on the beam intensity (Ω_n) are shown in Fig. 4.2. It indicates that depending on the ratios $\Omega_n/\delta\omega$ and $\lambda_k/\delta\omega$ some modes can be stable and some unstable. The comparison of the stability diagram for the Lorentz distribution shows (see Fig. 4.3) that it is wider, but still differs not very much from the stability diagram for more realistic Gaussian momentum distribution functions. In the last case the stability diagram is determined by the equations

$$(\Omega_n/\delta\omega) = \left[\frac{w-d}{\mathcal{P}(w)} \right]^{1/3}, \quad w = \Delta\omega_n/\delta\omega, \quad d = \Delta/\delta\omega,$$

$$(\lambda_k/\delta\omega) = -\sqrt{\pi/2} w(w-d) \frac{e^{-w^2/2}}{\mathcal{P}^2(w)}, \quad (4.1.14)$$

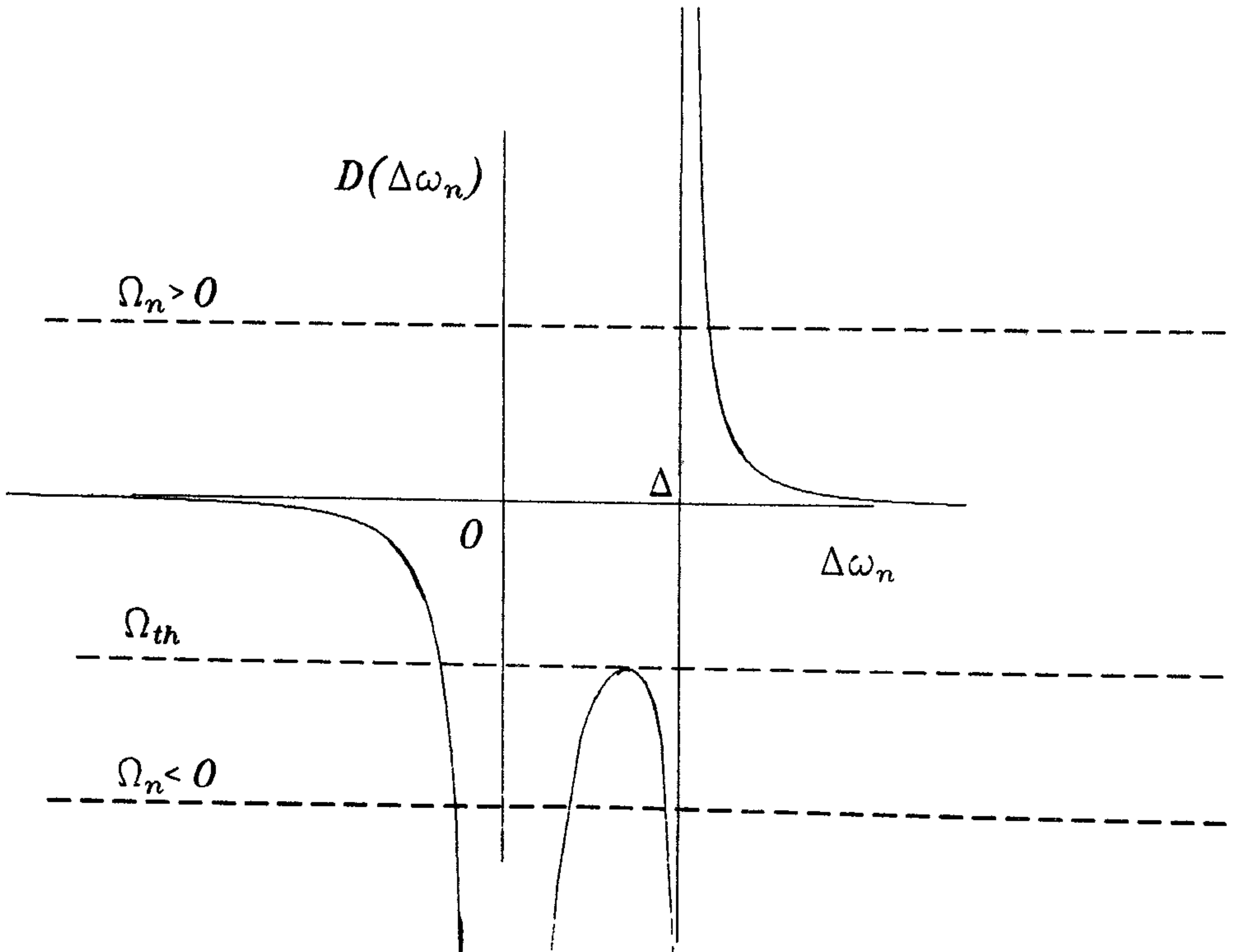


FIG. 4.1. Schematic behavior of $D(\Delta\omega_n)$ from Eq. (4.1.5).

$$\mathcal{P}(w) = \mathcal{P} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \frac{y}{w-y} \exp(-y^2/2) \quad (4.1.15)$$

and $\mathcal{P}$ again means the calculation of the principal value of the integral.

4.2. RESONANT TRANSVERSE INSTABILITY DUE TO INTERACTION WITH A HIGH Q-VALUE CAVITY

As it was pointed out in §3.3 the fields induced in a high Q -value cavity oscillate with frequencies ω_k and decay with decrements λ_k ($\lambda_k \ll \omega_k$). In this case it is most suitable to describe the fields by the set of field oscillators. Here we shall study the stability of transverse coherent oscillations of the beam interacting with RF-cavity assuming $\lambda_k \ll \omega_s$, when the fields induced by the beam decay during many revolution periods. Since ω_k is inversely proportional to sizes of the cavity, which typically are not big, in any case ω_s does not exceed ω_k . Below we shall assume that the number of the field mode k is not too high.

For the beam interacting with one mode of the cavity we can describe the induced fields by the potential

$$\mathbf{A}(\mathbf{r}, t) = c \mathbf{A}_k(\mathbf{r}) Q_k(t), \quad \text{div} \mathbf{A}_k = 0. \quad (4.2.1)$$

Moreover, to simplify the problem we shall assume for the Green function of the electrostatic problem the boundary condition

$$G(1, 2)|_{\Sigma} = 0$$

and neglect the surface effects. This replaces Eq. (3.3.44) for the field oscillator $Q_k(t)$ by

$$\ddot{Q}_k + 2\lambda_k \dot{Q}_k + \omega_k^2 Q_k = e j_k(t), \quad (4.2.2)$$

$$j_k(t) = \int d^3r \mathbf{A}_k(\mathbf{r}) \mathbf{j}(\mathbf{r}, t). \quad (4.2.3)$$

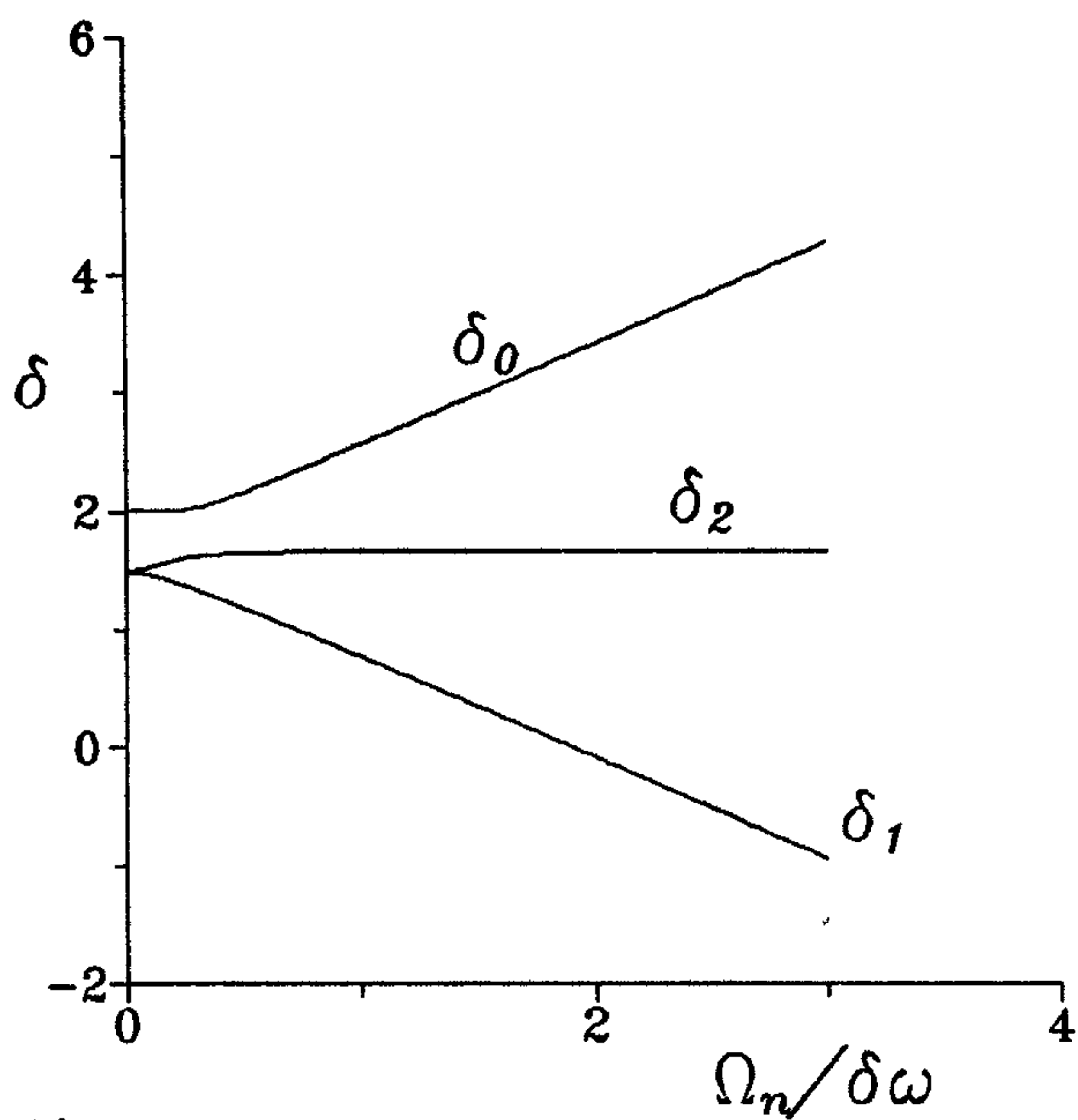
Due to cyclic motion of particles one can write

$$j_k(t) = \sum_{n=-\infty}^{\infty} j_k(n, \omega) \exp(-in\omega_s t - i\omega t) \quad (4.2.4)$$

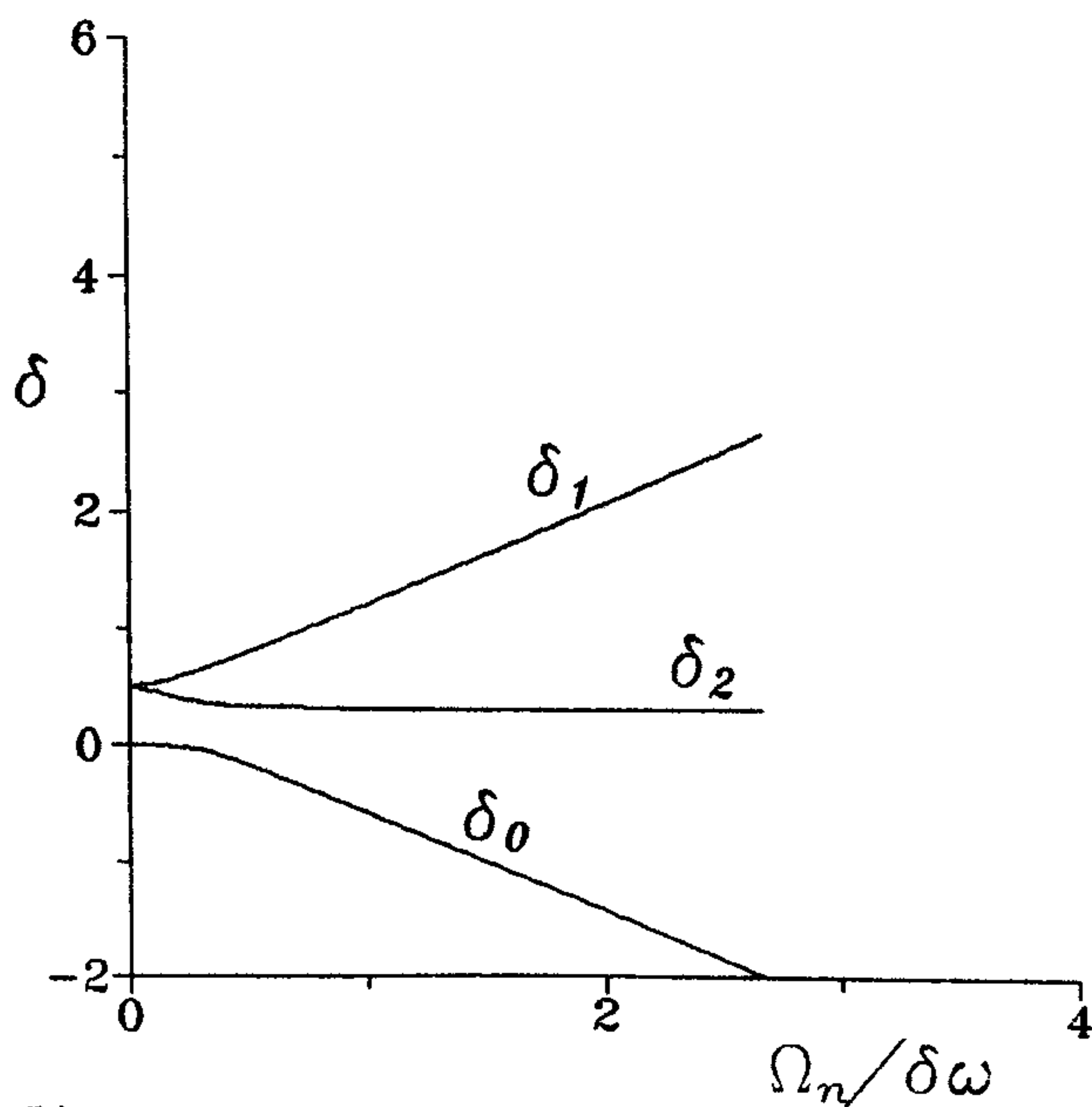
and therefore find the solution of Eq. (4.2.2) in the form

$$Q_k(t) = \sum_{n=-\infty}^{\infty} Q_k(n, \omega) \exp(-in\omega_s t - i\omega t). \quad (4.2.5)$$

The substitution of Eq. (4.2.5) in Eq. (4.2.2) yields



(a)



(b)

FIG. 4.2. Dependencies of resonant decrements of the coasting beam on its intensity at exact resonance ($\Delta=0$); (a) $\lambda_k/\delta\omega=1.5$, (b) $\lambda_k/\delta\omega=0.5$.

$$Q_k(n, \omega) = \frac{e \int d\Gamma (\mathbf{v} \mathbf{A}_k)_{\mathbf{m},n}^* f_{\mathbf{m},\omega}}{\omega_k^2 - (\omega + n\omega_s + i\lambda_k)^2}. \quad (4.2.6)$$

As shown in §3.1, the r.h.s. of Eq. (3.1.11) is proportional to

$$\bar{L}_{\mathbf{m},\omega} = \frac{e}{c} \overline{\mathbf{v} \mathbf{A}_{\mathbf{m},\omega}} = e \sum_{n=-\infty}^{\infty} (\mathbf{v} \mathbf{A}_k)_{\mathbf{m},n} Q_k(n, \omega). \quad (4.2.7)$$

Substituting here $Q_k(n, \omega)$ from Eq. (4.2.6) we can rewrite the integral equation (3.1.11) in the form

$$(\omega - \mathbf{m}\omega) f_{\mathbf{m},\omega} = N e^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \sum_{n=-\infty}^{\infty} \frac{(\mathbf{v} \mathbf{A}_k)_{\mathbf{m},n}}{\omega_k^2 - (\omega + n\omega_s + i\lambda_k)^2} \times \int d\Gamma' (\mathbf{v} \mathbf{A}_k)_{\mathbf{m},n}^* f_{\mathbf{m},\omega}(\mathbf{I}'). \quad (4.2.8)$$

Provided that $\omega_s \gg \lambda_k$ and

$$\omega_k \simeq \mathbf{m}\omega + n\omega_s \quad (4.2.9)$$

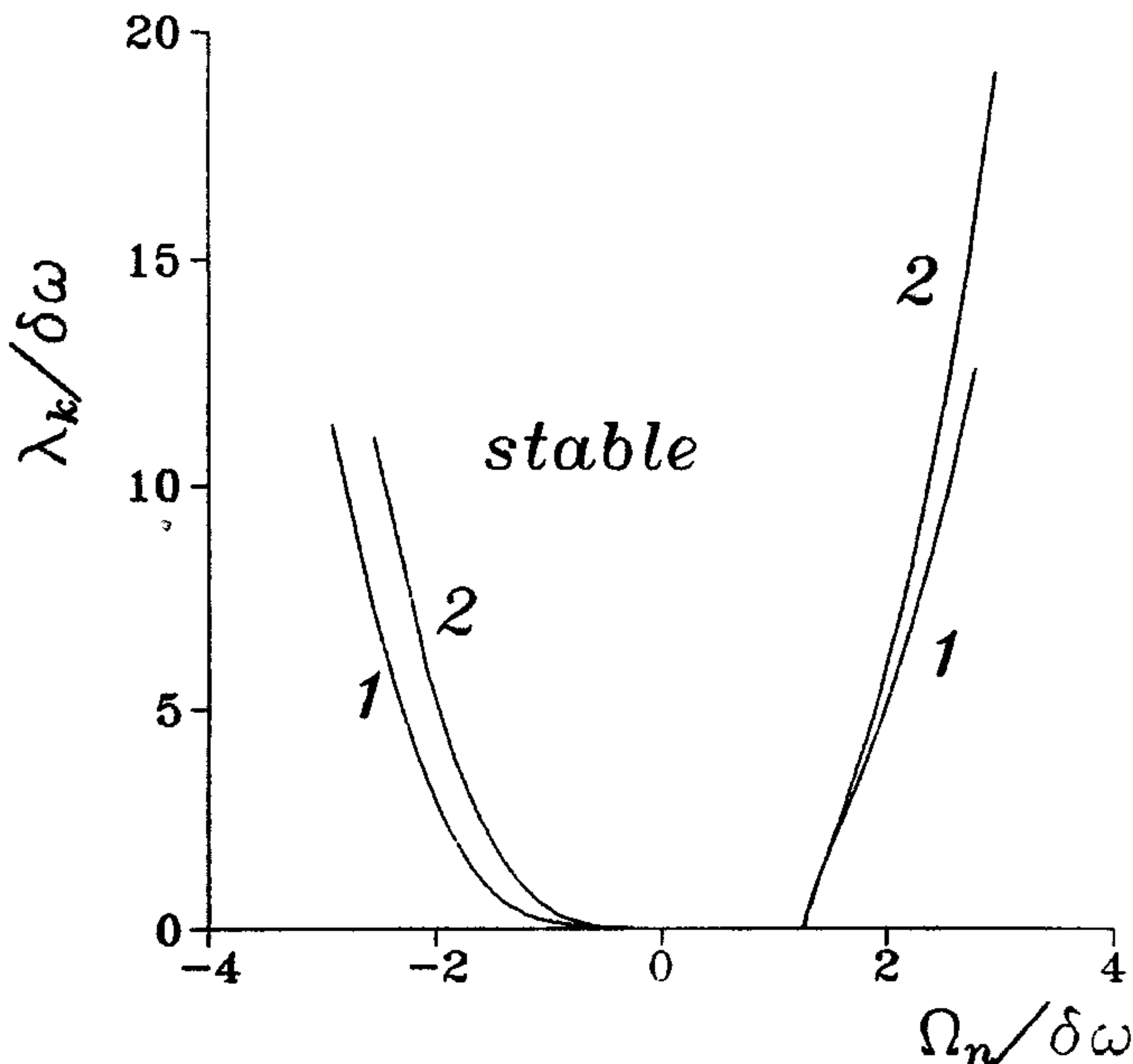


FIG. 4.3. Stability diagrams for the Lorentz (1) and Gaussian (2) momentum distribution functions.

Eq. (4.2.8) can be simplified neglecting in the sum over n all off-resonant terms. The resulting integral equation is

$$(\omega - m\omega)f_{m,\omega} = Ne^2 \left(m \frac{\partial f_0}{\partial I} \right) \frac{(\mathbf{vA}_k)_{m,n}}{\omega_k^2 - (\omega + n\omega_s + i\lambda_k)^2} \int d\Gamma' (\mathbf{vA}_k)_{m,n}^* f_{m,\omega}. \quad (4.2.10)$$

It has the same form for both coasting and bunched beam. (The only difference is in the distribution function of the stationary state f_0 .) This means that general properties of resonant transversal instabilities are very close.²⁸ In both cases the instability is caused by fields that decay for many turns. We define the detuning from the resonance $\Delta = \omega_k - m\omega - n\omega_s$, and $\Delta\omega_m = \omega - m\omega$ and rewrite Eq. (4.2.10) as

$$\Delta\omega_m f_{m,\omega} = \frac{Ne^2}{2\omega_k} \left(m \frac{\partial f_0}{\partial I} \right) \frac{(\mathbf{vA}_k)_{m,n}}{\Delta - \Delta\omega_m - i\lambda_k} \int d\Gamma' (\mathbf{vA}_k)_{m,n}^* f_{m,\omega}. \quad (4.2.11)$$

Below we give the calculations for one-dimensional betatron oscillations^{28,46} when $\mathbf{m} = (m_z, 0, 0)$. More general results can be obtained easily. Using the obvious substitution

$$f_{m,\omega} = \left(m \frac{\partial f_0}{\partial I} \right) \frac{(\mathbf{vA}_k)_{m,n}}{\omega - m_z\omega_z(I)},$$

one can get from Eq. (4.2.11) the dispersion equation

$$1 = - \frac{1}{\Delta\omega_m - \Delta + i\lambda_k} \int d\Gamma \frac{V^2(I) (m_z \partial f_0 / \partial I_z)}{\Delta\omega_m - m_z\omega_z(I)}, \quad (4.2.12)$$

where by definition

$$V^2(I) = \frac{Ne^2}{2\omega_k} |(\mathbf{vA}_k)_{m,n}|^2.$$

Remember that Eq. (4.2.12) is true only for the cases when the beam cooling is weak enough. ($\lambda_\perp \ll |m_z \delta\omega_z|$, $\lambda_\parallel \ll |n \delta\omega_0|$, and $\delta\omega_z$ and $\delta\omega_0$ are respectively the beam frequency spreads of betatron and revolution frequencies.) If on the contrary the beam cooling is strong, Eq. (4.2.12) must be modified. For instance, in the region $\lambda_\perp \gg \lambda_\parallel$ as well as $\lambda_\perp \gg \delta\omega_z \gg \delta\omega_0$ it reads (see §3.8)

$$1 = \frac{\Omega_{m,n}^2}{(\Delta\omega_m + i\lambda_m)(\Delta\omega_m - \Delta + i\lambda_k)}. \quad (4.2.13)$$

Here $\lambda_m = |m_z| \lambda_\perp / 2$ and

$$\Omega_{m,n}^2 = m_z \left\langle \frac{\partial V^2}{\partial I_z} \right\rangle$$

is the value giving the strength of the coupling of the beam and the cavity modes. Using the expansion of the eigenfunctions $\mathbf{A}_k(\mathbf{r})$ into the multipole series and assuming the smoothed focusing approximation $f_a(\vartheta) \simeq 1/\sqrt{v_a}$ as well as a Gaussian distribution

$$f_0 \propto \exp[-a_z^2/(2\sigma_z^2)],$$

we can write

$$\Omega_{m,n}^2 = \frac{\text{sgn}(m_z)}{(|m_z| - 1)!} \frac{Nr_0 c^2 R_0^2}{4\nu_k \nu_z} \left(\frac{\sigma_z^2}{2}\right)^{|m_z| - 1} \times \left| \frac{\partial^{|m_z| - 1}}{\partial z^{|m_z| - 1}} \left(\frac{\partial A_{k,\vartheta}}{\partial z} + \frac{im_z \nu_z}{R_0} A_{k,z} \right)_{\mathbf{r}=\mathbf{r}_0(\vartheta)} \right|^2 \quad (4.2.14)$$

$$r_0 = e^2/mc^2, \quad \nu_k = \omega_k/\omega_s.$$

For a cylindrical cavity with the axes directed along the closed orbit Eq. (4.2.14) can be simplified even more. In this case the field oscillations can be selected onto transversal electric (TE: $E_{k,\vartheta} \equiv 0$) and transversal magnetic (TM: $H_{k,\vartheta} \equiv 0$) ones. Defining

$$\omega_k^2/c^2 = k^2 = k_\perp^2 + k_\parallel^2,$$

where $k_\perp$ and $k_\parallel$ are respectively transversal and longitudinal eigennumbers of oscillations, one can write⁹ for TE-oscillations

$$A_{k,\vartheta} = 0, \quad \mathbf{A}_{k,\perp} = \frac{\sqrt{4\pi}}{k_\perp} \boldsymbol{\tau} \times \nabla_\perp \Psi_{k,M}(\mathbf{r}),$$

$$\int d^3r |\Psi_{k,M}(\mathbf{r})|^2 = 1$$

and for TM-oscillations

$$A_{k,\vartheta} = \sqrt{4\pi} \Psi_{k,E}(\mathbf{r}),$$

$$\mathbf{A}_{k,\perp} = \frac{ik_\parallel}{k_\perp^2} \sqrt{4\pi} \nabla_\perp \Psi_{k,E}(\mathbf{r}),$$

$$\int d^3r |\Psi_{k,M}(\mathbf{r})|^2 = \frac{k_\perp^2}{k^2},$$

where functions $\Psi_{k,E}$ and $\Psi_{k,M}$ satisfy the equation

$$(\Delta + k^2) \Psi_k(\mathbf{r}) = 0$$

and the boundary conditions

$$\Psi_E(\mathbf{r})|_{\Sigma}=0, \quad (\partial\Psi_M/\partial n)|_{\Sigma}=0$$

on the electrodes. ($\partial/\partial n \equiv \mathbf{n}\nabla$, $\mathbf{n}$ is the unit normal vector to the surface of the cavity.) The substitution of these expressions in Eq. (4.2.14) yields for the interaction with TE-oscillations

$$\Omega_{m,n}^2 = \frac{\pi m_z N r_0 c^2 v_z (\sigma_z^2/2)^{|m_z|-1}}{\gamma v_k k_{\perp}^2 (|m_z|-1)!} \left| \frac{\partial^{|m_z|-1}}{\partial z^{|m_z|-1}} (\nabla_x \Psi_{k,M})_n \right|^2, \quad (4.2.15)$$

and with TM-oscillations

$$\Omega_{m,n}^2 \approx \frac{\pi m_z N r_0 c^2 R_0^2 (\sigma_z^2/2)^{|m_z|-1}}{\gamma v_k v_z (|m_z|)!} \left| \left(\frac{\partial^{|m_z|}}{\partial z^{|m_z|}} \Psi_{k,E} \right)_n \right|^2. \quad (4.2.16)$$

In particular, for the most important (dipole $|m_z|=1$) coherent oscillations the values $\Omega_{m,n}^2$ do not depend on the beam transversal sizes

$$\Omega_{m,n}^2(TE) = \frac{m_z N r_0 c^2 \pi v_z}{\gamma k_{\perp}^2 v_k} |(\nabla_x \Psi_{k,M})_n|^2,$$

$$\Omega_{m,n}^2(TM) = \frac{m_z N r_0 c^2 \pi R_0^2}{\gamma v_z v_k} |(\nabla_z \Psi_{k,E})_n|^2.$$

Returning back to dispersion equations we have to note that Eqs. (4.2.12) and (4.2.13) are alternatives for the calculation of normal frequencies ω . For a cooled beam with small frequency spread the low frequency spectrum

$$1 \leq |m_z| < \lambda_{\perp} / \delta\omega_z$$

is determined by Eq. (4.2.13), while the high-frequency one

$$|m_z| > \lambda_{\perp} / \delta\omega_z \gg 1$$

by Eq. (4.2.12). Practically, these conditions can be realized, for instance, in electron-positron high-energy storage rings.

Let us now proceed to the calculation of eigenfrequencies ω and stability criteria. We shall start with the simplest case, when both friction and the beam frequency spreads are negligible ($\lambda_k, \lambda, \delta\omega \rightarrow 0$) and, therefore, the dispersion equation gets the form

$$1 = \frac{\Omega_{m,n}^2}{\Delta\omega_m(\Delta\omega_m - \Delta)}.$$

It has the roots

$$\Delta\omega_m^{\pm} = \frac{\Delta}{2} \pm \frac{1}{2} \sqrt{\Delta^2 + 4\Omega_{mn}^2} = \frac{\Delta \pm \Omega(\Delta)}{2}, \quad (4.2.17)$$

indicating that solutions will be unstable if $m_z < 0$ (the sum-type resonance) and

$$\Delta^2 < 4|\Omega_{m,n}^2|. \quad (4.2.18)$$

In the resonance stopband the oscillations have the increments

$$\delta_m = -\text{Im}\omega = \pm \frac{|\Delta|}{2} \sqrt{\frac{N}{N_{th}} - 1}, \quad (4.2.19)$$

where

$$N_{th} = \left\{ \frac{|m_z|e^2}{\omega_k \Delta^2} \int d^3I f_0(I) \frac{\partial}{\partial I_z} |(\mathbf{v} \mathbf{A}_k)_{\mathbf{m},n}|^2 \right\}^{-1} \quad (4.2.20)$$

is the threshold number of particles. In the exact resonance one has $\delta_m \propto N^{1/2}$, which describes the tendency for the beam bunching in the betatron phase space due to resonance instability. Following Ref. 28 we shall call such instabilities dynamic-type ones. It also is easy to verify that the result of the previous section ($\delta_m \propto N^{1/3}$) corresponds to hollow beam distribution

$$f_0 \propto \delta(I_z - I_z^{(0)}).$$

A dissipation of the cavity mode or the beam cooling can result in the so-called dissipative-type instabilities of coherent oscillations. In the most general case $\lambda_k \neq 0$, $\lambda_m \neq 0$, and $\lambda_z \gg |m_z \delta \omega_z|$, Eq. (4.2.13) has the roots

$$\Delta \omega_{\mathbf{m}}^{\pm} = \frac{1}{2} \{ \Delta - i\Lambda_{+} \pm \sqrt{(\Delta - \Lambda_{-})^2 + 4\Omega_{mn}} \}, \quad (4.2.21)$$

$\Lambda_{\pm} = \lambda_k \pm \lambda_m$. This yields the decrements of coherent oscillations in the form

$$\delta_{\mathbf{m}}^{\pm} = \frac{1}{2} \left\{ \Lambda_{+} \pm \sqrt{\frac{1}{2} [\sqrt{X^2 + 4\Delta^2 \Lambda_{-}^2} - X]} \right\}, \quad (4.2.22)$$

$$X = 4\Omega_{m,n}^2 + \Delta^2 - \Lambda_{-}^2.$$

After simple but tedious transformations one can easily get the stability condition:

$$\lambda_k \lambda_m (1 + \Delta^2 / \Lambda_{+}^2) + \Omega_{m,n}^2 > 0. \quad (4.2.23)$$

For differential-type resonances ($m_z > 0$) the condition (4.2.23) holds automatically. For sum-type resonances the oscillations will be stable if

$$|\Omega_{m,n}^2| \leq \lambda_k \lambda_m (1 + \Delta^2 / \Lambda_{+}^2). \quad (4.2.24)$$

This indicates that for dissipative instabilities the threshold number of particles decreases when either λ_k or λ_m decreases. That is why condition (4.2.24) can be stronger than (4.2.18).

Using Eq. (4.2.22) one can easily calculate decrements of coherent oscillations numerically. Simple analytical expressions for decrements can be obtained in two limiting cases. In the first case, $\Delta^2 + \Lambda_-^2 = \Delta_\Sigma^2 \gg |\Omega_{m,n}^2|$, the number of particles in the beam is small. Expanding r.h.s in Eq. (4.2.22) in the power series in $\Omega_{m,n}^2/\Delta_\Sigma^2$ and taking into account the leading terms one can get

$$\delta_m^+ = \lambda_k - \Lambda_- \frac{\Omega_{m,n}^2}{2\Delta_\Sigma^2}, \quad (4.2.25)$$

$$\delta_m^- = \lambda_m + \Lambda_- \frac{\Omega_{m,n}^2}{2\Delta_\Sigma^2}. \quad (4.2.26)$$

These equations are written for the case $\lambda_k > \lambda_m$. In the inverse case $\lambda_k < \lambda_m$ modes with δ^+ and δ^- replace each other.

$\Delta_\Sigma^2 \ll |\Omega_{m,n}^2|$, the number of particles in the beam is high. In the sum-type resonance

$$\delta_m^\pm = \pm (|\Omega_{m,n}^2|)^{1/2} \quad (4.2.27)$$

one of the solutions is unstable. In the differential resonance coherent oscillations decay with decrements

$$\delta_m^\pm = \frac{1}{2} \{ \Lambda_+ \pm \Lambda_- \sqrt{(\Delta^2/4\Omega_{mn}^2)} \}. \quad (4.2.28)$$

Let us now discuss in more detail the region $\lambda \ll |m_z \delta \omega_z|$, when the beam cooling does not significantly affect the damping of coherent oscillations and, hence, the eigenfrequencies ω are determined by Eq. (4.2.12) whereas the damping of these oscillations can be associated with the dissipation in the cavity or with Landau damping.

For the monochromatic beam the dispersion equation coincides with Eq. (4.2.13):

$$1 = \frac{\Omega_{m,n}^2}{\Delta \omega_m (\Delta \omega_m - \Delta + i\lambda_k)}.$$

Since here we have $\lambda_m = 0$, coherent oscillations will be stable only in the differential resonance. The decrement, which gets the coherent motion, depends on the coupling coefficient $\Omega_{m,n}^2/\Delta^2$. It is obvious that the maximum decrement will be limited by the value $\lambda_k/2$, which can be reached in the region $|\Delta| < 2|\Omega_{m,n}|$ when the coupling is strong and the energy of coherent motion is completely dissipated by the cavity. In the inverse case of the weak coupling $|\Delta| \gg 2|\Omega_{m,n}|$ the decrement linearly depends on the number of particles in the beam following the equation

$$\delta_m \simeq \frac{N\lambda_k m_z}{\Delta^2} \left\langle \frac{\partial V^2}{\partial I} \right\rangle. \quad (4.2.29)$$

These dependencies of δ on both λ_k and N are illustrated in Fig. 4.4.

Additional decay of coherent oscillations can be caused by Landau damping, which is associated with oscillation frequency spreads. (In the coasting beam the revolution frequency spread can give the contribution as well.) But as Eq. (4.2.12) contains the derivation of the distribution function, the contributions in the decrements from Landau damping generally depend on the shape of f_0 . For instance, for the so-called water-bag distribution

$$f_0(I_z) = \begin{cases} 1/I_0, & I_z < I_0, \\ 0, & I_z > I_0, \end{cases}$$

the betatron frequency spread is $\delta\omega_z = |\omega'_z| I_0$, whereas the normal frequencies of coherent oscillations $\Delta\omega_m$ obviously are contributed only by the surface particles $f'_0 \propto \delta(I_z - I_0)$. The next important parameter of the problem is the ratio of coherent frequency shift $\Delta\omega_m^\pm$ to the frequency spread in the beam. In the region

$$\lambda_k \ll |\operatorname{Re}(\Delta\omega_m)| \ll \delta\omega_z$$

Landau damping of oscillations mainly will be determined by the shape of the distribution function and will be almost independent on the coupling of the beam and the cavity.

The simple expressions for decrements can be obtained in the inverse region

$$|\operatorname{Re}(\Delta\omega_m)| \gg |m_z| \delta\omega_z \gg |\operatorname{Im}(\Delta\omega_m)|,$$

where the contributions into decrements from the frequency spread sufficiently depend on the behavior of the tail of the distribution function. In this region Eq. (4.2.12) can be rewritten in the form

$$\Delta\omega_m - \Delta + i\lambda_k = \frac{\Omega_{m,n}^2}{\Delta\omega_m} + i\pi m_z N \int d\Gamma V^2 \frac{\partial f_0}{\partial I_z} \delta(\Delta\omega_m - m_z \omega'_z I_z), \quad (4.2.30)$$

which corresponds to the redefining of λ_k :

$$\lambda_k^{eff} = \lambda_k - \pi m_z N \int d\Gamma V^2 \frac{\partial f_0}{\partial I_z} \delta(\Delta\omega_m - m_z \omega'_z I_z). \quad (4.2.31)$$

Since for monotonous distributions one has $\partial f_0 / \partial I_z < 0$, λ_k^{eff} will decrease with the beam intensity for sum resonances and will increase for differential ones. Moreover, as seen from Eqs. (4.2.30) and (4.2.31), the effect from the frequency spread can be important only in the case when the sign of the coherent tune shift coincides with the sign of the derivation $\partial\omega_z / \partial I_z$. For differential resonances, for instance, the decrement δ_m^- remains unaltered, while δ_m^+ can be estimated by the modified Eq. (4.2.28):

$$\delta_m^+ = \frac{\lambda_k^{eff}}{2} \left(1 + \frac{|\Delta|}{\Omega} \right) = \frac{1}{2} \left(1 + \frac{|\Delta|}{\Omega} \right) \left\{ \lambda_k - \frac{\pi N}{|\partial \omega_z / \partial I_z|} \left(V^2 \frac{\partial f_0}{\partial I_z} \right)_{I_z = \Omega / (m_z \omega_z')} \right\}. \quad (4.2.32)$$

For sum resonances the expansion of r.h.s in Eq. (4.2.21) in the powers of (λ_k^{eff}/Ω) yields the decrements

$$\delta_m^- = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 - N/N_{th}}} \right) \left\{ \lambda_k + \frac{\pi N}{|\partial \omega_z / \partial I_z|} \left(V^2 \frac{\partial f_0}{\partial I_z} \right)_{I_z = \Delta \omega_m^- / (m_z \omega_z')} \right\},$$

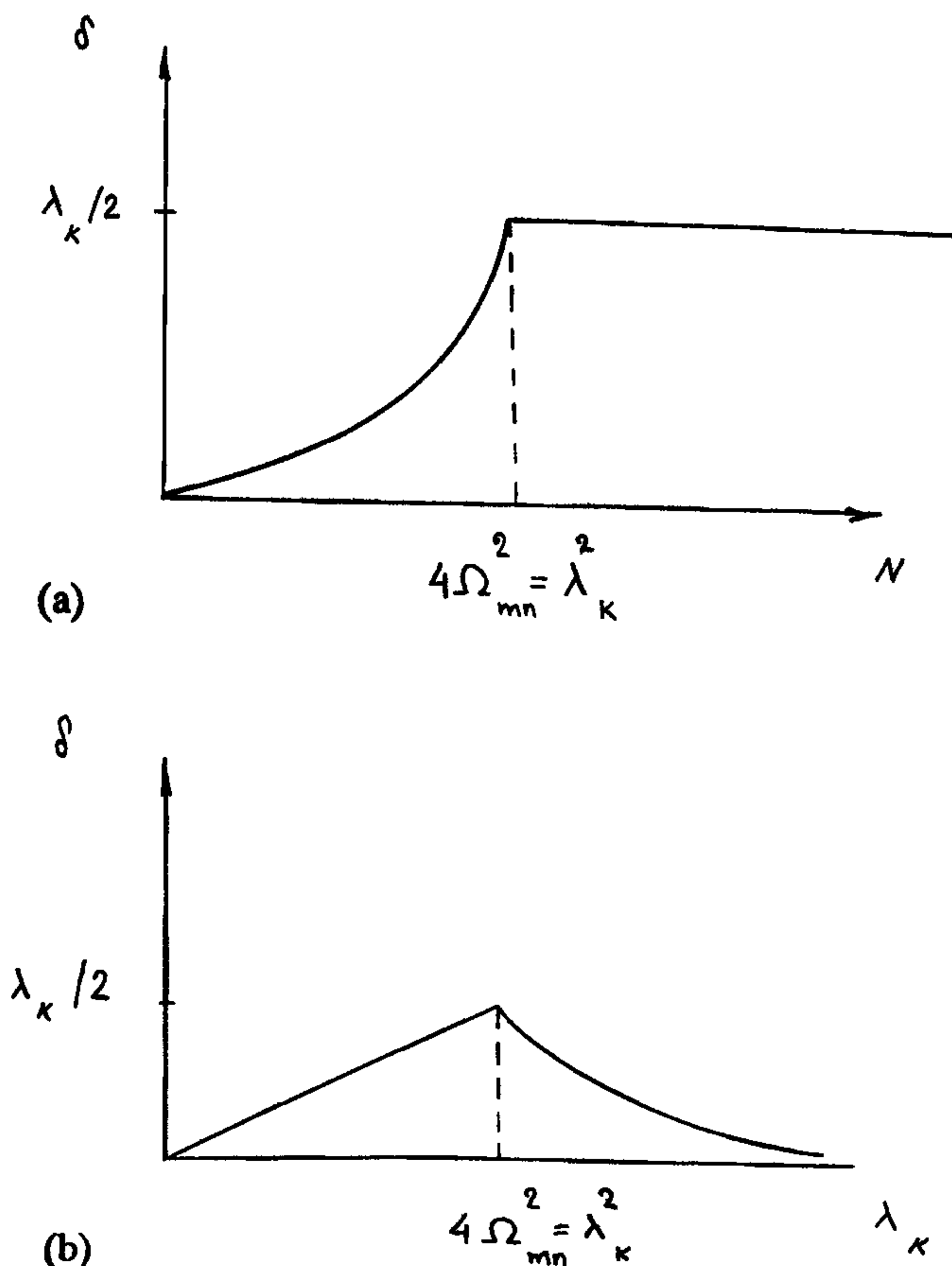


FIG. 4.4. The dependence of the decrement of betatron coherent oscillations on λ_k (a) and on the beam intensity (b) in the differential resonance; $\lambda_m=0$, $\Delta=0$.

$$\delta_m^+ = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 - N/N_{th}}} \right) \left\{ \lambda_k + \frac{\pi N}{|\partial \omega_z / \partial I_z|} \left(V^2 \frac{\partial f_0}{\partial I_z} \right)_{I_z = \Delta \omega_m^+ / (m_z \omega_z')} \right\},$$

$$\Delta \omega \pm m_z \omega_z' > 0, \quad |\Delta \omega_m^\pm| \gg |m_z| \delta \omega_z. \quad (4.2.33)$$

Therefore, coherent oscillations will be stable, provided that

$$\frac{\pi N}{|\partial \omega_z / \partial I_z|} \left| V^2 \frac{\partial f_0}{\partial I_z} \right|_{I_z = \Delta \omega_m^+ / (m_z \omega_z')} < \lambda_k < \frac{\pi N}{|\partial \omega_z / \partial I_z|} \left| V^2 \frac{\partial f_0}{\partial I_z} \right|_{I_z = \Delta \omega_m^- / (m_z \omega_z')}. \quad (4.2.34)$$

This indicates that, apart from the damping, the frequency spread can cause the antidamping of coherent oscillations, which can take place even in the case $\lambda_k = 0$. The physical reason for such an instability is clarified by Eqs. (4.2.32) and (4.2.23). Resonant particles from the tail of the distribution get the energy from the cavity, whereas the oscillations of the main part of the beam will grow or decay due to effective dissipation in the cavity. Examples of the beam stability diagrams are shown in Fig. 4.5.

4.3. BEAM-BEAM EFFECTS AND LIMITATIONS OF THE LUMINOSITY OF A COLLIDER

One of the most important parameters of a collider is its luminosity (the production rate per unit cross section of a reaction). If $\rho_{1,2}$ are densities in colliding bunches, the luminosity $\mathcal{L}$ reads

$$\mathcal{L} = \frac{2v}{T_s} \int d^2 r_1 ds dt \rho_1(\mathbf{r}_1, s+vt) \rho_2(\mathbf{r}_1, s-vt). \quad (4.3.1)$$

For a Gaussian distribution of particles in bunches and for head-on collisions the integration over transverse coordinates in Eq. (4.3.1) can be easily performed and results in

$$\mathcal{L} = \frac{N_1 N_2}{4\pi T_s \sqrt{\epsilon_x \epsilon_z}} \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi} \sigma_{\parallel}} \frac{\exp(-s^2 / \sigma_{\parallel}^2)}{\sqrt{\beta_x \beta_z}}. \quad (4.3.2)$$

Assuming that in the close vicinity of the interaction point (IP)

$$\beta_z(s) = \beta_z^* + \frac{s^2}{\beta_z^*}, \quad \beta_x(s) = \beta_x^* + \frac{s^2}{\beta_x^*}, \quad (4.3.3)$$

where $\beta_{x,z}^*$ is the value of the β -function at the (IP), we can rewrite Eq. (4.3.2) in the form

$$\mathcal{L} = \mathcal{L}_0 \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}\sigma_{\parallel}} \frac{\exp(-s^2/\sigma_{\parallel}^2)}{\sqrt{(1+s^2/\beta_x^{*2})(1+s^2/\beta_z^{*2})}}. \quad (4.3.4)$$

Here,

$$\mathcal{L}_0 = \frac{\omega_s}{2\pi} \frac{N_1 N_2}{4\pi\sigma_x\sigma_z} = \frac{N_1 N_2}{4ST_s} \quad (4.3.5)$$

is the luminosity of short bunches ($\sigma_{\parallel} \ll \beta_{x,z}^*$); $S = \pi\sigma_x\sigma_z$ is the bunch cross section area at the (IP), and T_s is the period of collisions. More or less simple expressions for $\mathcal{L}$ can be obtained in two limiting cases.

Case 1: If $\beta_x^* = \beta_z^* = \beta^*$, Eq. (4.3.4) yields

$$\mathcal{L} = \mathcal{L}_0 \frac{2\xi}{\sqrt{\pi}} \int_0^{\infty} \frac{du}{1+u^2} e^{-\xi^2 u^2} = \mathcal{L}_0 \sqrt{\pi} \xi e^{\xi^2} [1 - \text{erf}(\xi^2)],$$

$$\xi = \beta^*/\sigma_{\parallel}. \quad (4.3.6)$$

Here, $\text{erf}(x)$ is the error function.

Case 2: If, on the contrary, $\beta_x^* \gg \beta_z^*$, which typically corresponds to the flat cross section of colliding bunches ($\sigma_x \gg \sigma_z$), one can get from Eq. (4.3.4)

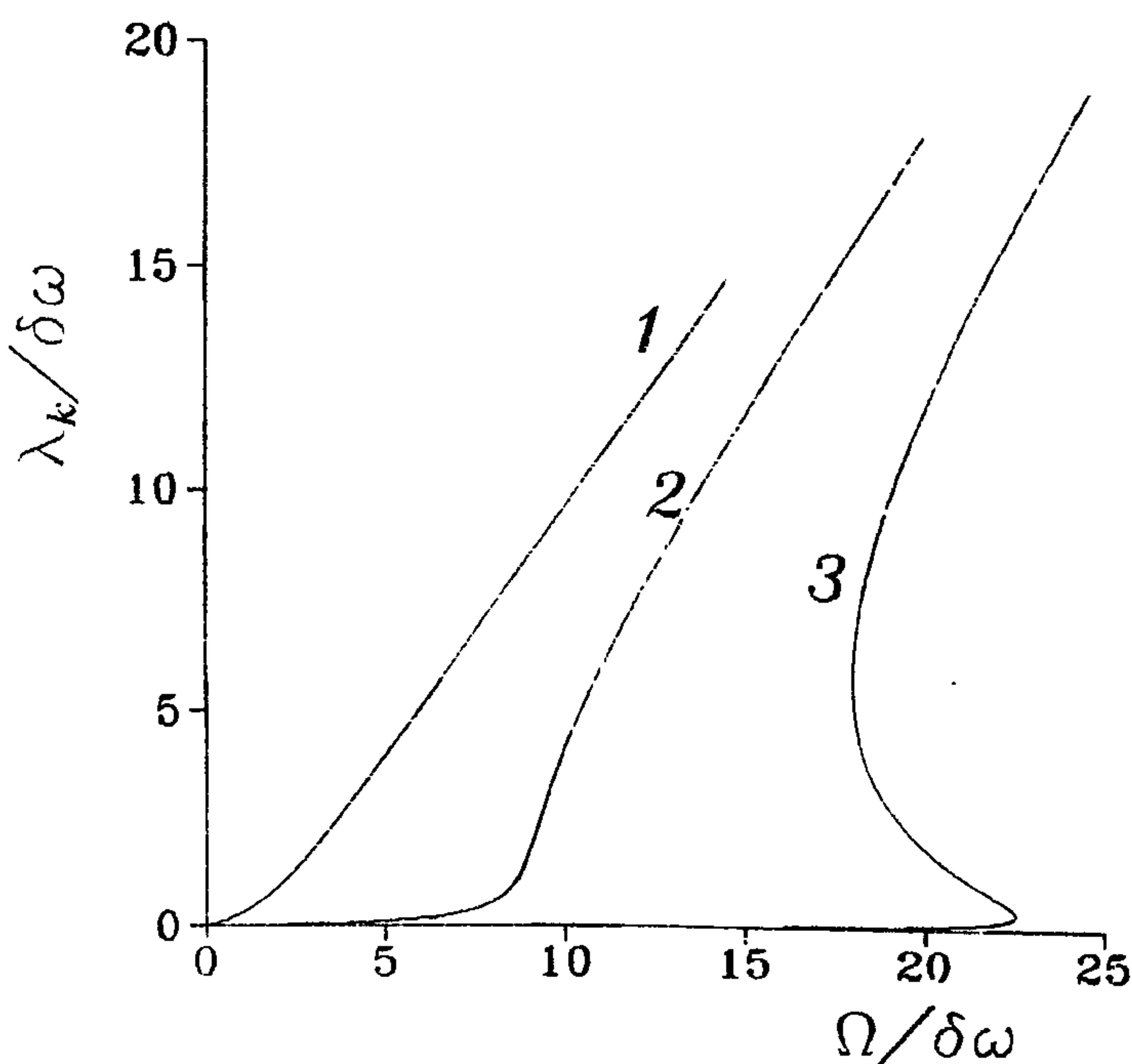


FIG. 4.5. Stability diagrams for resonant betatron coherent oscillations; $\Delta_1 < \Delta_2 < \Delta_3$.

$$\mathcal{L} = \mathcal{L}_0 \xi e^{\xi^2} K_0(\xi^2) / \sqrt{\pi}, \quad (4.3.7)$$

where $K_0(x)$ is the modified Bessel function of the 3rd kind. For long bunches ($\sigma_{\parallel} \gg \beta_{x,z}^*$) and fixed cross section areas S of bunches the asymptotes in Eqs. (4.3.6) and (4.3.7) differ by a logarithmic factor

$$\mathcal{L}_{\text{flat}} / \mathcal{L}_{\text{round}} \simeq 2 \ln(2/\xi), \quad \xi \ll 1.$$

(See also Fig. 4.6.)

All these equations indicate that the luminosity of the collider $\mathcal{L}$ increases with an increase in the intensities of colliding bunches and with a decrease in their transversal dimensions. We have seen before that coherent instabilities due to interaction of bunches with surrounding electrodes in principle can limit the achievable values of $N_{1,2}$. If these limitations are canceled (say, due to use of any damping systems), the achievable value of the luminosity $\mathcal{L}$ can be limited by electromagnetic interaction of colliding bunches (the so-called beam-beam effects). The importance of these phenomena is indicated by numerous experimental data.

Although the study of this phenomenon was started a long time ago (see, for instance, Refs. 8 and 48), a satisfactory theory describing the mutual influence of colliding bunches has not been constructed up to now. It is obvious that this problem is self-consistent—the perturbation from the countermoving beam depends on its state. Moreover, the problem is complicated by strong nonlinearity of the perturbing fields as well as by the interference with stochastic problems. That is why the approaches developed to date in this field are mainly directed to the study of particular effects as well as associated limitations, and neglect others.

One of the strongly developed approaches is the model based on the weak-strong beam approximation, when the motion of the particle from the weak beam is traced in the given fields of the strong beam. More close inspection of this model manifests the strong influence of nonlinear resonances excited by the strong beam on the sizes of the weak beam. As a rule, computer simulation must be used to get particular results though the possibility to do more or less realistic predictions even within the framework of this simplified model sometimes can be limited by resources of modern computers. To simplify the calculations the model expressions for the bunch fields are used by some codes. With a large number of iterations this can store the errors and distort the predicted results. The development of this approach has strongly supported investigations of dynamic stochasticity in the beam-beam interaction.

The computing of the mutual influence of colliding bunches (the so-called strong-strong beam interaction) presents a more complicated problem since it needs the exact calculation of bunch fields, which correspond to the instantaneous distributions in the bunches. As the first approximation to the description of self-consistent behavior of colliding bunches one can study their collective stability. Within this approach both spectra of beam-beam collective oscillations

and the limitations on the position of the working point (v_x, v_z) can be found. Moreover, the calculations from the linear theory of collective oscillations can be used in the construction of the nonlinear theory (in the calculation of kinetic equations as well as in computer simulation of strong-strong beam interaction).

Since both approaches are complementary in many points, it seems quite reasonable to discuss here the basic predictions of the weak-strong model of the beam-beam instability. More details can be found, for instance, in reviews Refs. 49 and 50. The fields from the countermoving strong bunch can be calculated using Eqs. (3.1.2), which we write in the Lorentz gauge

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_0 = -4\pi e \rho(\mathbf{r}_\perp, s - vt), \quad (4.3.8)$$

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -4\pi e \frac{\mathbf{v}}{c} \rho(\mathbf{r}_\perp, s - vt). \quad (4.3.9)$$

Here, $v = \tau \mathbf{v}$ is the average velocity of the bunch in the interaction region (IR). Solutions of the Eqs. (4.3.8) and (4.3.9) can be written in the form

$$\mathbf{A} = \frac{\mathbf{v}}{c} A_0(\mathbf{r}, t), \quad A_0(\mathbf{r}, t) = A_0(\mathbf{r}_\perp, s - vt), \quad (4.3.10)$$

where $A_0(\mathbf{r}_\perp, s - vt) = A_0(\mathbf{r}_\perp, y)$ satisfies the equation

$$\left(\Delta_\perp + \frac{1}{\gamma^2} \frac{\partial^2}{\partial y^2} \right) A_0 = -4\pi e \rho(\mathbf{r}_\perp, y). \quad (4.3.11)$$

Since $\rho(\mathbf{r}_\perp, y)$ is the periodic function of y

$$\rho(\mathbf{r}_\perp, y) = \sum_n \rho_n(\mathbf{r}_\perp) e^{ik_n y}, \quad k_n = n/R_0, \quad (4.3.12)$$

we shall find the solution of Eq. (4.3.6) in the form

$$A_0(\mathbf{r}_\perp, y) = \sum_n A_n(\mathbf{r}_\perp) e^{ik_n y},$$

where $A_n(\mathbf{r}_\perp)$ satisfies the equation

$$\Delta_\perp A_n - \frac{k_n^2}{\gamma^2} A_n = -4\pi e \rho_n(\mathbf{r}_\perp). \quad (4.3.13)$$

Taking into account that

$$\Delta_\perp A_0 \propto k_\perp^2 A_0 > A_0/l_\perp^2,$$

where $l_\perp$ is the typical transverse size of the vacuum chamber at the IP, for the harmonics $|n| \leq \gamma R_0/l_\perp$ the second term in the left-hand side in Eq. (4.3.13) is

negligible compared with the first and, thus, can be omitted. As shall be seen, the main contribution in the bunch fields yields the region $k_{\perp} a \gg 1$ (a is the bunch radius), where one can neglect the influence of the walls of the vacuum chamber and write

$$A_0(\mathbf{r}_{\perp}, y) = 4\pi N e \int \frac{d^2 k_{\perp}}{(2\pi)^2 k_{\perp}^2} \exp(i\mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp}) \rho(\mathbf{k}_{\perp}, y), \quad (4.3.14)$$

$$\rho(\mathbf{k}_{\perp}, y) = \int d^2 r_{\perp} \exp(-i\mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp}) \rho(\mathbf{r}_{\perp}, y). \quad (4.3.15)$$

Using these formulas one can describe the interaction of countermoving bunches ($\mathbf{v}_1 = -\mathbf{v}_2$) by the Lagrangian

$$L = \frac{e}{c} \mathbf{v} \mathbf{A} - e A_0 = -e A_0 (1 + v^2/c^2), \quad (4.3.16)$$

or, with the relativistic accuracy ($\gamma \gg 1$),

$$L(1,2) = -2e A_0(\mathbf{r}_{\perp}, s + vt)$$

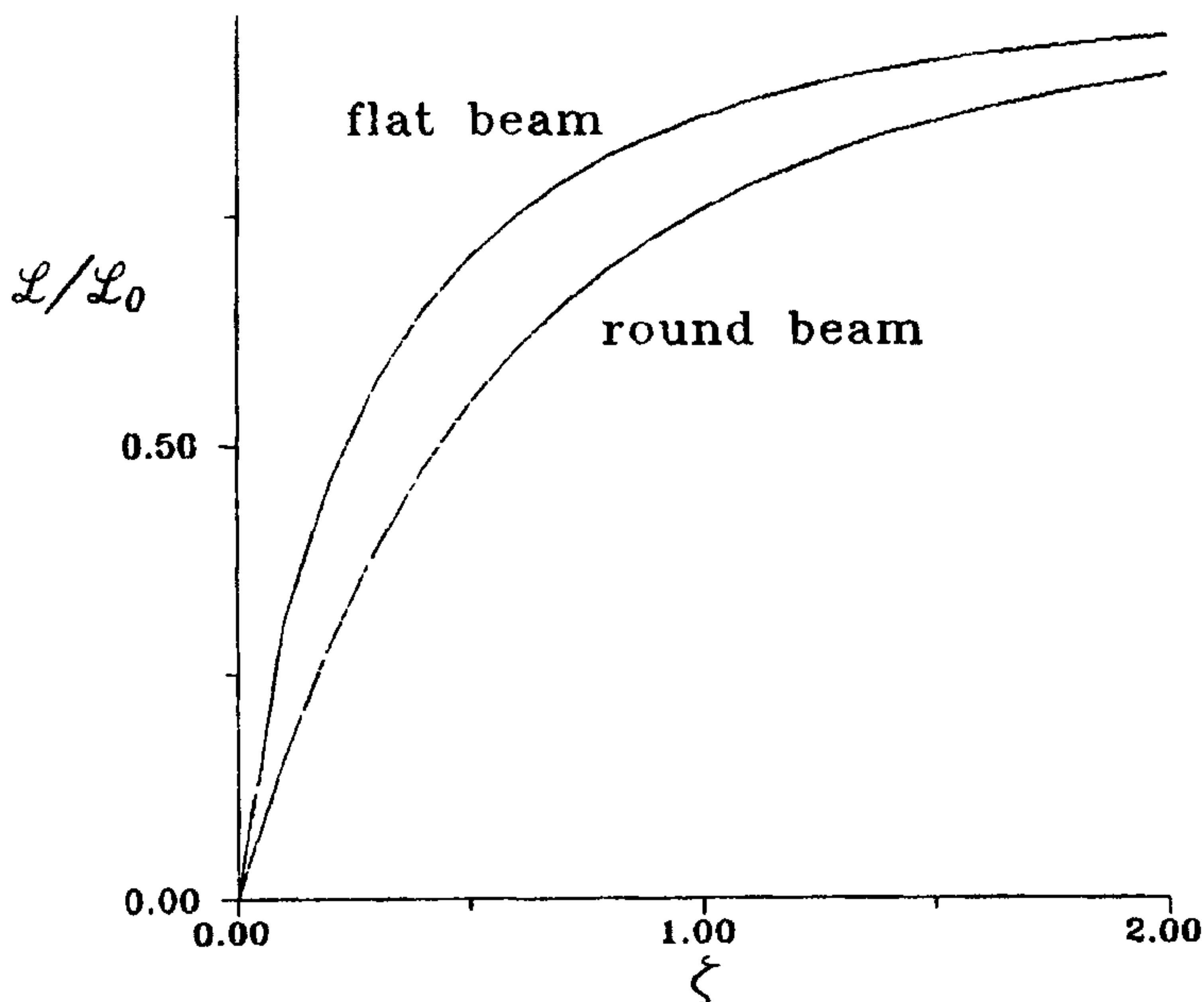


FIG. 4.6. Dependencies of the luminosities of the flat and round beams on the bunch length.

$$= -\frac{2Ne_1e_2}{\pi} \int \frac{d^2k_1}{k^2} \int d^2r'_1 \exp[i\mathbf{k}_1 (\mathbf{r}_1 - \mathbf{r}'_1)] \rho(\mathbf{r}'_1, s+vt). \quad (4.3.17)$$

Here, $e_{1,2}$ are the charges of particles in bunches 1 and 2. Since in each bunch the particles are distributed around the synchronous phase following, for instance, $\rho(\mathbf{r}) = \rho_0(s)\rho(\mathbf{r}_1)$

$$\rho_0(s-vt) = \frac{1}{\sqrt{2\pi}\sigma_{\parallel}} \exp[-(s-vt)^2/2\sigma_{\parallel}^2], \quad (4.3.18)$$

the particles passing the interaction region feel the short (during $\Delta t = \sigma_{\parallel}/2v$) periodic kicks. For two colliding bunches the period of such kicks is $T_s/2$, which means that there are two interaction points (IP) in this case. Below we assume that colliding bunches are well separated in one IP and that the spreading of the interaction over time interval $\sigma_{\parallel}/2v$ can be neglected. Then, we can rewrite the resulting Lagrangian (4.3.17) in the form

$$L(1,2) = -\frac{Ne_1e_2}{\pi v} \delta_T(t) \int \frac{d^2k_1}{k_1^2} \exp(i\mathbf{k}_1 \mathbf{r}_1) \rho(\mathbf{k}_1), \quad (4.3.19)$$

where $\delta_T(t)$ is the periodic δ -function

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \exp(-in\omega_s t). \quad (4.3.20)$$

As we saw in Chapter 1 such periodic perturbations can result in two important effects: the tune shift of the particle oscillations as well as the variation of the lattice functions; and the excitation of nonlinear resonances if the working point of the ring occurs close to the resonant values $m\nu = n$. Let us first estimate the tune shifts and associated stability limitations due to beam-beam interaction in the linear (in $\mathbf{r}_1$) approximation. Assuming

$$\rho(\mathbf{r}_1) = \frac{1}{2\pi\sigma_x\sigma_z} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right), \quad (4.3.21)$$

and substituting

$$\exp(i\mathbf{k}_1 \mathbf{r}_1) \simeq 1 + i\mathbf{k}_1 \mathbf{r}_1 - \frac{(\mathbf{k}_1 \mathbf{r}_1)^2}{2} + \dots,$$

we write

$$L(1,2) \simeq \frac{Ne_1e_2}{2\pi R_0} \delta_T(\vartheta_s) \int \frac{d^2k_1}{k_1^2} (k_x^2 x^2 + k_z^2 z^2) \exp\left(-\frac{k_x^2 \sigma_x^2 + k_z^2 \sigma_z^2}{2}\right)$$

or, using*

$$\Phi_z = \int \frac{d^2 k_1}{k_1^2} k_z^2 \exp\left(-\frac{k_x^2 \sigma_x^2 + k_z^2 \sigma_z^2}{2}\right) = \frac{2\pi}{\sigma_z(\sigma_x + \sigma_z)},$$

and, therefore,

$$L = -4\pi p_s \omega_s \delta_T(\vartheta_s) \left(\frac{\xi_x x^2}{2\beta_x^*} + \frac{\xi_z z^2}{2\beta_z^*} \right). \quad (4.3.22)$$

Here, we have introduced the so-called beam-beam parameters

$$\xi_z = -\frac{Ne_1 e_2 \beta_z^*}{2\pi p v \sigma_z(\sigma_x + \sigma_z)}, \quad (4.3.23)$$

$$\xi_x = -\frac{Ne_1 e_2 \beta_x^*}{2\pi p v \sigma_x(\sigma_x + \sigma_z)}. \quad (4.3.24)$$

The Lagrangian (4.3.22) modifies the Hamiltonian describing unperturbed motion of particles in Eq. (1.3.9) by the addition

$$\Delta h = -\frac{L}{\omega_s} = 4\pi p_s \delta(\vartheta_s) \left(\frac{\xi_x x^2}{2\beta_x^*} + \frac{\xi_z z^2}{2\beta_z^*} \right) \quad (4.3.25)$$

describing the beam-beam interaction like the action of an equivalent thin lens that is placed at the IP and has the focusing length

$$F_\alpha = \beta_\alpha^* / (4\pi \xi_\alpha). \quad (4.3.26)$$

If the betatron phase advance per turn is μ_0 , simple calculations result in

$$\cos \mu = \cos \mu_0 - 2\pi \xi \sin \mu_0, \quad \mu = 2\pi \nu. \quad (4.3.27)$$

Equation (4.3.27) clarifies the physical sense of parameters ξ_α as the tune shift per IP for weak perturbations. The equation also yields the stability condition $|\cos \mu| \leq 1$ in the form ($e_1 e_2 < 0$ is assumed)

*Here the contributing integrals can be calculated easily. Indeed, we can write

$$\Phi_z = - \int_0^\infty ds I_x(s) \frac{d}{ds} I_z(s),$$

and

$$I_\alpha(s) = \int_{-\infty}^\infty dk_\alpha \exp[-k_\alpha^2(\sigma_\alpha^2/2 + s)] = \left(\frac{2\pi}{\sigma_\alpha^2 + 2s} \right)^{1/2}.$$

$$\xi < \frac{1}{2\pi} \begin{cases} \cot(\pi\nu), & n < \nu < n + 1/2, \\ -\tan(\pi\nu), & n + 1/2 < \nu < n + 1. \end{cases} \quad (4.3.28)$$

The plot of the stability diagram (see Fig. 4.7) indicates the limitation of ξ below the resonances $2\nu = n$.

In fact, the described beam-beam instability is strongly complicated by the nonlinearity of the perturbing fields. As mentioned before, this nonlinearity results in the dependences of the tunes (ν_x , ν_z , and, generally, ν_c) on the amplitudes of oscillations and in the excitation of nonlinear resonances. This can be illustrated using some more or less simple models. As an example, we consider the model where the strong bunch perturbs the weak one by the force

$$\mathcal{F} = -2Ne^2\delta_T(ct)\frac{r_\perp}{d^2 + r_\perp^2}, \quad (4.3.29)$$

which corresponds to the density in the strong bunch:

$$\rho(r_\perp) = \frac{Nd^2}{\pi(d^2 + r_\perp^2)^2}. \quad (4.3.30)$$

The perturbed motion of particles from the weak bunch is described by the equations

$$J'_\alpha = \frac{2R_0}{pc} \frac{\partial r_\perp}{\partial \psi_\alpha} \mathcal{F}, \quad \psi'_\alpha = \nu_\alpha - \frac{2R_0}{pc} \frac{\partial r_\perp}{\partial J_\alpha} \mathcal{F}. \quad (4.3.31)$$

One can easily see that Eqs. (4.3.31) are generated by the Hamiltonian

$$\mathcal{H} = \nu_\alpha J_\alpha + 2Ne^2\delta_T(ct) \ln\left(\frac{d^2 + r_\perp^2}{d^2}\right). \quad (4.3.32)$$

If the working point of the machine (ν_x, ν_z) is tuned to some particular resonance value ($m_x\nu_x + m_z\nu_z = n$) the averaging method can be employed to study the motion of a particle in the slow variables. For vertical oscillations ($\nu_z \simeq n/m$) this yields

$$\mathcal{H} = \Delta J_z + U_{0,0} + |U_{m,n}| \cos(m\chi),$$

$$\Delta = \nu_z - n/m, \quad \chi = \psi - \vartheta(n/m) + \arg(U_{m,n}), \quad (4.3.33)$$

$$U_{m,n} = \frac{2Ne^2}{\pi} \int_0^{2\pi} \frac{d\psi}{2\pi} \ln\left(\frac{d^2 + z^2}{d^2}\right) \exp(-im\psi), \quad m = 2k. \quad (4.3.34)$$

The harmonics $U_{m,n}$ can be calculated directly. Using the substitution $2\psi = \alpha$ and after the integration by parts, $U_{m,n}$ can be rewritten in the form

$$U_{m,n} = (2Ne^2/\pi)I_k,$$

$$I_k = (i/k) \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{e^{-ik\alpha} \sin \alpha}{q + \cos \alpha}, \quad q = 1 + \frac{2d^2}{J\beta}.$$

Substituting here $z = e^{-i\alpha}$, we write

$$I_k = (1/k) \oint_{|z|=1} \frac{dz}{-2\pi i} z^{k-1} \frac{1-z^2}{z^2 + 2qz + 1},$$

or

$$I_k = \frac{(-1)^k}{k} (q - \sqrt{q^2 - 1})^k.$$

This results in

$$\mathcal{H} = \mathcal{H}_0 + |U_m| \cos(m\chi), \quad (4.3.35)$$

$$\mathcal{H}_0 = \Delta J + \xi \frac{2d^2}{\beta} \ln \left(1 + \frac{J\beta}{2d^2} + \sqrt{1 + J\beta/d^2} \right), \quad (4.3.36)$$

$$|U_m| = \begin{cases} 0, & m = 2k + 1 \\ \xi \frac{2d^2}{k\beta} \left(1 + \frac{2d^2}{J\beta} [1 - \sqrt{1 + J\beta/d^2}] \right)^k, & m = 2k. \end{cases} \quad (4.3.37)$$

Calculating the tune shift

$$\Delta\nu(J) = \frac{\partial \mathcal{H}_0}{\partial J} - \Delta = \frac{2\xi}{1 + J\beta/d^2 + \sqrt{1 + J\beta/d^2}}, \quad (4.3.38)$$

we find that the true resonant condition ($\partial \mathcal{H}_0 / \partial J = 0$) can be satisfied only within the stopband

$$-\xi \leq \Delta \leq 0. \quad (4.3.39)$$

The main specific properties of the particle phase space with the beam-beam instability can be found by examining the behavior of the Hamiltonians $\mathcal{H}_{\pm} = \mathcal{H}_0 \pm |U_m|$ introduced in §1.5. Since $\mathcal{H}_0$ and $|U_m|$ respectively have asymptotes

$$\mathcal{H}_0 = -|\Delta|J + \xi \frac{2d^2}{\beta} \ln(J/2\varepsilon), \quad J \gg \varepsilon = \frac{d^2}{\beta},$$

$$|U_m| = \xi \frac{2d^2}{\beta k} \begin{cases} 1, & J \gg \varepsilon, \\ (J/4\varepsilon)^k, & J \ll \varepsilon, \end{cases}$$

the Hamiltonians $\mathcal{H}_{\pm}$ are closed and, therefore, the oscillations are finite in J . In the phase χ the motion is finite inside the buckets centered at the fixed-points:

$$\frac{\partial \mathcal{H}_+}{\partial J} = 0, \quad \chi_j = \frac{2\pi j}{m}. \quad (4.3.40)$$

If $|\Delta| \ll \xi$, the value J_s occurs well outside the beam:

$$J_s = \varepsilon 2\xi / |\Delta| \gg \varepsilon,$$

and the sizes of the buckets as well as the frequency of the phase oscillations in these buckets can be obtained using the Hamiltonian

$$\Delta \mathcal{H}(J, \chi) = -\frac{\xi}{2u_s^2} (\delta u)^2 + \frac{\xi}{k} \cos(m\chi), \quad u = J/2\varepsilon, \quad u_s = \xi/|\Delta|. \quad (4.3.41)$$

Simple calculations yield for the bucket size (δJ_b)

$$\delta J_b = \frac{8\xi}{|\Delta| \sqrt{k}} \varepsilon, \quad k \gg 1 \quad (4.3.42)$$

or, in tunes,

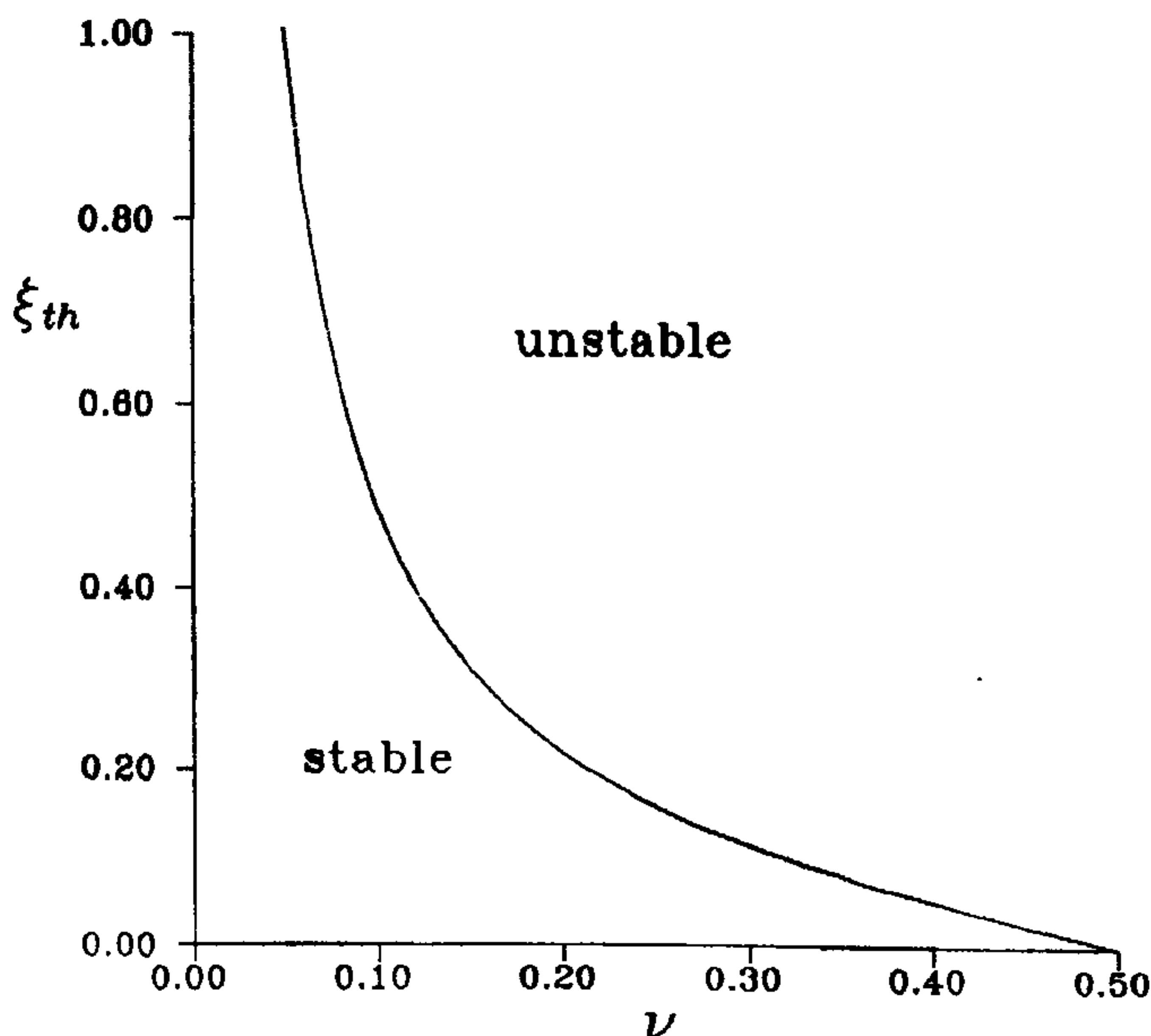


FIG. 4.7. The first period of the stability diagram of incoherent betatron oscillations for short colliding bunches.

$$\Delta\nu_b = 2|\Delta|/\sqrt{k}, \quad k \gg 1. \quad (4.3.43)$$

Since the distance between the neighbor resonances is $\delta\nu \simeq \nu/m$, after

$$k \gg k_{th} = (\nu/8\Delta)^2 \gg 1$$

these resonances will overlap and cannot be considered isolated anymore.

Below we shall also discuss the model studying horizontal (coherent or incoherent) oscillations in the ribbon beam with the transversal density

$$\rho(\mathbf{r}_\perp) = \frac{1}{\sqrt{2\pi}\sigma_x} \delta(z) \exp\left(-\frac{x^2}{2\sigma_x^2}\right). \quad (4.3.44)$$

In this model many characteristics of the beam-beam instability can be easily calculated. For instance, for $\Delta\nu_x$ one has

$$\begin{aligned} \Delta\nu_x &= -\frac{1}{\omega_s} \frac{\partial \bar{L}}{\partial I_x} = -\frac{2}{p_s \omega_s} \frac{\partial}{\partial J_x} L_{00} \\ &= \frac{Ne_1 e_2}{2\pi p_s \nu} \frac{\partial}{\partial J_x} \frac{2}{\pi} \int \frac{dk_x dk_z}{k_x^2 + k_z^2} J_0[k_x \sqrt{J_x \beta_x^*}] \exp(-k_x^2 \sigma_x^2/2). \end{aligned} \quad (4.3.45)$$

The direct integration in Eq. (4.3.45) yields

$$\begin{aligned} \Delta\nu_x(J_x) &= \xi_x \frac{2\varepsilon}{J_x} [1 - \exp(-J_x/2\varepsilon)], \\ \xi_x &= -\frac{Ne_1 e_2}{2\pi p_s \nu \varepsilon}, \quad \sigma_x^2 = \varepsilon \beta_x^*. \end{aligned} \quad (4.3.46)$$

The powers of the beam-beam resonances in the model are determined by the harmonics

$$V_{mn} = \frac{2L_{mn}}{p_s} = \xi_x \varepsilon \frac{(J_x/\varepsilon)^{|m_x|/2} \Gamma(|m_x|/2)}{|m_x|!} F(|m_x|/2, |m_x|+1, -J_x/2\varepsilon), \quad (4.3.47)$$

where $F(\alpha, \gamma, x)$ is the confluent hypergeometric function.⁵² All these equations indicate that the perturbation of particles by the beam-beam resonances generally transports particles toward the large amplitudes ($J \gg \varepsilon$). Since the powers of beam-beam resonances slowly decay when both $J_x \rightarrow \infty$ and $m_x \rightarrow \infty$, in real cases the resonances of higher orders can give important contribution to beam-beam instability. As some of them can overlap, the perturbations from these resonances generally cannot be considered independent. As a result, the densities in colliding bunches will dilute, limiting the luminosity of the collider.

The detailed description of this phenomenon is a very complicated problem. Therefore, for practical estimations of the limitations due to beam-beam effects

the critical value $\xi = \xi^*$ is introduced. Though generally it varies from one machine to another, the world averaged value is not high. Say, for electron-positron colliders $\xi^* \leq 0.05$ and approximately one order of magnitude less for proton-antiproton machines. Once some ξ^* is adopted for the collider, its luminosity will be limited by the value (round, identical bunches are assumed)

$$\mathcal{L} \leq \mathcal{L}^* = \frac{N\gamma\xi^*}{4T_s\beta^*}. \quad (4.3.48)$$

4.4. INCOHERENT BEAM-BEAM INTERACTION OF LONG BUNCHES

To conclude the discussion of the weak-strong beam approximation let us briefly describe the main specific features of the beam-beam instability caused by the finite length of colliding bunches $\sigma_{\parallel}$. As R. Siemann and R. Krishnaghopal⁵¹ pointed out, in colliders with $\beta^* \leq \sigma_{\parallel}$ the variation of the phase advance per IR of betatron oscillations cannot be considered a small number. The calculation related to beam-beam resonances must take this fact into account. Since in the case of interest, particles from one bunch at some betatron phases can cross the countermoving partner at angles $\phi \sim \sqrt{J/\beta^*}$, we may expect at least two effects: the suppression of the beam-beam kick for those phases and, hence, the powers of beam-beam resonances (the so-called phase averaging effect); and the excitation of the synchrobetatron beam-beam resonances. Both phenomena are caused by the multiple passings of the IR and, therefore, present the multiturn effects.

Using the variables action-phase (J, ψ) , we can write the most general expression for the Hamiltonian describing the beam-beam interaction in the following form:

$$H(J, \psi, \theta) = \nu_x J_x + \nu_z J_z + \nu_c J_c - V_{bb}(J, \psi, \theta). \quad (4.4.1)$$

Here, the term V_{bb} describes perturbations due to beam-beam interactions. For a Gaussian distribution over transverse coordinates in the strong bunch (4.3.21) and $e_1 e_2 = -e^2$ we can find from Eq. (4.3.17)

$$V_{bb} = \frac{4Ne^2 R_0}{pc} \rho_0(s+ct) \int \frac{d^2 k}{\pi k^2} \exp \left\{ i \mathbf{k} \mathbf{r}_1 - \frac{k_x^2 \sigma_x^2 + k_z^2 \sigma_z^2}{2} \right\}. \quad (4.4.2)$$

The strength of a resonance

$$m_x \nu_x + m_z \nu_z + m_c \nu_c = n, \quad (4.4.3)$$

where $m_{x,z,c}$ and n are integers, is determined by the Fourier harmonic

$$V_{m,n} = \int_0^{2\pi} \frac{d\psi_x d\psi_z d\psi_c}{(2\pi)^3} \int_0^{2\pi} \frac{d\theta}{2\pi} V(J, \psi, \theta) \exp(in\theta - im_x \psi_x - im_z \psi_z - im_c \psi_c). \quad (4.4.4)$$

For this reason, evaluation of relevant harmonics of V_{bb} enables conclusions concerning the importance of various beam-beam perturbations.

The calculation of the harmonics $V_{m,n}$ in Eq. (4.4.2) over phases of betatron oscillations can be done using the expansion¹⁴

$$\exp(ikx \cos \phi) = \sum_{m=-\infty}^{\infty} i^m J_m(kx) e^{im\phi}. \quad (4.4.5)$$

This yields

$$\begin{aligned} V_{m,n} = & \frac{2Ne^2}{\pi pc} \int_0^\pi ds [\rho_0(s+ct)]_{m_c} \exp[im_x \chi_x(s) + im_z \chi_z(s) + ins/R_0] \\ & \times \int \frac{d^2 k}{\pi k^2} J_{m_x}[k_x \sqrt{J_x \beta_x(s)}] J_{m_z}[k_z \sqrt{J_z \beta_z(s)}] \exp\left(-\frac{k_x^2 \sigma_x^2 + k_z^2 \sigma_z^2}{2}\right), \end{aligned} \quad (4.4.6)$$

where

$$[\rho_0(s+ct)]_{m_c} = \int_0^{2\pi} \frac{d\psi_c}{2\pi} \rho_0(s+ct) e^{-im_c \psi_c}.$$

Using the relationships $\sigma_x^2 = \epsilon_x \beta_x(s)$ and $\sigma_z^2 = \epsilon_z \beta_z(s)$, where $\epsilon_{x,z}$ are the horizontal and vertical emittances of the strong bunch, as well as the substitutions $k_x = u/\sqrt{\beta_x(s)}$ and $k_z = v/\sqrt{\beta_z(s)}$, we rewrite Eq. (4.4.6) in the form

$$\begin{aligned} V_{m,n} = & \frac{2Ne^2}{\pi pc} \int_0^\pi ds [\rho_0(s+ct)]_{m_c} \exp[im_x \chi_x(s) + im_z \chi_z(s) + ins/R_0] \\ & \times \int \frac{dudv}{\pi} \frac{J_{m_x}(u \sqrt{J_x}) J_{m_z}(v \sqrt{J_z})}{u^2 \sqrt{\beta_z(s)/\beta_x(s)} + v^2 \sqrt{\beta_x(s)/\beta_z(s)}} \exp\left(-\frac{u^2 \epsilon_x + v^2 \epsilon_z}{2}\right). \end{aligned} \quad (4.4.7)$$

If in the close vicinity of the IP holds $\beta_x(s) = \beta_z(s)$, which means that the bunches have the two-dimensional envelope with a round cross section,* the harmonic $V_{m,n}$ in Eq. (4.4.7) can be factored:

$$V_{m,n} = V_m^{(0)} Y_{m,n}. \quad (4.4.8)$$

The first factor

$$V_m^{(0)} = \frac{Ne^2}{\pi pc} \int \frac{dudv}{\pi(u^2 + v^2)} J_{m_x}(u \sqrt{J_x}) J_{m_z}(v \sqrt{J_z}) \exp\left(-\frac{u^2 \epsilon_x + v^2 \epsilon_z}{2}\right) \quad (4.4.9)$$

*The cross sections of the bunches still can be round or flat depending on the ratio of the vertical and horizontal emittances.

yields the strengths of the beam-beam resonances of the synchronous particle in the thin-lens approximation. For this reason, it does not depend on the azimuthal harmonic number n . Introducing the so-called beam-beam parameter ξ ,

$$\xi = \frac{Ne^2}{2\pi pc \sqrt{\epsilon_x \epsilon_z} (1+r)}, \quad (4.4.10)$$

where $r = \sqrt{\epsilon_z/\epsilon_x}$ is the beam aspect ratio, we can also rewrite Eq. (4.4.9) in the form

$$V_m^{(0)} = 2\xi(1+r) \int \frac{dudv}{\pi \left(\frac{u^2}{\epsilon_x} + \frac{v^2}{\epsilon_z} \right)} J_{m_x} [u \sqrt{(J_x/\epsilon_x)}] J_{m_z} [v \sqrt{(J_z/\epsilon_z)}] \exp \left(-\frac{u^2 + v^2}{2} \right). \quad (4.4.11)$$

Using the substitution

$$\frac{1}{\alpha} = \int_0^\infty ds e^{-\alpha s},$$

and the relationship¹⁴

$$\int_{-\infty}^\infty du J_m(u \sqrt{J/\epsilon}) \exp \left(-\frac{u^2}{2} \left[1 + \frac{2s}{\epsilon} \right] \right) = \begin{cases} 0, & m=2l+1, \\ \frac{\sqrt{2\pi}}{\sqrt{1+(2s/\epsilon)}} \mathcal{J}_{m/2} \left(\frac{J}{4[\epsilon+2s]} \right), & m=2l, \end{cases}$$

where $\mathcal{J}_m(x) = e^{-x} I_m(x)$, $I_m(x)$ is the Bessel function of the imaginary argument,¹⁴ we can rewrite Eq. (4.4.11) as an one-dimensional integral:

$$V_m^{(0)} = 2\xi(1+r) \int_0^\infty \frac{dt}{\sqrt{(1+t/\epsilon_x)(1+t/\epsilon_z)}} \mathcal{J}_{m_x/2} \left(\frac{J_x}{4[\epsilon_x+t]} \right) \mathcal{J}_{m_z/2} \left(\frac{J_z}{4[\epsilon_z+t]} \right). \quad (4.4.12)$$

The second factor in Eq. (4.4.8),

$$Y_{m,n} = 2 \int_0^\Pi ds [\rho_0(s+ct)]_{m_c} \exp [im_x \chi_x(s) + im_z \chi_z(s) + ins/R_0] \quad (4.4.13)$$

describes the modulation of the beam-beam interaction by the synchrotron oscillations as well as the dependencies of the powers of the beam-beam resonances $|V_{m,n}|$ on the bunch length. Without variations of $\chi_{x,z}(s)$ along the IR harmonics $V_{m,n}$ in Eq. (4.4.8) describe the betatron resonances only. The excitation of synchrobetatron resonances becomes possible essentially due to a modulation of $\chi_{x,z}$ along the IR. Such a modulation describes a possibility for particles from a weak beam to cross the strong beam at the angle

($\phi \sim \sqrt{J/\beta^*}$). For example, in the case of conventional collisions at angle ϕ , this causes a longitudinal kick ($\delta p \propto \sqrt{J/\beta^*}$), which is modulated by betatron oscillations.

The resulting expression for $V_{m,n}$ reads

$$V_{m,n} = 2\xi_{eff}(1+r) \int_0^\infty \frac{dt}{\sqrt{(1+t/\varepsilon_x)(1+t/\varepsilon_z)}} \mathcal{J}_{m_x/2}\left(\frac{J_x}{4[\varepsilon_x+t]}\right) \mathcal{J}_{m_z/2}\left(\frac{J_z}{4[\varepsilon_z+t]}\right). \quad (4.4.14)$$

It differs from Eq. (4.4.12) by the redefinition of ξ due to longitudinal effects

$$\xi_{eff} = \xi Y_{m,n}. \quad (4.4.15)$$

Eq. (4.4.14) describes the excitation of both betatron and synchrobetatron resonances of colliding bunches with a round beam envelope ($\beta_z^* = \beta^*$).

Remember that by the equation

$$\Delta\nu_\alpha = -\frac{\partial V_0^{(0)}}{\partial J_\alpha} \quad (4.4.16)$$

the harmonic $V_0^{(0)}$ gives the tune shifts of the oscillations of particles due to the beam-beam interaction. For instance, for vertical oscillations Eq. (4.4.16) results in

$$\begin{aligned} \Delta\nu_z(J_x, J_z) = & \frac{\xi(1+r)}{2\varepsilon_z} \int_0^\infty \frac{dt}{(1+t/\varepsilon_x)^{1/2}(1+t/\varepsilon_z)^{3/2}} \\ & \times \left[\mathcal{J}_0\left(\frac{J_z}{4[\varepsilon_z+t]}\right) - \mathcal{J}_1\left(\frac{J_z}{4[\varepsilon_z+t]}\right) \right] \mathcal{J}_0\left(\frac{J_x}{4[\varepsilon_x+t]}\right), \end{aligned} \quad (4.4.17)$$

Note that the tune shift in Eq. (4.4.17) is not affected by the phase averaging and coincides with a value calculated for a short strong bunch. In the region $J_z \ll \varepsilon_z$ and $J_x \ll \varepsilon_x$ this equation yields $\Delta\nu_z(0) = \xi$, but, if amplitudes' oscillations increase, the tunes shifts ($\Delta\nu_z, \Delta\nu_x$) generally decrease. If we may neglect the dependence of the synchrotron tune on $a_c = R_0\varphi_c$, a resonance condition

$$m_x\nu_x(J_x, J_z) + m_z\nu_z(J_x, J_z) + m_c\nu_c = n$$

defines a resonance line [$J_z = J_z(J_x)$] in the space (J_x, J_z) .

If the values of β_z^* and β^* differ, harmonics of the beam-beam Lagrangian $V_{m,n}$ can be factored only in special cases. Here, we calculate the strengths of resonances with the excitation of vertical betatron oscillations due to collisions with a very flat strong beam [$\beta_x(s) \gg \beta_z(s)$ and $\varepsilon_x \gg \varepsilon_z$]. For these resonances within the region [see in Eq. (4.4.7)]

$$\frac{1}{\sqrt{\epsilon_z}} > v \gg \frac{\sqrt{\beta_z}}{\sigma_x} \quad (4.4.18)$$

and

$$\sqrt{J_z \beta_z} \ll m_z \sigma_x \quad (4.4.19)$$

we can omit in the denominator of Eq. (4.4.7) the term $u^2 \sqrt{\beta_z(s)/\beta_x(s)}$. This yields

$$V_{m,n} = Y_{m,n} \left(\frac{Ne^2}{\pi p c \sqrt{\epsilon_x}} \right) \mathcal{J}_{m_x/2} \left(\frac{J_x}{4\epsilon_x} \right) \int_0^\infty \frac{dt}{\sqrt{\epsilon_z + t}} \mathcal{J}_{m_z/2} \left(\frac{J_z}{4[\epsilon_z + t]} \right), \quad (4.4.20)$$

where

$$Y_{m,n} = 2 \int_0^\Pi ds [\rho_0(s + ct)]_{m_c} \exp[im_x \chi_x(s) + im_z \chi_z(s) + ins/R_0] \sqrt{\frac{\beta_z(s)}{\beta_x(s)}}. \quad (4.4.21)$$

If apart from $\beta_x \gg \beta_z$, also $\beta_x \gg \sigma_\parallel$, which is typical for many colliders, we can neglect in the exponent in Eq. (4.4.21) the term $m_x \chi_x(s)$ as well as factor out of the integral the value $1/\sqrt{\beta_x}$. This results in

$$V_{m,n} = Y_{m,n}^f \xi_z \sqrt{\epsilon_z J_z} \mathcal{J}_{m_x/2} \left(\frac{J_x}{4\epsilon_x} \right) \int_0^{J_z/(4\epsilon_z)} \frac{dt}{t^{3/2}} e^{-t} I_{m_z/2}(t), \quad J_z \ll \sigma_x^2 / \beta_z^*. \quad (4.4.22)$$

Here,

$$Y_{m,n}^f = 2 \int_0^\Pi ds [\rho_0(s + ct)]_{m_c} \exp[im_z \chi_z(s) + ins/R_0] \sqrt{\beta_z(s)/\beta_z^*}, \quad (4.4.23)$$

and

$$\xi_z = \frac{Ne^2 \beta_z^*}{2\pi p c \sigma^* \sigma_x}.$$

Note that $Y_{m,n}^f$ does not depend on m_x .

The factors $Y_{m,n}$ in Eqs. (4.4.14) and (4.4.22) describe the effect of the finite length of the strong bunch as well as the effect of the Floquet-modulation on the strengths of the beam-beam resonances and the excitation of synchrotron resonances. As far as these effects can be described by a redefinition of the beam-beam parameter, below we can focus on the dependencies of Y_m and Y_m^f on the bunch length and on the amplitude of the synchrotron oscillations. Specific features of the resonance suppression factors (Y_m and Y_m^f) strongly depend on the ratio of β_z and β_x at the (IP). Below we discuss separately the

cases when $\beta_x(\vartheta) = \beta_z(\vartheta)$ and when $\beta_x(\vartheta) \gg \beta_z(\vartheta)$, $\varepsilon_x \gg \varepsilon_z$.

In the first case, taking into account that in the close vicinity of the (IP) β -functions are described by Eqs. (4.3.3) so the phase advances of betatron oscillations are

$$\psi_x(s) = \psi_z(s) = \arctan(s/\beta^*), \quad (4.4.24)$$

we rewrite Y_m from Eq. (4.4.13) as

$$Y_{m_1}(\varphi_c) = 2R_0 \int_{-\infty}^{\infty} d\vartheta_s [\rho_0(2\vartheta_s - \varphi)]_{m_c} \exp[i2q \arctan(\vartheta_s R_0/\beta^*)]. \quad (4.4.25)$$

$$2q = m_x + m_z$$

Substituting here as ρ_0 a Gaussian distribution and $x = a_c/(2\sigma_{\parallel})$ ($a_c = R_0\varphi_c$) as well as

$$\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}, \quad \sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}},$$

we can rewrite Eq. (4.4.25) in the form

$$Y_m = \xi \sqrt{2/\pi} \int_{-\infty}^{\infty} du \left(\frac{1+iu}{1-iu} \right)^q \int_0^{2\pi} \frac{d\psi_c}{2\pi} \exp[-2\xi^2(u - x\cos\psi_c)^2 - im_c\psi_c]. \quad (4.4.26)$$

Eq. (4.4.26) is very difficult for analytic calculation. However, it can be transformed to simplify both the numerical calculations and the calculation of the asymptotes of Y_m . First, we rewrite it in the form

$$\begin{aligned} Y_m &= (-1)^q \xi \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \frac{d\psi_c}{2\pi} e^{-im_c\psi_c} \int_{-\infty}^{\infty} du e^{-2\xi^2 u^2} \left(1 - \frac{2}{1-iu-ix\cos\psi_c} \right)^q \\ &= (-1)^q \left\{ \delta_{m_c,0} + \xi \sqrt{2/\pi} \int_0^{2\pi} \frac{d\psi_c}{2\pi} e^{-im_c\psi_c} \right. \\ &\quad \times \left. \int_{-\infty}^{\infty} du \exp(-2u^2\xi^2) \sum_{\mu=1}^q \frac{l}{\mu!(l-\mu)!} \frac{(-2)^\mu}{(1-iu-ix\cos\psi_c)^\mu} \right\}. \quad (4.4.27) \end{aligned}$$

Now, using the substitution

$$\frac{1}{(1-iu-ix\cos\psi_c)^\mu} = \frac{1}{(\mu-1)!} \int_0^\infty ds s^{\mu-1} \exp[-s(1-iu-ix\cos\psi_c)],$$

we can easily calculate the phase harmonic

$$[\exp(isx\cos\psi_c)]_{m_c} = i^{m_c} J_{m_c}(sx) \quad (4.4.28)$$

and the integral over u

$$\xi \int_{-\infty}^{\infty} du \exp(-2u^2\xi^2 + isu) = \sqrt{\pi/2} \exp[-s^2/(8\xi^2)]. \quad (4.4.29)$$

After these calculations we find that the sum in Eq. (4.4.27)

$$\sum_{\mu=1}^q \frac{q!(-2s)^{(\mu-1)}}{\mu!(\mu-1)!(l-\mu)!}$$

coincides with the associated Laguerre polynomial $L_{q-1}^1(2s)$. (See, for instance, Ref. 14.) The resulting expression for Y_m reads

$$Y_m = (-1)^q \left[\delta_{m_c,0} - 2i^{m_c} \int_0^{\infty} ds \exp\left(-s - \frac{s^2}{8\xi^2}\right) J_{m_c}(sx) L_{q-1}^1(2s) \right]. \quad (4.4.30)$$

The functions $Y_m(\xi, x)$ in Eq. (4.4.30) coincide with similar functions calculated in Ref. 51 assuming a constant β -function along the IR, only in the region $\xi \gg 1$ and $x \ll 1$. In the asymptotic region ($\xi \ll 1$, and/or $a_c \gg \beta_z^*$) the dependencies of $Y_m(\xi, x)$ in Eq. (4.4.30) on the bunch length and on the amplitude of synchrotron oscillations strongly differ from those predicted in Ref. 51. For instance, for large amplitudes of synchrotron oscillations ($x \rightarrow \infty$) Eq. (4.4.30) predicts $Y_m \rightarrow (-1)^q$ for betatron resonances ($m_c = 0$), and $Y_m \simeq -2i^{m_c}/x$ for synchrobetatron resonances ($m_c \neq 0$). This discrepancy is a result of the modulation of β -functions along the IR, which is taken into account in Eq. (4.4.30). Simple asymptotic expressions can be obtained for long bunches $\xi \ll 1$. In this case, we may neglect in the integrand of Eq. (4.4.30) the variation of the combination $\exp(-s) L_{q-1}^1(2s)$ within the interval $0 \leq s < 8\xi^2 \ll 1$. Remaining integration in Eq. (4.4.30) results in

$$Y_m \simeq (-1)^q [\delta_{m_c,0} - 2q\xi i^{m_c} \sqrt{2\pi} \mathcal{J}_{m_c/2}(x^2\xi^2)], \quad x \gg q\xi. \quad (4.4.31)$$

For the first synchrobetatron sidebands ($|m_c| = 1$) this equation, for example, yields

$$|Y_m| \simeq \frac{2q\xi}{x} [1 - \exp(-2x^2\xi^2)] \simeq \begin{cases} 4lx\xi^3, & a_c \ll \sigma_{\parallel}, \\ 2l(\beta_z^*)^2/(\sigma_{\parallel}a_c), & a_c \gg \sigma_{\parallel}. \end{cases} \quad (4.4.32)$$

In the region $\xi \sim 1$ the factor Y_{q,m_c} must be calculated numerically. These calculations indicate (see Fig. 4.8) that Y_{q,m_c} can significantly decrease when amplitudes of synchrotron oscillations of particles are small, and when ξ does not differ very much from one. Outside these regions the strengths of these resonances do not differ very much from those calculated in the zero bunch length approximation. Among others, this means that in the region ($\sigma_{\parallel} \simeq \beta^*$) the influence of beam-beam effects on the lifetime of the beam will be mainly deter-

mined by the tails of its longitudinal distribution. These qualitative results have been confirmed by numerical simulations of the phase averaging effect in Refs. 51 and 53. Note also that the suppression of beam-beam resonances in the region of small amplitudes of synchrotron oscillations ($a_c \leq \sigma_{\parallel}$) generally increases, when the order of the resonance q increases. (See Fig. 4.9.) Due to

$$q = \frac{|m_x + m_z|}{2} < \frac{|m_x| + |m_z|}{2}, \quad m_x m_z < 0,$$

the strengths of two-dimensional differential-type betatron resonances can be higher than that of the sum-type resonances.

For flat bunches ($\beta^* \gg \beta_z^*$, $\varepsilon_x \gg \varepsilon_z$) the integrals in Eq. (4.4.23),

$$Y_m^f = 2 \int_{-\infty}^{\infty} ds [\rho(s+ct)]_{m_c} \exp[im_z \phi_z(s)] \sqrt{\beta_z(s)/\beta_z^*}, \quad (4.4.33)$$

do not enable any significant simplifications. For this reason, it was calculated numerically. Results of these calculations (see Fig. 4.10) show the suppression of both betatron and synchrobetatron resonances in the region $a_c < \beta_z^*$ and $\xi \simeq 1$, as well as the enhancement of this suppression with the increase of the betatron order of a resonance ($l = m_z/2$). At small amplitudes of synchrotron oscillations ($a_c < \beta_z^*$) the behavior of the functions $Y_m^f(\xi, a_c)$ and $Y_m(\xi, a_c)$ is very similar. One can see, however, that for large amplitudes of synchrotron oscillations the strengths of resonances in flat bunches ($\propto |Y_m^f|$) can exceed the nominal value ($|Y_m| = 1$). The difference in the behavior of Y_m and Y_m^f is caused by an additional factor $\sqrt{\beta_z(s)/\beta_z^*}$ in Eq. (4.4.33), describing the mismatching of the beam-beam kick and of the response of betatron oscillations in flat bunches.

For flat colliding bunches the tune shift of vertical oscillations also depends both on the length of the strong bunch and on the amplitudes of synchrotron oscillations. Using Eq. (4.4.22), we write

$$\Delta \nu_z(J_x, J_z, J_c) = Y_0^f(\xi, a_c) \Delta \nu_z^{(0)}(J_x, J_z), \quad (4.4.34)$$

where

$$\Delta \nu_z^{(0)}(J_x, J_z) = \xi \mathcal{J}_0 \left(\frac{J_x}{4\varepsilon_x} \right) \sqrt{\frac{\varepsilon_z}{J_z}} \int_0^{J_z/(4\varepsilon_z)} \frac{dt}{\sqrt{t}} e^{-t} [I_0(t) - I_1(t)] \quad (4.4.35)$$

is the tune shift of the vertical betatron oscillations due to collisions with a short strong bunch. For a Gaussian linear density the function $Y_0^f(\xi, a_c)$ generally increases when either $\sigma_{\parallel}$ or a_c increase.

Apart from a modification of the lattice functions, this dependence results in at least two important consequences. First, a resonance condition

$$m_x \nu_x(J_x, J_z) + m_z \nu_z(J_x, J_z, J_c) + m_c \nu_c = n \quad (4.4.36)$$

can now be held in three-dimensional space (J_x, J_z, J_c) and, therefore, determines a resonance surface instead of the resonance line. Generally, this can enhance the perturbations of the bunch distribution function due to the so-called effect of the phase convection.⁵⁴

Second, a common growth of Y_m^f and Y_0^f at large amplitudes of the synchrotron oscillations ($a_c \gg \beta_z^*$) generally increases the widths of resonances. Simple expressions estimating these widths can be obtained, for example, in an asymptotic region ($J_z \gg \varepsilon_z$) for a pure vertical one-dimensional resonance [$v_z(J_z, J_c) = n/m_z$, $m_c = 0$, and $J_x = 0$] when Eqs. (4.4.22) and (4.4.35) can be rewritten in the form

$$V_{m,n} \simeq Y_{m,n}^f \frac{4\xi_z \sqrt{\varepsilon_z J_z}}{m_z^2}, \quad \varepsilon_z \ll J_z \ll \sigma_x^2 / \beta_z^* \quad (4.4.37)$$

and

$$\Delta v_z \simeq \frac{CY_0^f \xi_z}{2} \sqrt{\frac{\varepsilon_z}{J_z}}, \quad C = 2 \int_0^\infty \frac{dt}{\sqrt{t}} e^{-t} [I_0(t) - I_1(t)] \simeq 3.19. \quad (4.4.38)$$

Using these expressions and $\Delta = v_z - n/m_z$ as the detuning from the resonance, we can write a reduced Hamiltonian describing the motion in the slow variables (J_z , $\chi = \psi_z - n\theta/m_z$):

$$h = \frac{\Delta}{\xi_z} x^2 + CY_0^f x + \frac{4}{m_z^2} |Y_m^f| x \cos(m\chi), \quad x = \sqrt{J_z / \varepsilon_z}. \quad (4.4.39)$$

Without synchrotron radiation damping the fixed points of the resonance are given by the equations

$$\partial h / \partial \chi = 0, \quad \partial h / \partial J_z = 0. \quad (4.4.40)$$

The first equation yields the stationary phases [$\sin(m_z \chi) = 0$], which after their substitution in Eq. (4.4.39) determine the so-called $h_\pm$ -Hamiltonians:

$$h_\pm = \frac{\Delta}{\xi_z} x^2 + CY_0^f x \pm \frac{4}{m_z^2} |Y_m^f| x. \quad (4.4.41)$$

Then, the second equation in (4.4.40) gives two stationary amplitudes

$$\frac{\partial h_\pm}{\partial x} = 2 \frac{\Delta}{\xi_z} x + CY_0^f \pm \frac{4 |Y_m^f|}{m_z^2} = 0. \quad (4.4.42)$$

If $\Delta < 0$, these equations have solutions

$$x_0^\pm = \frac{\xi_z Q_\pm}{2 |\Delta|}, \quad Q_\pm = (CY_0^f \pm 4 |Y_m^f| / m_z^2). \quad (4.4.43)$$

The root x_0^+ gives the position of the center of the bucket, while the roots of the equation

$$h_-(x_0^-) = \frac{\xi_z}{4|\Delta|} Q_-^2 = h_+(x) \quad (4.4.44)$$

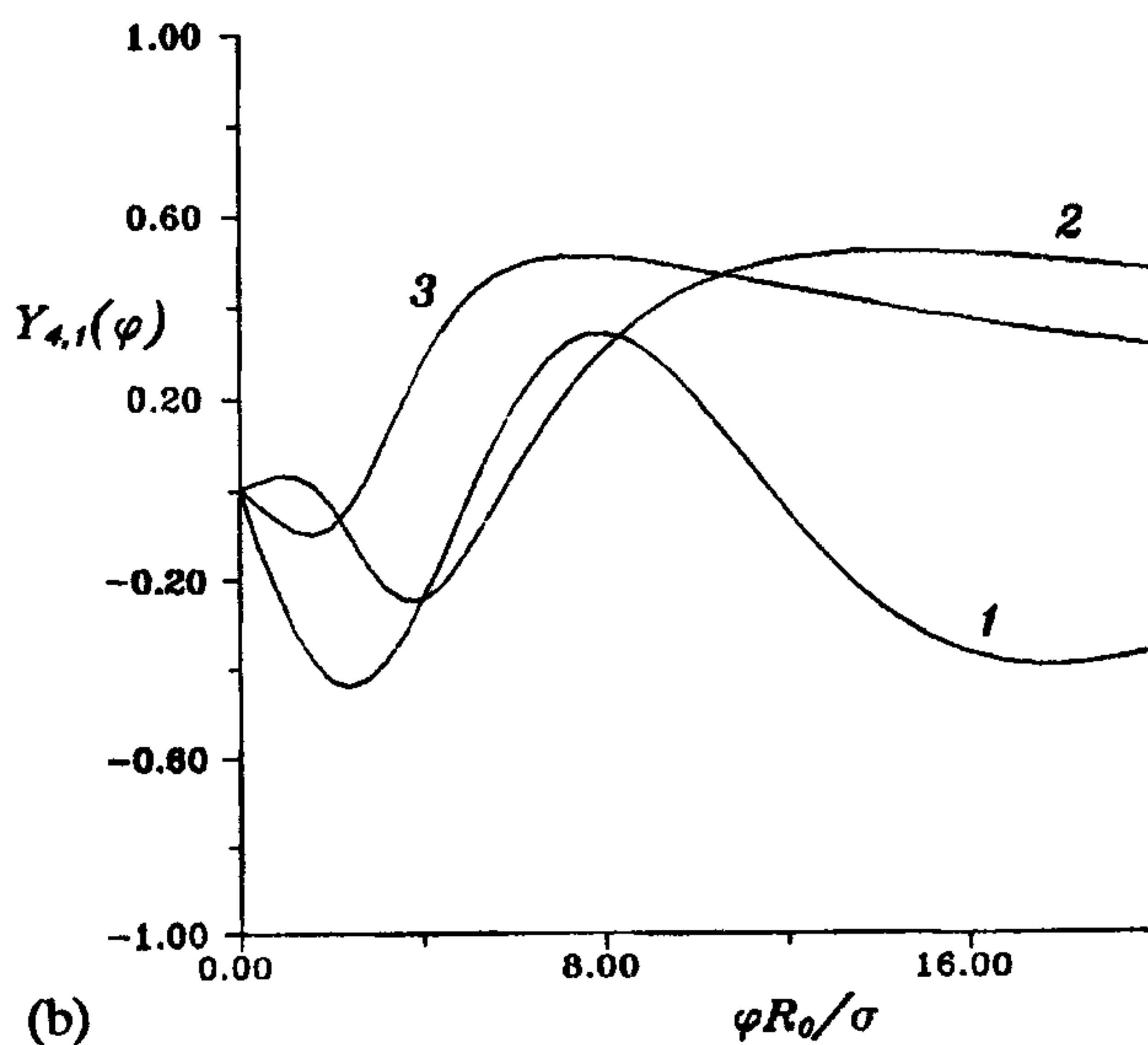
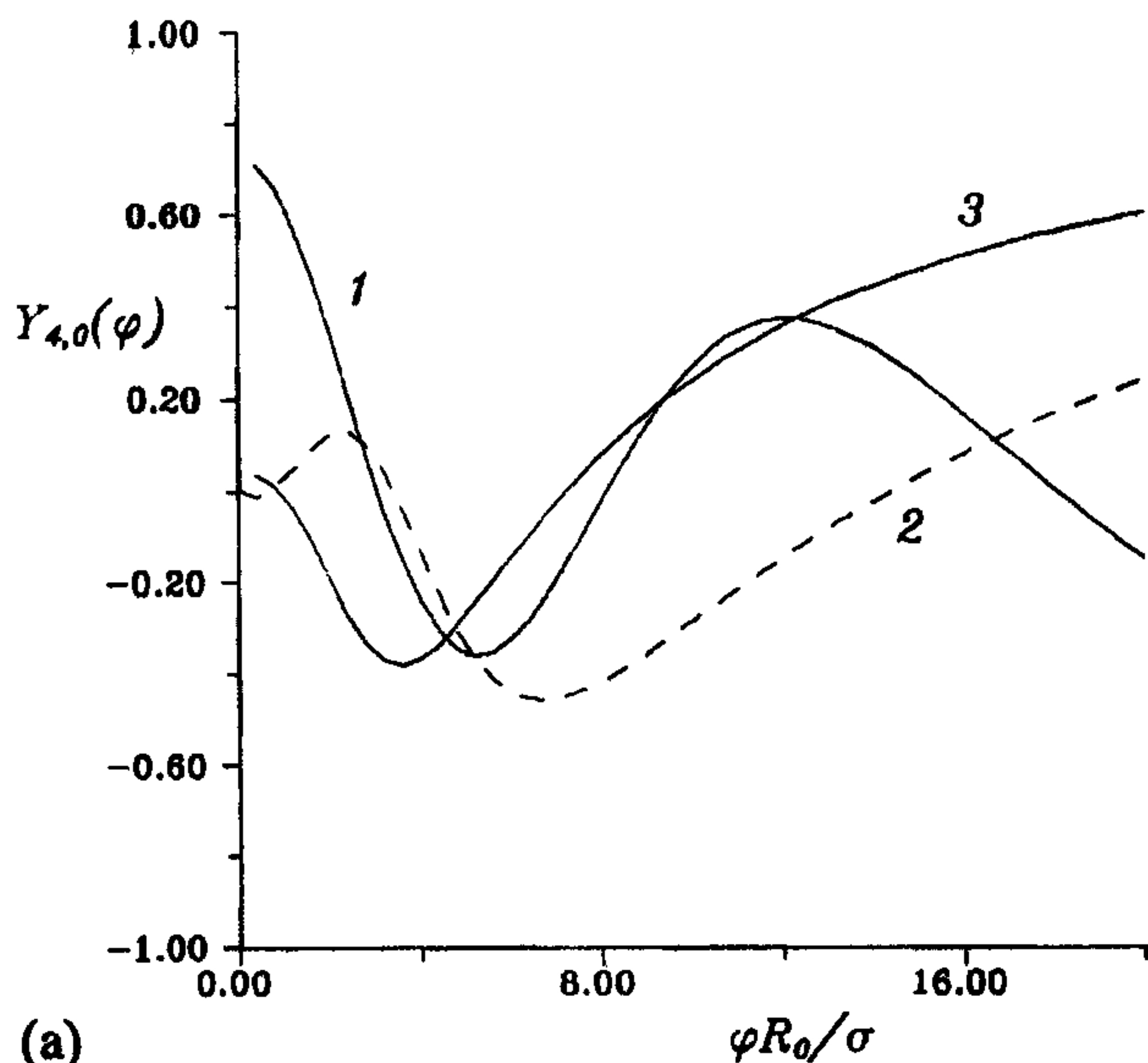


FIG. 4.8. Dependencies of the suppression factors Y_{q,m_c} for betatron (a) and for the first synchrobetatron sideband (b) beam-beam resonances on the amplitude of synchrotron oscillations; $q=4$; 1. $\xi=5$; 2. $\xi=1$; 3. $\xi=0.5$.

yield the width of the bucket in J_z -space. Simple calculations result in

$$x_{1,2} = \frac{\xi_z}{2|\Delta|} (Q_+ \pm \sqrt{Q_+^2 - Q_-^2}) \quad (4.4.45)$$

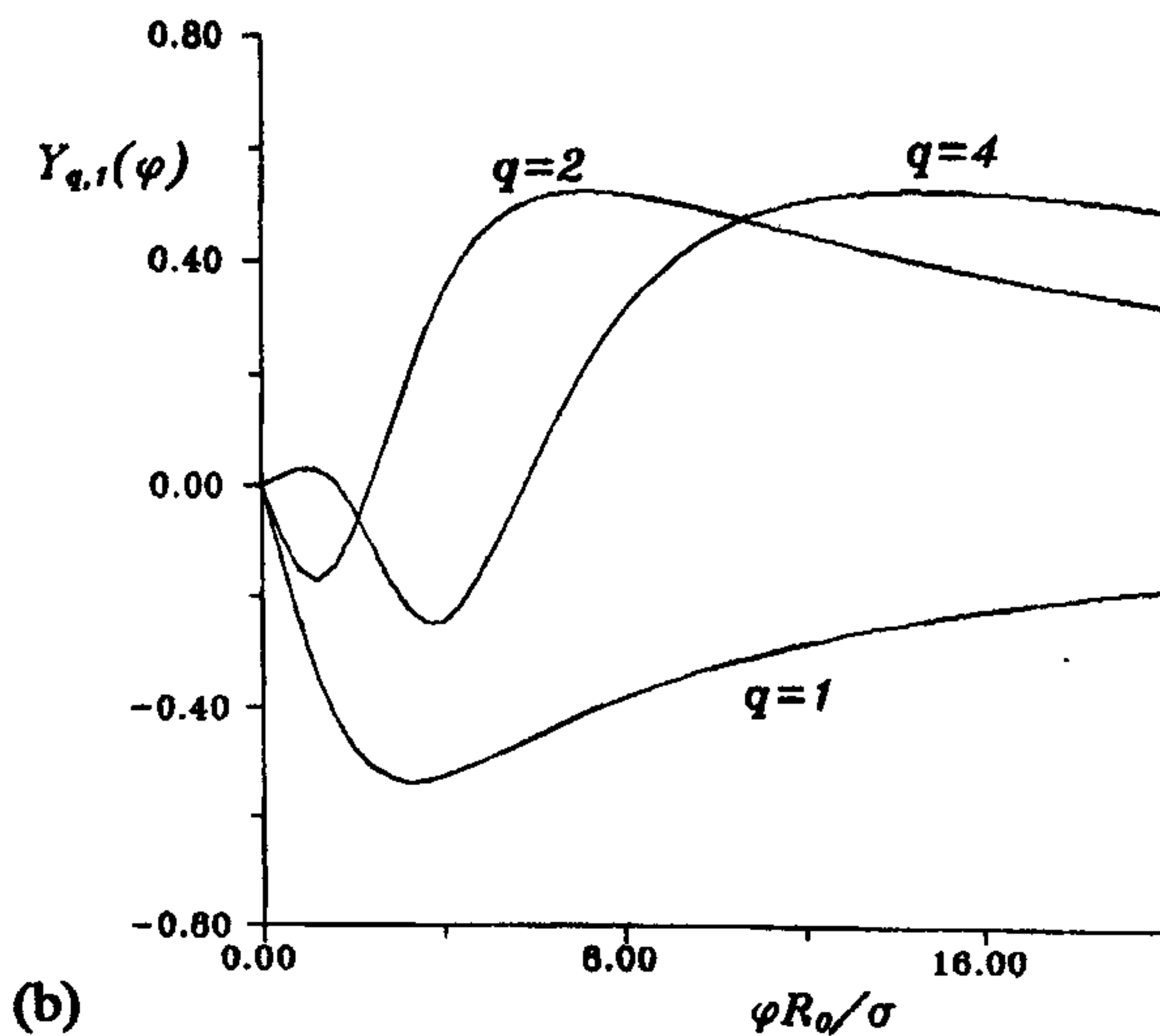
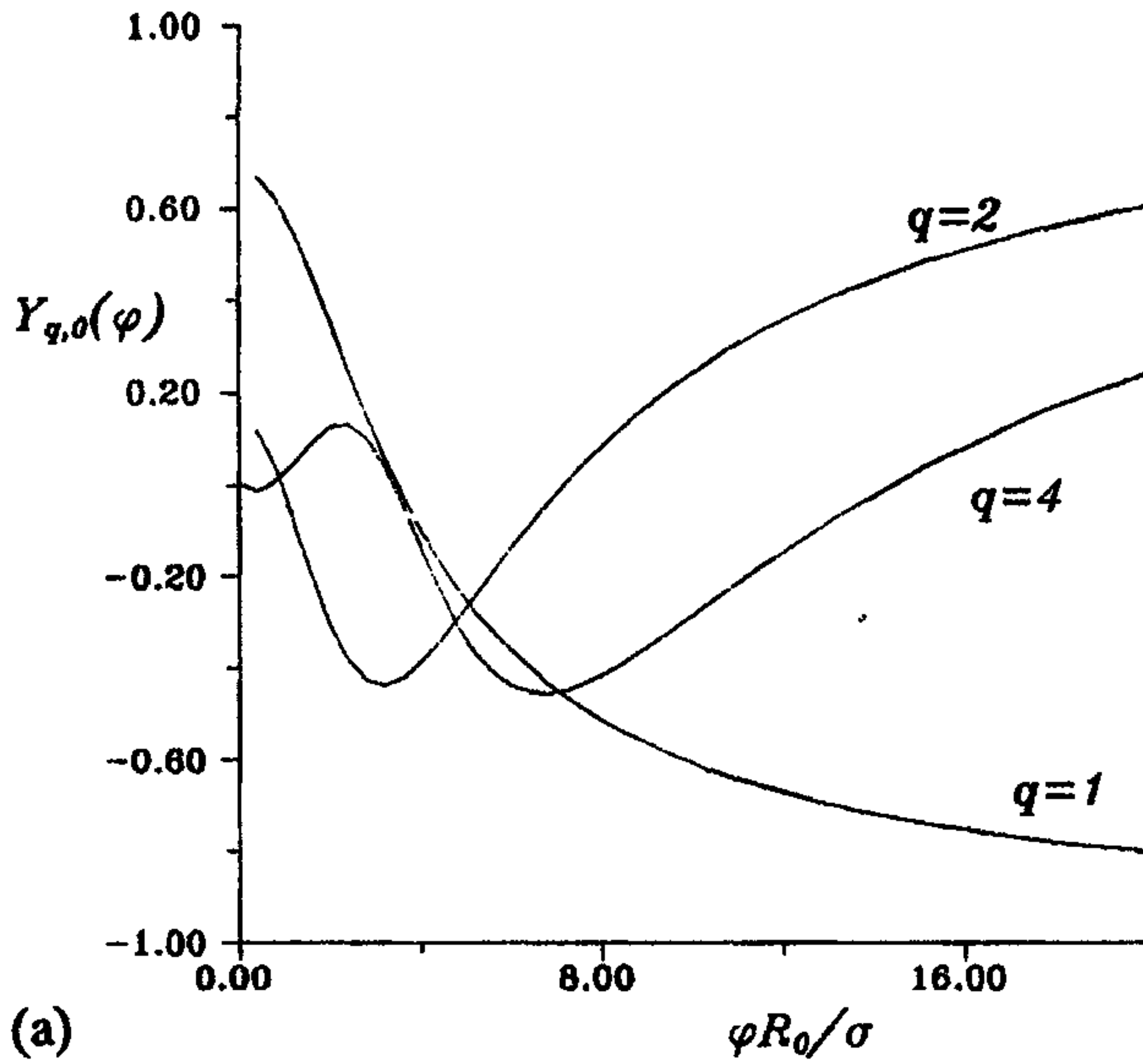


FIG. 4.9. Dependencies of the suppression factors Y_{q,m_c} for betatron (a) and for the first synchrobetatron sideband (b) beam-beam resonances on the amplitude of synchrotron oscillations and on the order of the resonance q ; $\zeta = 1$.

or

$$\Delta x = x_1 - x_2 = \frac{\xi_z}{2|\Delta|} \sqrt{Q_+^2 - Q_-^2} = \frac{4\xi_z}{m_z|\Delta|} \sqrt{CY_0^f |Y_m^f|}. \quad (4.4.46)$$

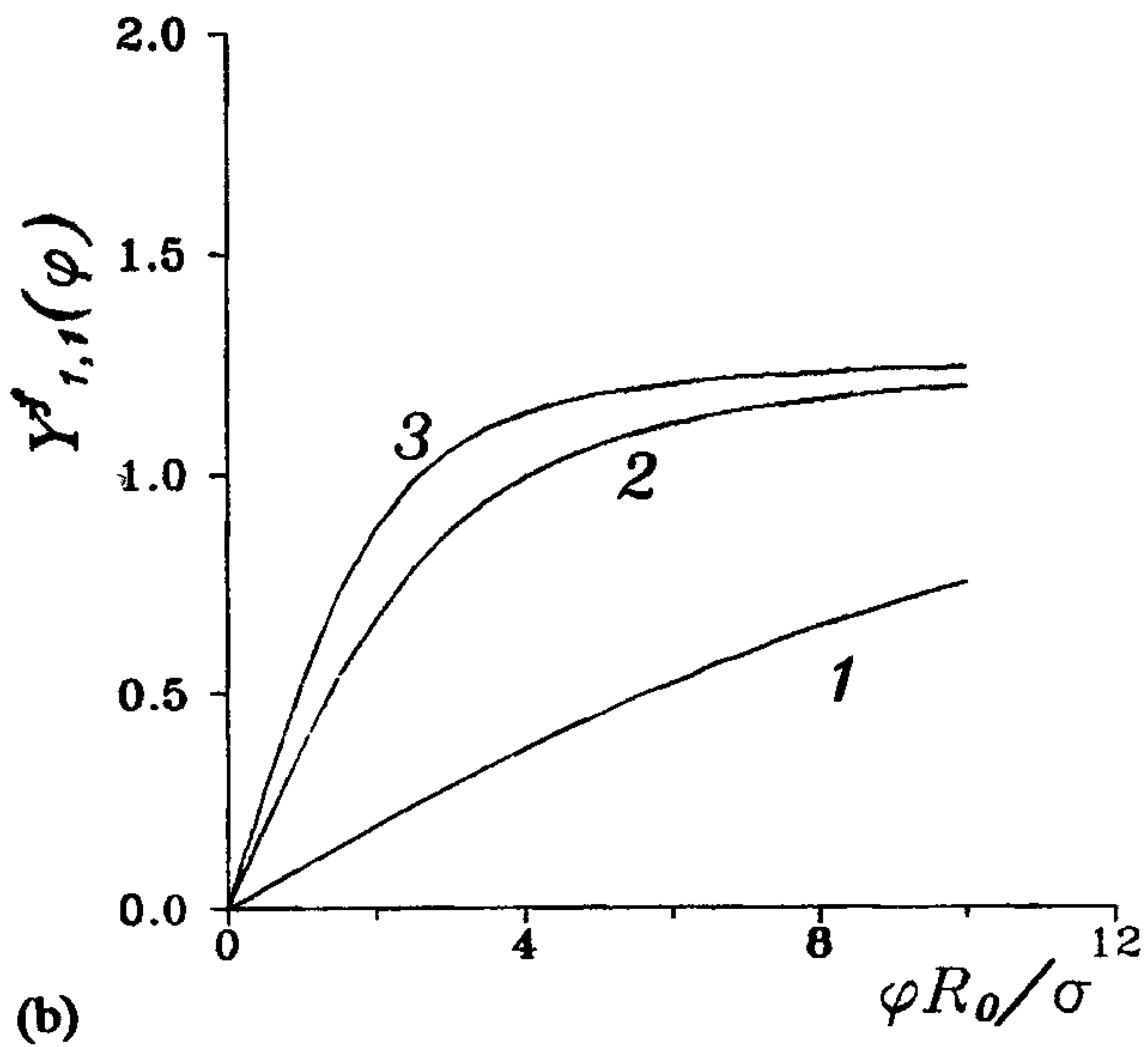
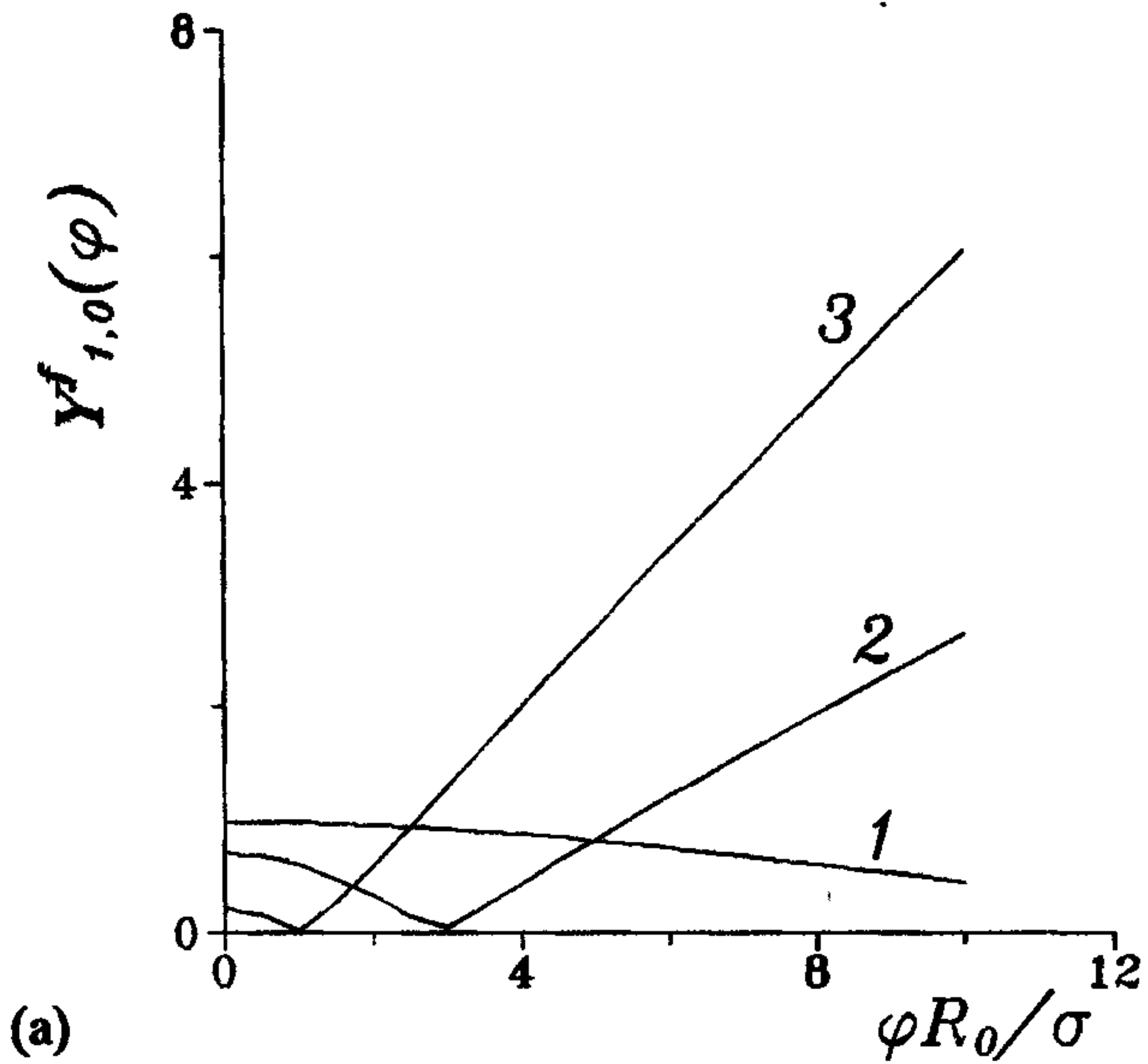


FIG. 4.10. Dependencies of the suppression factors Y_{q,m_c}^f for the first betatron (a) and for the first synchrobetatron sideband (b) beam-beam resonances on the amplitude of synchrotron oscillations; 1. $\xi=5$; 2. $\xi=1$; 3. $\xi=0.5$.

The (dimensionless) frequency of small phase oscillations near the center of the bucket is

$$\delta\nu_b = 2|\Delta| \sqrt{\frac{|Y_m^f|}{Q_+}}. \quad (4.4.47)$$

The synchrotron radiation damping does not destroy the resonance and, hence, during the cooling particles can be collected in the bucket, if $\delta\nu_b \gg \Lambda_z$. Here, Λ_z is a dimensionless decrement of vertical betatron oscillations. The widths of these resonances in $\Delta\nu$ -space reads

$$\delta\nu = \Delta\nu_z(x_2) - \Delta\nu_z(x_1) = \frac{8|\Delta|(CY_0^f)^{3/2}}{m_z Q_-^2} \sqrt{|Y_m^f|}. \quad (4.4.48)$$

For higher-order resonances ($Q_{\pm} \simeq CY_0^f$) Eqs. (4.4.47) and (4.4.48) can be simplified:

$$\delta\nu \simeq \frac{4\delta\nu_b}{m_z}, \quad \delta\nu_b = 2|\Delta| \sqrt{\frac{|Y_m^f|}{CY_0^f}}. \quad (4.4.49)$$

Equations (4.4.46)–(4.4.49) clearly indicate the suppression of the widths of the discussed resonances in the regions where Y_m^f is small ($a_c \leq \beta_z^*$ and $\sigma_{\parallel} \simeq \beta_z^*$), as well as a general increase of the widths in J -space and a displacement of the centers of the buckets toward higher amplitudes in the region $a_c \gg \beta_z^*$. Since the widths of higher-order resonances in $\Delta\nu$ -space are proportional to $\sqrt{Y_m^f/Y_0^f}$, they do not have such a growth.

Described dependences of factors Y_m and Y_m^f on the bunch length and on the amplitude of the synchrotron oscillations exhibit several phenomena, which differ in terms of specific features of the beam–beam instability in long and short bunches. First, Ref. 51 predicts the suppression of the beam–beam resonances with the increase of the bunch length. Although asymptotic dependencies of suppression factors differ from that found in Ref. 51 and (for $\beta_z^* = \beta^*$)

$$|Y_m| \propto \begin{cases} 1, & m_c = 0, \\ (\beta_z^*)^2 / (\sigma_{\parallel} a_c), & m_c \neq 0, \end{cases} \quad \sigma_{\parallel} \gg \beta_z^*, a_c \gg \beta_z^*,$$

the strengths of both betatron and synchrobetatron resonances strongly decrease in the region $\sigma_{\parallel} \simeq \beta_z^*$ and $a_c \leq \beta_z^*$. Since these resonances are responsible for the blow-up of the beam, we may expect an increase in the threshold value of ξ and associated increase of the peak luminosity in the region $\sigma_{\parallel} \simeq \beta_z^*$. The suppression of the beam–beam resonances occurs due to a common effect of the finite bunch length and of the Floquet-modulation of phases of betatron oscillations along the IR. This result holds for collisions of bunches both with equal vertical and horizontal β -functions and with flat geometry. The suppression of the

beam-beam resonances in the region $\sigma_{\parallel} \simeq \beta_z^*$ qualitatively agrees with results of measurements at CESR.⁵⁵

Second, due to Floquet-modulation of the betatron phases, the collisions of long bunches is accompanied by the excitation of synchrobetatron resonances. At large amplitudes of synchrotron oscillations the strengths of these resonances become comparable with the strengths of the betatron resonances in short bunches. These resonances blow up the tails of colliding bunches and, therefore, can limit the lifetime of the beam. As was mentioned, in the case of head-on collisions the synchrobetatron resonances are excited since a particle crosses the counter-moving bunch at the angle $\sqrt{J/\beta_z^*} \sin \psi$. However, in contrast to the case of the conventional crossing at the angle, these resonances do not select the parity of the synchrotron order of the resonance (m_c). The increase in the strengths of synchrobetatron resonances for large amplitudes of synchrotron oscillations with an increase in the modulation of β -functions in the IR is in general agreement with results of simulations in Ref. 56.

Specific features of the beam-beam resonances of long bunches essentially depend on the ratio of the vertical and horizontal β -functions in the IR. In the case of the round beam envelope ($\beta_z^* = \beta^*$) the beam-beam tune shifts do not depend on the longitudinal variables (ξ, a_c) and the strengths of the beam-beam resonances never exceed the nominal value ($|Y_m| = 1$). On the contrary, for flat colliding bunches both the tune shift of vertical oscillations and the strengths of resonances with the excitation of vertical betatron oscillations (Y_m^f) exceed the nominal value for large a_c . In the region $\epsilon_z \ll J_z \ll \sigma_x^2/\beta_z^*$ this causes a general increase in the widths of resonances in the J -space and displaces the centers of resonances toward the higher amplitudes of betatron oscillations.

The fact that both the strengths of synchrobetatron resonances and the tune shifts do not blow up at large amplitudes of synchrotron oscillations presents a big advantage of the schemes with $\beta_z^* = \beta^*$. As a disadvantage of such schemes we must mention a weak suppression of two-dimensional resonances with $m_x m_z < 0$ and $|m_x - m_z| = 2$. Although the schemes of the interaction regions with a round beam envelope (equal horizontal and vertical β -functions) are not typical for the future high energy colliders, the difference in the strengths of the synchro-betatron resonances in the tail regions gives more arguments for more careful inspection of such schemes.

4.5. LINEAR COHERENT BEAM-BEAM OSCILLATIONS IN THE RIGID BUNCH MODEL

In spite of very important predictions the weak-strong model still gives a limiting view of the beam-beam phenomenon. Once the interaction perturbs the motion of particles in both colliding beams they evolve in a self-consistent way. This fact becomes especially important in the case of a strong-strong interaction, when it yields some additional effects. Since the general description of the self-

consistent beam-beam interaction presents a very complicated problem, we shall discuss here some special cases and models when it can be simplified. One of the most widely known and simplest is the so-called rigid bunch model, which is used to study the beam coherent dipole oscillations neglecting the dilution of the beam distribution in phases. Below we shall follow paper Ref. 57. In the rigid bunch model coherent oscillations of a bunch are described by the displacement of its distribution function $f(\mathbf{r}, \mathbf{p}, t)$ as a whole. If $\mathcal{F}$ is the total force acting on a particle and the evolution of the distribution function is governed by Vlasov's equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \mathcal{F} \frac{\partial f}{\partial \mathbf{p}} = 0, \quad (4.5.1)$$

one can easily obtain the equations

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{r} \rangle &= - \int d\Gamma \mathbf{r} \left(\mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \mathcal{F} \frac{\partial f}{\partial \mathbf{p}} \right) = \langle \mathbf{v} \rangle, \\ \frac{d}{dt} \langle \mathbf{p} \rangle &= - \int d\Gamma \mathbf{p} \left(\mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \mathcal{F} \frac{\partial f}{\partial \mathbf{p}} \right) = \langle \mathcal{F} \rangle, \end{aligned} \quad (4.5.2)$$

for the dipole momenta of $f(\mathbf{r}, \mathbf{p}, t)$

$$\langle \mathbf{r} \rangle = \int d\Gamma \mathbf{r} f(\mathbf{r}, \mathbf{p}, t), \quad \langle \mathbf{p} \rangle = \int d\Gamma \mathbf{p} f(\mathbf{r}, \mathbf{p}, t). \quad (4.5.3)$$

Since $\langle \mathcal{F} \rangle$ generally depends on the higher order momenta of the distribution function

$$\langle \mathcal{F} \rangle = \mathcal{F}(0) + \frac{\partial \mathcal{F}}{\partial r_\alpha} \langle r_\alpha \rangle + \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial r_\alpha \partial r_\beta} \langle r_\alpha r_\beta \rangle + \dots$$

Eqs. (4.5.2) are not closed. For instance, one can also write

$$\begin{aligned} \frac{d}{dt} \langle z^2 \rangle &= 2 \langle z v_z \rangle, \quad \frac{d}{dt} \langle z v_z \rangle = \langle v_z^2 \rangle + \frac{1}{\gamma m} \langle z \mathcal{F}_z \rangle; \\ \frac{d}{dt} \langle v_z^2 \rangle &= \frac{2}{\gamma m} \langle v_z \mathcal{F}_z \rangle \end{aligned} \quad (4.5.4)$$

and therefore

$$\begin{aligned} \frac{d}{dt} \sigma^2 &= 2(\langle z v_z \rangle - \langle z \rangle \langle v_z \rangle), \\ \frac{d}{dt} \sigma_p^2 &= 2(\langle p_z \mathcal{F}_z \rangle - \langle p_z \rangle \langle \mathcal{F}_z \rangle), \end{aligned}$$

$$\sigma^2 = \langle z^2 \rangle - \langle z \rangle^2, \quad \sigma_p^2 = \langle p_z^2 \rangle - \langle p_z \rangle^2.$$

The continuation of this process gives the infinite set of generally coupled equations, where higher-order momenta are expressed in terms of the lower-order ones.* However, if the distribution function has the form specific for the rigid bunch approximation

$$f(\mathbf{r}, \mathbf{p}, t) = f_0\{[\mathbf{r} - \langle \mathbf{r}(t) \rangle]^2, [\mathbf{p} - \langle \mathbf{p}(t) \rangle]^2\}, \quad (4.5.5)$$

the values σ_z^2 and σ_p^2 (as well as higher-order spreads) will be conserved, whereas higher-order momenta of f can be calculated using $\langle \mathbf{r} \rangle$, $\langle \mathbf{p} \rangle$, and those constant spreads. Therefore, by means of Eqs. (4.5.2) the rigid bunch model describes coherent oscillations of the bunch with constant size and shape. It is certainly clear that the distribution (4.5.5) is quite unrealistic. Due to nonlinearity of forces disturbing the motion of particles, once centered around $\langle z(t=0) \rangle$ and $\langle p_z(t=0) \rangle$, the distribution will dilute in phases of oscillations with corresponding decay of $\langle z \rangle$, $\langle p_z \rangle$ and enlargement of the bunch effective emittance. This means that this model will give an adequate description of coherent oscillations only for the time intervals that are limited

$$\Delta t \ll 1/\Delta\omega$$

by the frequency spread in the beam $\Delta\omega$ (or probably by the rise time of the instability).

If for the sake of simplicity we describe the motion of a particle in the storage ring in the smooth focusing approximation, the force $F_{x,z}$ can be written in the form

$$\mathcal{F}_x = -\gamma m \omega_x^2 x + \delta \mathcal{F}_x, \quad \mathcal{F}_z = -\gamma m \omega_z^2 z + \delta \mathcal{F}_z.$$

The substitution of these expressions in Eqs. (4.5.2) yields the following equations describing coherent betatron oscillations in the rigid bunch model:

$$\begin{aligned} \langle \ddot{x} \rangle + \omega_x^2 \langle x \rangle &= \langle \delta \mathcal{F}_x \rangle / \gamma m, \\ \langle \ddot{z} \rangle + \omega_z^2 \langle z \rangle &= \langle \delta \mathcal{F}_z \rangle / \gamma m. \end{aligned} \quad (4.5.6)$$

For the beam-beam interaction the perturbing force $\delta \mathcal{F}$ is determined by the Lagrangian (4.3.19)

*In some sense this calculation of higher-order momenta is equivalent to the calculation of higher-order correlation functions discussed in Chapter 2 and it is so if we replace Eq. (4.5.1) by the equation for fluctuations δF . The resulting system of equations for momenta will describe both the development of coherent oscillations and the heating of the beam. Provided that the conditions for dynamic stochasticity⁵⁸ are not valid, Vlasov Eq. (4.5.1) can result only in adiabatic processes (evolution of coherent oscillations and mismatching of the beam phase space with collective fields), but no beam heating could be expected.

$$\delta \mathcal{F}_\alpha^{(1,2)} = \frac{\partial}{\partial r_\alpha} L(1,2). \quad (4.5.7)$$

The averaging of this equation with the distribution function (4.5.5) yields

$$\begin{aligned} \langle \delta \mathcal{F}_\alpha^{(1,2)} \rangle &= \int d^3r \rho_0^{(1)} [\mathbf{r}_\perp - \langle \mathbf{r}_\perp(t) \rangle] \frac{\partial}{\partial r_\alpha} L(1,2) \\ &= N_2 e^2 \delta(ct) \int \frac{d^2k}{\pi k^2} \rho_0^{(2)}(\mathbf{k}) \rho_0^{(1)}(-\mathbf{k}) \exp[i\mathbf{k}(\langle \mathbf{r}_\perp^{(1)} \rangle - \langle \mathbf{r}_\perp^{(2)} \rangle)] \\ &= N_2 e^2 \delta(ct) \frac{\partial}{\partial \mathbf{b}} \int \frac{d^2k}{\pi k^2} \rho_0^{(2)}(\mathbf{k}) \rho_0^{(1)}(-\mathbf{k}) \exp(i\mathbf{k}\mathbf{b}), \end{aligned} \quad (4.5.8)$$

where we used that for the distribution function in Eq. (4.5.5) holds the relationship

$$\rho(\mathbf{k}) = \exp[-i\mathbf{k}\langle \mathbf{r}_\perp(t) \rangle] \rho_0(\mathbf{k})$$

and we took

$$\mathbf{b} = \langle \mathbf{r}_\perp^{(1)} \rangle - \langle \mathbf{r}_\perp^{(2)} \rangle, \quad e_1 e_2 = -e^2.$$

Using Eq. (4.5.8) we can rewrite Eqs. (4.5.6) for, say, vertical betatron coherent oscillations in the form

$$\langle \ddot{z}^{(1)} \rangle + \omega_z^2 \langle z^{(1)} \rangle = \frac{N_2 e^2}{\gamma m} \delta_T(ct) \frac{\partial}{\partial b_z} U_{1,2}(b), \quad (4.5.9)$$

$$\langle \ddot{z}^{(2)} \rangle + \omega_z^2 \langle z^{(2)} \rangle = -\frac{N_1 e^2}{\gamma m} \delta_T(ct) \frac{\partial}{\partial b_z} U_{2,1}(b), \quad (4.5.10)$$

$$U_{1,2} = \int \frac{d^2k}{\pi k^2} \rho_0^{(2)}(\mathbf{k}) \rho_0^{(1)}(-\mathbf{k}) \exp(i\mathbf{k}\mathbf{b}). \quad (4.5.11)$$

These equations indicate that for identical colliding beams the beam-beam interaction affects only their relative coherent oscillations. Such oscillations (π -mode) are described by the impact parameter $\mathbf{b}$, which satisfies the equation

$$\ddot{b}_z + \omega_z^2 b_z = \frac{2N_2 e^2}{\gamma m} \delta_T(ct) \frac{\partial}{\partial b_z} U(b), \quad (4.5.12)$$

where

$$U(b) = \frac{1}{2N} (N_1 U_{1,2} + N_2 U_{2,1}), \quad N = (N_1 + N_2)/2. \quad (4.5.13)$$

It is remarkable that Eq. (4.5.12) describing the interaction of two generally strong-strong colliding bunches has exactly the same form as the equation of motion of a single particle in the weak-strong beam model, which is written for the special distribution of particles in the strong beam

$$\rho_{eff}(\mathbf{k}) = \frac{1}{2N} [N_2 \rho_0^{(2)}(\mathbf{k}) \rho_0^{(1)}(-\mathbf{k}) + N_1 \rho_0^{(2)}(-\mathbf{k}) \rho_0^{(1)}(\mathbf{k})]. \quad (4.5.14)$$

Hence, all the results of the weak-strong theory can be applied for the description of coherent beam-beam oscillations of strong-strong beams within the framework of the rigid beam model. In the case of Gaussian distributions in colliding bunches

$$\rho_0^{(1)}(k_x, k_z) = \exp\left(-\frac{k_x^2 \sigma_{1x}^2 + k_z^2 \sigma_{1z}^2}{2}\right),$$

$$\rho_0^{(2)}(k_x, k_z) = \exp\left(-\frac{k_x^2 \sigma_{2x}^2 + k_z^2 \sigma_{2z}^2}{2}\right),$$

$\rho_{eff}(\mathbf{k})$ also will be a Gaussian

$$\rho_{eff}(\mathbf{k}) = \exp\left(-\frac{k_x^2 \Sigma_x^2 + k_z^2 \Sigma_z^2}{2}\right),$$

$$\Sigma_x^2 = \sigma_{1x}^2 + \sigma_{2x}^2, \quad \Sigma_z^2 = \sigma_{1z}^2 + \sigma_{2z}^2$$

In this case predictions for the strong-strong case can be obtained from weak-strong calculations using the simple scale transformation.

We can note also that Eq. (4.5.12) is generated by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} [\dot{b}_z^2 + \omega_z^2 b_z^2] - \frac{2Ne^2}{\gamma m} \delta(ct) U(\mathbf{b}). \quad (4.5.15)$$

In the analogy with Eq. (1.6.23) we can describe the unperturbed coherent oscillations by the formulas

$$b_z = (J\tilde{\beta})^{1/2} \cos\psi, \quad b'_z = db_z/d\vartheta = -v_z (J\tilde{\beta})^{1/2} \sin\psi, \\ \psi' = v_z, \quad \tilde{\beta} = R_0/v_z, \quad (4.5.16)$$

which generate the canonical transformation from variables (b_z, b'_z) to action-phase variables (J, ψ) . In these variables the Hamiltonian (4.5.15) reads

$$\mathcal{H}(J, \psi) = v_z J - \frac{4Nr_0}{\gamma} \delta_T(\vartheta) U(\mathbf{b}). \quad (4.5.17)$$

Obtained equations can be used to study both linear and nonlinear coherent oscillations. In this section we detail the case when colliding beams with Gaus-

sian transversal distributions execute small vertical coherent oscillations ($b_x=0, |b_z| \ll \Sigma$) and thus, Eq. (4.5.12) can be linearized in b_z . Replacing $U(b)$ in Eq. (4.5.17) by the expression

$$U(b_z) = \int_0^\infty ds \frac{\exp\left\{-\frac{b_z^2}{4(s + \Sigma_z^2/2)}\right\}}{\sqrt{(s + \Sigma_z^2/2)(s + \Sigma_x^2/2)}} \quad (4.5.18)$$

and expanding it in the powers of b_z/Σ , one can get in the lowest-order approximation

$$U(b_z) \simeq U(0) - \frac{b_z^2}{\Sigma_z(\Sigma_x + \Sigma_z)} \quad (4.5.19)$$

and, thus,

$$\mathcal{H} = \nu_z J + \xi_z^\Sigma J(1 + \cos 2\psi) \sum_{n=-\infty}^{\infty} \exp(-in\vartheta). \quad (4.5.20)$$

Here

$$\xi_z^\Sigma = \frac{Nr_0\tilde{\beta}}{2\pi\gamma\Sigma_z(\Sigma_x + \Sigma_z)/2} \quad (4.5.21)$$

is the beam-beam parameter for vertical coherent oscillations. For identical beams ($\Sigma_\alpha = \sqrt{2}\sigma_\alpha$) it coincides with that for individual particles.⁶⁰ But we have to mention that the twice-larger value for ξ_z^Σ will be obtained if the linearization of equations of motion for individual particles will be done before the averaging. (See, for instance, Refs. 57 and 59.) This discrepancy is caused by the contribution into Eq. (4.5.17) from the nonlinear part of the force acting on a particle at the IP.⁶¹

The Hamiltonian (4.5.20) indicates nontrivial behavior of coherent oscillations if the tune ν_z is close to the resonant value $\nu_z = n/2 (n=1,2,\dots)$. In the resonant case the usage of the first approximation of the averaging method gives a new Hamiltonian, which in slow variables J and $\varphi = \psi - n\vartheta/2$ has the form

$$\mathcal{H}(J, \varphi) = \Delta J + \xi_z^\Sigma J(1 + \cos 2\varphi), \quad \Delta = \nu_z - n/2. \quad (4.5.22)$$

The inspection of Hamiltonians $h_\pm(J)$,

$$\mathcal{H}_+(J) = (\Delta + 2\xi_z^\Sigma)J, \quad \mathcal{H}_-(J) = \Delta J, \quad (4.5.23)$$

shows that oscillations will be stable outside the stopband*

$$-2\xi_z^\Sigma < \Delta < 0. \quad (4.5.24)$$

*For $e_1 e_2 = e^2$ obviously $0 < \Delta < 2\xi_z^\Sigma$.

If the widths of the stopband are fixed for any reason, this stability condition will limit the luminosity of the collider.

Let us now discuss the interaction of bunches that move in different rings and thus may have different parameters. The importance of this problem was recently understood^{62,63} referring to the analysis of coherent beam-beam limitation in the so-called asymmetric B-factories. Primarily, such installations were discussed as containing the existing large ring which had to be complemented by a small one matching the desirable energy range. Due to the difference in ring perimeters and, thus, difference in revolution frequencies, a particular bunch from one ring could collide with different countermoving bunches. This circumstance causes the propagation of the oscillations along the beams in both rings. Therefore, to find stability conditions one has to study many bunch problems. Here we shall briefly discuss the stability of dipole coherent beam-beam oscillations for the scheme where one bunch from the small ring collides with q bunches moving in the large ring. The ratio of the revolution frequencies in these rings is

$$\omega_s^{(1)}/\omega_s^{(2)}=q. \quad (4.5.25)$$

Following Ref. 63 we use the rigid bunch approximation with the linearization of equations before their averaging. Then, if $\vartheta = \omega_s^{(2)}t$ is chosen as the independent variable, coherent beam-beam oscillations are described by equations

$$\begin{aligned} z_1'' + q^2 g_1(q\vartheta) z_1 &= -\frac{4\pi q \xi_1}{f_1^2} \sum_{a=1}^{q-1} \delta(\vartheta - 2\pi a/q) (z_1 - z_2^a), \\ z_2'' + g_2(\vartheta) z_2 &= \frac{4\pi \xi_2}{f_2^2} \delta(\vartheta - 2\pi a/q) (z_1 - z_2^a). \end{aligned} \quad (4.5.26)$$

(for the sake of simplicity no beam cooling is assumed; superscript Σ is omitted for short.) Here, $g_{1,2}$ are rigidity coefficients in the rings and $f_{1,2}$ are the Floquet functions at the IP. Let us find the solutions of Eqs. (4.5.26) in the form

$$\begin{aligned} z_1 &= f_1(q\vartheta) \sum_n z_1(n) \exp[-i(\nu + n)\hat{\vartheta}_1], \quad \hat{\vartheta}_1 = \vartheta + \frac{\chi_1(q\vartheta)}{q\nu_1}, \\ z_2^a &= f_2(\vartheta) \sum_n z_2^a(n) \exp[-i(\nu + n)\hat{\vartheta}_2], \quad \hat{\vartheta}_2 = \vartheta + \frac{\chi_2(\vartheta)}{\nu_2}, \end{aligned} \quad (4.5.27)$$

where $\nu_{1,2}$ are tunes in the rings, ν is the unknown tune of coherent oscillations. Substituting z_1 and z_2 from Eq. (4.5.27) in Eq. (4.5.26) and using equations

$$\begin{aligned} \chi_1'(q\vartheta) + q\nu_1 &= q/f_1^2(q\vartheta), \quad f_1'' + g_1 f_1 = 1/f_1^3, \\ \chi_2'(\vartheta) + \nu_2 &= 1/f_2^2(\vartheta), \quad f_2'' + g_2 f_2 = 1/f_2^3, \end{aligned}$$

we obtain

$$\begin{aligned}
 \sum_n z_1(n) \exp(-in\hat{\vartheta}_1) \frac{(qv_1)^2 - (\nu+n)^2}{v_1^2 f_1^3} &= -\frac{4\pi q \xi_1}{f_1^2} \sum_{a=0}^{q-1} \sum_n \{ f_1 z_1(n) \exp(-in\hat{\vartheta}_1) \\
 &\quad - f_2 z_2^a(n) \exp(-in\hat{\vartheta}_2) \times \exp[-i\nu(\hat{\vartheta}_2 - \hat{\vartheta}_1)] \delta\left(\vartheta - \frac{2\pi a}{q}\right), \\
 \sum_n z_2^a(n) \exp(-in\hat{\vartheta}_2) \frac{v_1^2 - (\nu+n)^2}{v_2^2 f_2^3} &= -\frac{4\pi \xi_2}{f_2^2} \delta\left(\vartheta - \frac{2\pi a}{q}\right) \\
 &\quad \times \sum_n \{ f_2 z_2^a(n) \exp(-in\hat{\vartheta}_2) - f_1 z_1(n) \exp[-in\hat{\vartheta}_1 - i\nu(\hat{\vartheta}_2 - \hat{\vartheta}_1)] \}.
 \end{aligned} \tag{4.5.28}$$

Since

$$\int_0^{2\pi} \frac{d\vartheta}{v_1 f_1^2} \exp[-i(n-n')\hat{\vartheta}_1] = \int_0^{2\pi} d\vartheta_1 \exp[-i(n-n')\hat{\vartheta}_1] = 2\pi \delta_{nn'},$$

one can rewrite Eqs. (4.5.28) in the form

$$\begin{aligned}
 z_1(n) &= -\frac{2q\xi_1 v_1}{(qv_1)^2 - (\nu+n)^2} \sum_{a=0}^{q-1} \sum_{n'} \left\{ z_1(n') \exp\left[i(n-n')\frac{2\pi a}{q}\right] \right. \\
 &\quad \left. - \frac{f_2}{f_1} z_2^a(n') \exp\left(i(n-n')\frac{2\pi a}{q} - \frac{i(n'+\nu)}{v_2} \chi_2^a\right) \right\}, \\
 z_2^a(n) &= \frac{2\xi_2 v_2}{v_1^2 - (\nu+n)^2} \frac{f_1}{f_2} \sum_{n'} \left\{ z_1(n') \right. \\
 &\quad \left. - \frac{f_2}{f_1} z_2^a(n') \exp\left[-\frac{i(n'+\nu)}{v_2} \chi_2^a\right] \right\} \exp\left[-in'\frac{2\pi a}{q}\right], \tag{4.5.29} \\
 \chi_2^a &= \chi_2(2\pi a/q).
 \end{aligned}$$

Using a definition

$$b_a = \sum_{n'} \left(z_1(n') - \frac{f_2}{f_1} z_2^a(n') \exp\left[-\frac{i(n'+\nu)}{v_2} \chi_2^a\right] \right) \exp\left[-in'\frac{2\pi a}{q}\right],$$

one can get from Eqs. (4.5.29)

$$\begin{aligned}
 b_a \left(1 + \sum_n \frac{2\xi_2 v_2}{v_2^2 - (\nu+n)^2} \right) &= - \sum_n \frac{2\xi_1 q v_1}{(qv_1)^2 - (\nu+n)^2} \\
 &\quad \times \sum_{a'=0}^{q-1} b_{a'} \exp\left[-in\frac{2\pi(a-a')}{q}\right]. \tag{4.5.30}
 \end{aligned}$$

The substitution

$$\sum_{n=-\infty}^{\infty} \alpha_n = \sum_{l=-\infty}^{\infty} \sum_{p=1}^q \alpha_{lq+p}, \quad (4.5.31)$$

in the right-hand side of this equation yields the eigenvectors

$$b_p = \sum_{a=0}^{q-1} b_a \exp\left(\frac{2\pi i p a}{q}\right), \quad p=0,1,\dots,q-1 \quad (4.5.32)$$

as well as the dispersion equation of the problem

$$1 + \xi \sum_{n=-\infty}^{\infty} \frac{2\nu_2}{\nu_2^2 - (\nu + n)^2} + \xi \sum_{n=-\infty}^{\infty} \frac{2q^2\nu_1}{(q\nu_1)^2 - (\nu + p + qn)^2} = 0. \quad (4.5.33)$$

In Chapter 6 we shall see that equations of this kind are specific for multi-bunch instabilities. The value p has the sense of the longitudinal wave number of the eigensolution. Using the formula¹⁴

$$\sum_{n=-\infty}^{\infty} \frac{1}{x+n} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} = \pi \cot \pi x,$$

we can also rewrite Eq. (4.5.33) in the form

$$1 + 4\pi\xi_1 \frac{\xi_1 \sin \mu_1}{1 - 2\xi_1 \cos \mu_1 + \xi_1^2} + 4\pi\xi_2 \frac{\xi_2^q \sin \mu_2}{1 - 2\xi_2^q \cos \mu_2 + \xi_2^{2q}} = 0, \quad (4.5.34)$$

$$\xi_1 = \xi \exp(-2\pi i p/q), \quad \xi = \exp(-2\pi i \nu/q), \quad \mu_{1,2} = 2\pi \nu_{1,2}.$$

In particular for two identical colliding bunches ($q=1$, $p=0$, $\mu_1=\mu_2=\mu_0$, $\mu=2\pi\nu$) Eq. (4.5.34) yields the well known result:⁵⁹

$$\cos \mu = \cos \mu_0 - 4\pi\xi \sin \mu_0.$$

As could be seen from Eq. (4.5.33) the instability may occur if the tunes ν_1 and ν_2 are close to some resonant values. Let us consider first the specific for this problem coupling sum-type

$$q\nu_1 + \nu_2 = n \quad (4.5.35)$$

and difference-type

$$q\nu_1 - \nu_2 = n \quad (4.5.36)$$

resonances. Introducing here detunings $\Delta_{\pm} = q\nu_1 \pm \nu_2 - n$ and taking into account only resonant terms in Eq. (4.5.33) we can rewrite the dispersion equation in the form

$$1 = \frac{\xi_2}{x^\pm - \Delta_\pm/2} \mp \frac{q\xi_1}{x^\pm + \Delta_\pm/2},$$

$$x^\pm = \Delta\nu + \Delta_\pm/2, \quad \Delta\nu = \nu - \nu_2. \quad (4.5.37)$$

The roots of this equation are

$$x^+ = -\frac{\xi_-}{2} \pm \frac{1}{2} \sqrt{(\Delta_+ + \xi_+)^2 - (\xi_+^2 - \xi_-^2)}, \quad (4.5.38)$$

$$x^- = \frac{\xi_-}{2} \pm \frac{1}{2} \sqrt{(\Delta_+ + \xi_-)^2 + (\xi_+^2 - \xi_-^2)}, \quad (4.5.39)$$

$$\xi_\pm = q\xi_1 \pm \xi_2.$$

These formulas indicate no instability for the difference-type resonances (4.5.36), whereas near sum-type resonances (4.5.35) coherent oscillations can be unstable if the actual detuning

$$\Delta_+ + \xi_+ = q(\nu_1 + \xi_1) + \nu_2 + \xi_2 - n$$

occurs within the regions

$$-2\sqrt{q\xi_1\xi_2} \leq \Delta_+ + \xi_+ \leq 2\sqrt{q\xi_1\xi_2}. \quad (4.5.40)$$

Since n enters in Eq. (4.5.35) in the combination $n = lq + p$ ($p = 0, 1, \dots, q-1$), criterion (4.5.40) determines in the plane (ν_1, ν_2) q stopbands

$$l + \frac{p - (\sqrt{q\xi_1} + \sqrt{\xi_2})^2}{q} \leq \nu_1 + \frac{\nu_2}{q} \leq l + \frac{p - (\sqrt{q\xi_1} - \sqrt{\xi_2})^2}{q}. \quad (4.5.41)$$

Both the widths of the stopband (4.5.41) and the maximum value of the instability increment

$$\delta_{\max} = (-\text{Im}\Delta\nu)_{\max} = \sqrt{q\xi_1\xi_2} \quad (4.5.42)$$

grow like $\sqrt{q}$ when the number of bunches in larger ring q increases. Once the distance between the neighbor stopband is $1 - 4\sqrt{q\xi_1\xi_2}$, one may expect their overlapping when

$$q \simeq q_{\max} = \frac{1}{16\xi_1\xi_2}. \quad (4.5.43)$$

If, for instance, we shall adopt the values $\xi_1 = \xi_2 = 0.05$ as desirable, Eq. (4.5.43) yields $q_{\max} = 25$. Provided some threshold width of the stopband $\Delta_0 = 4\sqrt{q\xi_1\xi_2}$ is given, the luminosity of such a collider

$$\mathcal{L} = \frac{qN_1N_2}{2\pi T_s \Sigma^2} \quad (4.5.44)$$

will not depend on the number of bunches in the larger ring

$$\mathcal{L} < \mathcal{L}_{\max} = \frac{\Delta_0^2}{8} \frac{\pi \gamma_1 \gamma_2}{T_s \mathcal{J}_0^2} (\varepsilon_1 + \varepsilon_2). \quad (4.5.45)$$

This equation assumes certainly that q is not limited by the total energy losses (synchrotron radiation, etc.). More details concerning beam-beam instability in asymmetric rings can be found in Ref. 62 and 63.

The next interesting proposal, which was found to be limited by coherent beam-beam instabilities, is the so-called four-beam compensated scheme. Reference 64 suggests cancelling the single-particle beam-beam perturbations due to bringing into collision from both sides of two countercharged bunches, which compensate the space charge of each other. Reference 65 pointed out that spontaneous separation of bunches at the (IP) will break the local compensation and therefore can excite coherent oscillations of colliding bunches. The stability criterion for those oscillations can be found using previous results and the following arguments. At the (IP) any bunch collides with both same-charged and countercharged bunches. The interaction with the same-charged bunch will cause the stopbands above resonances $\nu_\alpha = n$. This limits the allowed values of ξ^Σ by

$$\xi^\Sigma < \frac{1}{4\pi} \tan \pi \nu. \quad (4.5.46)$$

The interaction with the countercharged bunch will cause the resonances below resonances $\nu_\alpha = n$ and will limit ξ^Σ by the value

$$\xi^\Sigma < \frac{1}{4\pi} \cot \pi \nu. \quad (4.5.47)$$

The total stability diagram is placed below both curves (see Fig. 4.11) and generally is narrower than that for the noncompensated scheme. Within the framework of the rigid bunch model this was recently calculated in Ref. 66.

Using the analogy in the description of coherent oscillations of colliding bunches in the rigid bunch model with the predictions of the weak-strong bunch approximation, we can expect strong effect of the bunch length ($\sigma_{||}$) on the increments of coherent modes. As was shown in §4.4, in this case the strengths of coherent beam-beam resonances [$\propto \xi$] are multiplied by some suppressing factor [$\xi \rightarrow \xi Y_1$]. The requirement of high luminosity generally demands small amplitudes of the synchrotron coherent oscillations, when Y_1 depends only on the ratio $\beta^*/\sigma_{||}$. For dipole oscillations this dependence [Fig. 4.12] indicates strong suppression of the dipole instability in the region

$$1/4 < \beta^*/\sigma_{\parallel} < 1.$$

4.6. MULTIPOLE COHERENT BEAM-BEAM OSCILLATIONS

4.6.1 Let us now apply the general theory developed in §3.1 to the description of coherent beam-beam oscillations. In contrast with the rigid bunch model, the straightforward calculations based on Vlasov's equation enable one to study the stability of both dipole and multipole coherent oscillations.* As we shall see below, for the beam-beam interaction this can be especially important. We shall consider the simplest case, when two bunches collide at one IP, and assume zero dispersion at this point.

The self-consistent evolution of coherent oscillations of two colliding bunches is described by the pair of equations

$$\begin{aligned} \left(\frac{\partial}{\partial \vartheta_s} + \nu_1 \frac{\partial}{\partial \psi} \right) \tilde{f}^{(1)} + \frac{\partial \tilde{L}_{1,2}}{\partial \psi} \frac{\partial f_0^{(1)}}{\partial \mathbf{I}} &= 0, \\ \left(\frac{\partial}{\partial \vartheta_s} + \nu_2 \frac{\partial}{\partial \psi} \right) \tilde{f}^{(2)} + \frac{\partial \tilde{L}_{2,1}}{\partial \psi} \frac{\partial f_0^{(2)}}{\partial \mathbf{I}} &= 0. \end{aligned} \quad (4.6.1)$$

As was pointed out in §3.1, this form of linearized Vlasov equation implies that stationary distribution f_0 is uniform in phases ψ . For the beam-beam interaction it assumes the working point of the ring ν outside the stopbands of incoherent beam-beam instability. Otherwise the excitation of incoherent nonlinear resonances can interfere with the evolution of coherent oscillations. This process must be described by a more complicated equation than Eqs. (4.6.1). Outside incoherent stopbands the influence of the stationary part of the fields from the counter-moving bunch results in the beam-beam tune shift, which will be supposed below to be included in tunes ν .

As before we write the Lagrangians $L_{a,b}$ for relativistic particles ($\gamma \gg 1$) in the form

$$\begin{aligned} \tilde{L}_{1,2} &= -\frac{N_2 e_1 e_2}{\pi c} \int \frac{d^2 k}{k^2} \exp(i \mathbf{k} [\mathbf{r}_1^{(1)} - \mathbf{r}_1^{(2)}]) \\ &\quad \times \delta_T \left(\vartheta_s + \frac{\Delta \vartheta_1 - \Delta \vartheta_2}{2} \right) \tilde{f}^{(2)}(\mathbf{I}_2, \psi_2, \vartheta_s) d\Gamma_2. \end{aligned} \quad (4.6.2)$$

Here, $\Delta \vartheta(\mathbf{I}, \psi, \vartheta_s) = \vartheta - \vartheta_s$, $\vartheta_s = \omega_s t$. Since for smooth distribution functions $\partial f_0 / \partial I_\alpha \propto f_0 / p_s \varepsilon_\alpha$, one can see from Eqs. (4.6.1) and (4.6.2) that the rates of the evolution of $\tilde{f}^{(1,2)}$ generally are determined by the beam-beam interaction parameters ξ .

*Coherent beam-beam oscillations were studied using Vlasov's technique in many papers. (See, for instance, Refs. 65, 68, 69, and 70.)

Without interaction ($\xi \rightarrow 0$) the spectra of coherent oscillations $[\tilde{f}^{(1,2)} \propto \exp(-i\nu\vartheta_s)]$ for bunches 1 and 2 respectively are

$$\nu = m_1 \nu_1, \quad \nu = m_2 \nu_2 \quad (4.6.3)$$

and therefore, the systematic variations of amplitudes $\tilde{f}$ are associated with the tuning of the working point to resonances

$$m_1 \nu_1 - m_2 \nu_2 = n \quad (4.6.4)$$

or to machine resonances

$$m_1 \nu_1 = n_1, \quad m_2 \nu_2 = n_2.$$

In the last case the instability can be caused by the coupling of modes with multipole numbers m and $-m$.

Since the direct solution of Eqs. (4.6.1) generally presents a very complicated problem, below we shall consider some special cases when it can be significantly simplified but still demonstrates the main features of coherent beam-beam oscillations.

We shall start with the stability analysis for betatron coherent beam-beam oscillations of short bunches ($\sigma_{\parallel} \ll \beta^*$), when Eq. (4.6.2) can be rewritten in the form ($e_1 e_2 = -e^2$):

$$\tilde{L}_{1,2} = \frac{N_2 e^2}{c} \delta(\vartheta_s) U_{1,2}, \quad (4.6.5)$$

$$U_{1,2} = \int \frac{d^2 k}{\pi k^2} \int d\Gamma_2 \exp(i\mathbf{k}[\mathbf{r}_1^{(1)} - \mathbf{r}_1^{(2)}]) \tilde{f}^{(2)}(\mathbf{I}, \psi, \vartheta_s),$$

$$k^2 = k_x^2 + k_z^2.$$

Substituting this equation in Eqs. (4.6.1) and using the Fourier transformation

$$\tilde{f}^{(1,2)}(\mathbf{I}, \psi, \vartheta_s) = \sum_{\mathbf{m}} f_{\mathbf{m}}^{(1,2)}(\mathbf{I}, \vartheta_s) \exp(i\mathbf{m}\psi),$$

one can rewrite Eqs. (4.6.1) as

$$\begin{aligned} \frac{\partial f_{\mathbf{m}}^{(1)}}{\partial \vartheta_s} + i\mathbf{m}\nu_1 f_{\mathbf{m}}^{(1)} &= -i\mathbf{m} \frac{\partial f_0^{(1)}}{\partial \mathbf{I}} \frac{N_2 e^2}{c} \delta(\vartheta_s) [U_{1,2}]_{\mathbf{m}}, \\ \frac{\partial f_{\mathbf{m}}^{(2)}}{\partial \vartheta_s} + i\mathbf{m}\nu_2 f_{\mathbf{m}}^{(2)} &= -i\mathbf{m} \frac{\partial f_0^{(2)}}{\partial \mathbf{I}} \frac{N_1 e^2}{c} \delta(\vartheta_s) [U_{2,1}]_{\mathbf{m}}. \end{aligned} \quad (4.6.6)$$

To find the eigensolutions of these equations, let us note that after the crossing the IP amplitudes $f_{\mathbf{m}}$ get the variations

$$\delta f_{\mathbf{m}}^{(1)} = f_{\mathbf{m}}^{(1)}(0+) - f_{\mathbf{m}}^{(1)}(0-) = -i(m\partial f_0/\partial I) \frac{N_2 e^2}{c} [U_{1,2}]_{\mathbf{m}},$$

$$\delta f_{\mathbf{m}}^{(2)} = f_{\mathbf{m}}^{(2)}(0+) - f_{\mathbf{m}}^{(2)}(0-) = -i(m\partial f_0/\partial I) \frac{N_1 e^2}{c} [U_{2,1}]_{\mathbf{m}}. \quad (4.6.7)$$

Here, $f_{\mathbf{m}}(0+)$ and $f_{\mathbf{m}}(0-)$ are the amplitudes just after and before the IP. Between collisions particles execute free oscillations $f_{\mathbf{m}}(\vartheta_s) \propto \exp(-im\nu\vartheta_s)$. Then, due to periodicity of coefficients in Eq. (4.6.1) one can write for eigen-solutions

$$f_{\mathbf{m}}(0+) = \lambda e^{2\pi im\nu} f_{\mathbf{m}}(0-). \quad (4.6.8)$$

As Eq. (4.6.8) must be valid at any turn, for amplitudes $f_{\mathbf{m}}$ spaced by n subsequent turns it obviously yields

$$f_{\mathbf{m}}(2\pi n + 0) = \lambda^n e^{2\pi in m\nu} f_{\mathbf{m}}(0-).$$

Therefore, coherent oscillations will be stable if all eigenvalues of the problem λ satisfy the criterion $|\lambda| \leq 1$. The integral equations for eigenfunctions $f_{\mathbf{m}}$ will be obtained after the substitution of Eq. (4.6.8) in Eq. (4.6.7):

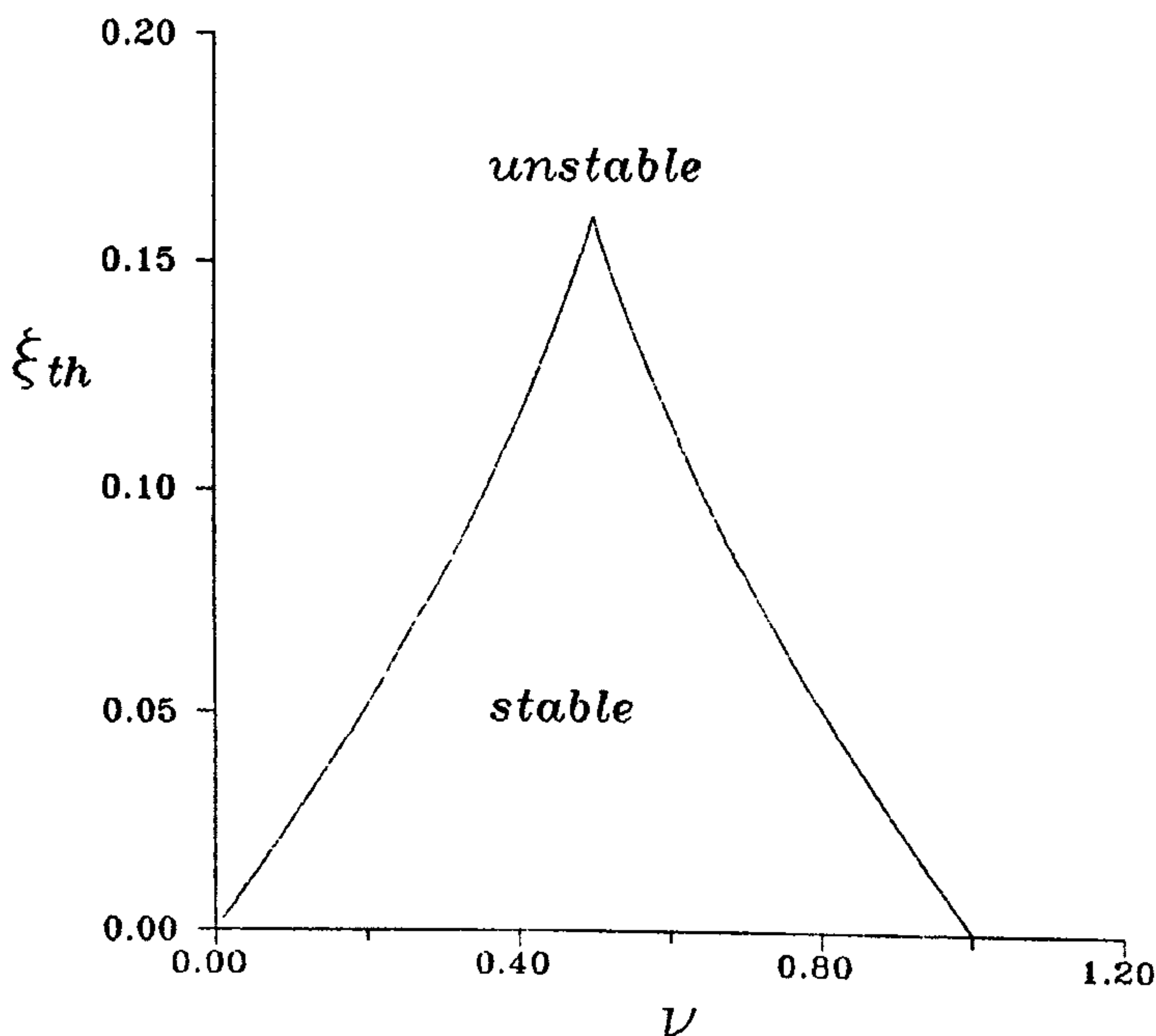


FIG. 4.11. Stability diagram of dipole coherent oscillations of the compensated colliding bunches.

$$\begin{aligned}
 f_{\mathbf{m}}^{(1)} &= \frac{N_2 e^2}{c} \frac{i(\mathbf{m} \partial f_0^{(1)} / \partial \mathbf{I}) [U_{1,2}]_{\mathbf{m}}}{1 - \lambda \exp(2\pi i \mathbf{m} \nu_1)}, \\
 f_{\mathbf{m}}^{(2)} &= \frac{N_1 e^2}{c} \frac{i(\mathbf{m} \partial f_0^{(2)} / \partial \mathbf{I}) [U_{1,2}]_{\mathbf{m}}}{1 - \lambda \exp(2\pi i \mathbf{m} \nu_2)}.
 \end{aligned}
 \tag{4.6.9}$$

These equations are still too complicated for their direct solution. For heuristic purposes as well as to illustrate some significant properties of the beam-beam collective modes, let us consider some model cases when the solution of Eqs. (4.6.9) can be more or less simplified.

4.6.2 First, we calculate the spectrum of radial coherent oscillations assuming identical colliding bunches with distributions

$$f_0(\mathbf{I}) = \delta(I_z) \theta(I_0 - I_x) / I_0, \quad I_0 = p\varepsilon/2, \tag{4.6.10}$$

which can (though very roughly) simulate the ribbon beam. Below we shall need the incoherent beam-beam tune shift for horizontal oscillations for this distribution. Using Eq. (4.3.45) one can easily find

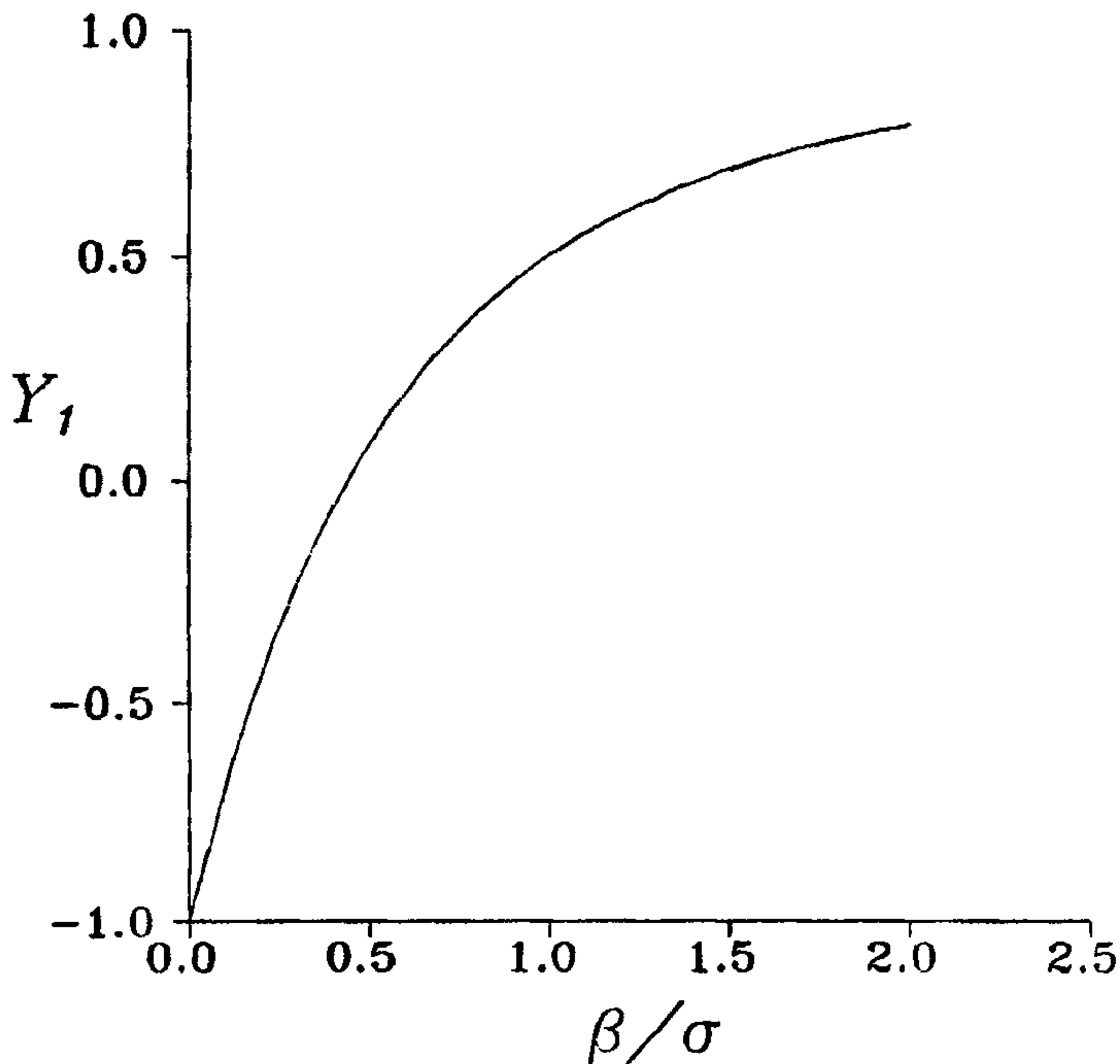


FIG. 4.12. Dependence of Y on ξ for dipole coherent oscillations of the rigid colliding bunches.

$$\Delta v_x = \xi \begin{cases} 1, & J \leq \varepsilon \\ \varepsilon/J, & J > \varepsilon, \end{cases}$$

where

$$\xi = \frac{Ne^2}{\pi p c \varepsilon}.$$

Once

$$U_m = \int \frac{d^2 k}{\pi k^2} J_{m_z}(k_z a_z) J_{m_x}(k_x a_x) \int d\Gamma_2 \exp(-i k r^{(2)}_{\perp}) \tilde{f}(\mathbf{I}, \psi, \vartheta_s), \quad (4.6.11)$$

where $J_m(x)$ is the Bessel function¹⁴ and $a_\alpha = \sqrt{J_\alpha \beta_\alpha^*}$ ($\alpha = x, z$), only radial multipole oscillations can be excited in bunches with distributions (4.6.10). For these oscillations $\mathbf{m} = \{m_z = 0, m_x = m, 0\}$ after the substitution

$$f_{\mathbf{m}}^{(1,2)} = \delta(I_z) \delta(I_x - I_0) C_m^{(1,2)}$$

Eqs. (4.6.9) can be transformed into the system of algebraic equations

$$\begin{aligned} [1 - \lambda \exp(im\mu_{x1})] C_m^{(1)} &= -2\pi i \xi_1 m \sum_{m'} M_{m,m'} C_m^{(2)}, \\ [1 - \lambda \exp(im\mu_{x2})] C_m^{(2)} &= -2\pi i \xi_2 m \sum_{m'} M_{m,m'} C_m^{(1)}, \end{aligned} \quad (4.6.12)$$

$$M_{m,m'} = \begin{cases} 0, & |m| + |m'| = 2l + 1, \\ 2 \int_0^\infty \frac{dk}{k} J_{|m|}(ka_1) J_{|m'|}(ka_2), & |m| + |m'| = 2l, \end{cases} \quad (4.6.13)$$

$$\xi_{1,2} = \frac{N_{2,1} e^2}{\pi p c \varepsilon_{1,2}}, \quad (4.6.14)$$

$$a_{1,2} = \sqrt{\varepsilon_{1,2} \beta^*} \quad (\mu_x)_{1,2} = (2\pi v_x)_{1,2}.$$

These equations indicate that for symmetric colliding bunches ($\varepsilon_1 = \varepsilon_2$, $v_{x1} = v_{x2}$) the instability can be due to resonances $m v_x = n$. For this case, the calculation of the integral in Eq. (4.6.13) yields¹⁴

$$M_{m,m'} = \frac{4}{\pi} \frac{\sin\left(\frac{\pi}{2}(|m| - |m'|)\right)}{|m|^2 - |m'|^2} = \frac{\delta_{m,m'}}{|m|}, \quad |m| + |m'| = 2l.$$

Using the substitution

$$C_m^a = \frac{A_a}{[1 - \lambda \exp(im\mu_x)]}, \quad a=1,2,$$

we can rewrite Eqs. (4.6.12) in the form

$$\begin{aligned} A_1(1 - 2\lambda \cos \mu_m + \lambda^2) &= 4\pi\xi\lambda \sin \mu_m A_2, \\ A_2(1 - 2\lambda \cos \mu_m + \lambda^2) &= 4\pi\xi\lambda \sin \mu_m A_1. \end{aligned} \quad (4.6.15)$$

Here, $\mu_m = 2\pi m \nu_x (J = \varepsilon)$. For small values of $\xi \ll 1$ one has $\mu_m = m\mu_0 + 2\pi m\xi$, where $\mu_0 = 2\pi \nu_x$. Let us consider first the case of dipole oscillations. The expansion of Eqs. (4.6.15) in powers of ξ with an accuracy up to ξ^2 transforms these equations into the following system ($m=1$):

$$\begin{aligned} A_1(1 - 2\lambda \cos \mu_0 + \lambda^2) &= -4\pi\xi\lambda \sin \mu_0 (A_1 - A_2), \\ A_2(1 - 2\lambda \cos \mu_0 + \lambda^2) &= 4\pi\xi\lambda \sin \mu_0 (A_1 - A_2). \end{aligned} \quad (4.6.16)$$

These equations have the same structure as the relevant equations in the rigid bunch model. (See §4.5.) In the agreement with predictions of this model Eqs. (4.6.16) predict no influence of the beam-beam kicks on o-modes ($A_1 + A_2 = 0$). For π -modes Eqs. (4.6.16) result in the dispersion equation

$$1 = - \frac{8\pi\xi\lambda \sin m\mu_0}{1 - 2\lambda \cos m\mu_0 + \lambda^2}. \quad (4.6.17)$$

Taking the roots of this equation in the form

$$\lambda = e^{-i\mu} = \cos \mu - i \sin \mu, \quad (4.6.18)$$

we can write

$$\cos \mu = \cos(\mu_x) - 4\pi\xi \sin(\mu_x). \quad (4.6.19)$$

The comparison of this equation and Eq. (4.3.27) shows that dipole coherent oscillations are twice as strongly affected by beam-beam forces than the motion of a single particle is. The same result was obtained in an early paper of A. Pivinski⁵⁹ (see also Ref. 67) using the rigid bunch calculations with linearization of the beam-beam force before averaging the equations.

For multipole oscillations ($m > 1$) it is more convenient to do similar calculations using Eqs. (4.6.15) directly. Near nonlinear resonances $\nu_x \simeq n/m$ this results in

$$(\nu/m)^2 = (\Delta + \xi)^2 + 2(\Delta + \xi)\xi/m$$

for π -modes and in

$$(\nu/m)^2(\Delta + \xi)^2 - 2(\Delta + \xi)\xi/m$$

for o-modes. Positions and widths of the stopbands are determined by stability condition $\nu^2 > 0$. For o-modes these are placed inside the stopbands of incoherent oscillations:

$$-\xi \leq \Delta \leq -\xi + 2\xi/m.$$

This violates initial assumptions and indicates that the presented analysis is not reliable any more. The studying of the stability of o-modes needs more complicated equations, taking into account the influence of nonlinear beam-beam resonances on the stationary distribution function.

For π -modes the stopbands are placed outside the stopbands of incoherent oscillations:

$$-\xi - 2\xi/m \leq \Delta \leq -\xi.$$

That is why below we shall mainly discuss the stability of π -modes.

Provided the working point of the ring is far enough from machine resonances ($\xi \ll |\Delta|$, $\Delta = \{\nu_x\} - n/m$, $\{\nu_x\}$ is the fractional part of ν_x , n and m are integers), Eq. (4.6.19) yields the coherent tune shift of oscillations

$$\mu = m\mu_x + \Delta\mu, \quad \Delta\mu = 2\pi\Delta\nu_m, \quad \Delta\nu_m = 2\xi.$$

This fact can be used for direct measurements of the beam-beam parameter ξ . (See Ref. 60 for more details.)

Let us yet briefly discuss the case when transverse sizes of colliding bunches are strongly different. (Say, $\varepsilon_1 \ll \varepsilon_2$.) Once the strength and nonlinearities of the perturbing force strongly depend on the transverse beam sizes of the counter-moving beam, one may expect strong suppression of instabilities of multipole oscillations. Calculation of the integral in Eq. (4.6.13) yields

$$M_{m,m'}(a_1, a_2) \simeq \frac{\kappa^{|m|}}{l} \binom{l}{|m|}, \quad |m| + |m'| = 2l, \quad \kappa = a_1/a_2 \ll 1. \quad (4.6.20)$$

With different bunch sizes we can separate two cases. In the first we have $\nu_1 = \nu_2$ and $\xi_1 = \xi_2$, which corresponds to strong asymmetry in the number of particles in bunches. Assuming that the working point is close to resonance value $\nu_x \simeq n/m$, we can rewrite Eqs. (4.6.12) in the form

$$\begin{aligned} C_m^{(1)} &= \left(\frac{\xi}{m}\right) \frac{\text{sgn}(m)\kappa^m}{\nu/m - \Delta} (C_m^{(2)} + C_{-m}^{(2)}), \quad \lambda = \exp(-2\pi i\nu), \\ C_m^{(2)} &= \left(\frac{\xi}{m}\right) \frac{\text{sgn}(m)\kappa^m}{\nu/m - \Delta} (C_m^{(1)} + C_{-m}^{(1)}), \quad \Delta = \nu_x - n/m. \end{aligned} \quad (4.6.21)$$

For π -modes this yields the dispersion equation

$$1 = \frac{2(\xi\Delta/m)\kappa^{|m|}}{(\nu/m)^2 - \Delta^2},$$

which has the roots

$$\nu = \pm \sqrt{\Delta^2 + 2\frac{\xi}{m}\kappa^{|m|}\Delta}. \quad (4.6.22)$$

Again, the oscillations will be unstable if the working point ν_x occurs below the resonances $\nu_x \simeq n/m$ within the stopbands

$$-2\frac{\xi}{m}\kappa^{|m|} \leq \Delta \leq 0, \quad (4.6.23)$$

though the widths of these stopbands as well as maximum increments

$$(-\text{Im}\nu)_{\max} = \pm \frac{\xi\kappa^{|m|}}{m} \quad (4.6.24)$$

exponentially decay with m .

If, on the contrary, the numbers of particles in bunches are close ($N_1 \simeq N_2$), due to $\xi_1 \gg \xi_2$ it may happen that $\nu_{x1} \neq \nu_{x2}$. For this case the instability can be caused by resonances*

$$m\nu_{x1} - m'\nu_{x2} \simeq n,$$

when Eqs. (4.6.12) get the form

$$\begin{aligned} \Delta\nu_m C_m^{(1)} &= \xi_1 m \Lambda_{mm'} C_{m'}^{(2)}, \\ (\Delta\nu_m - \Delta) C_m^{(2)} &= \xi_2 m' \Lambda_{mm'} C_m^{(1)}, \end{aligned} \quad (4.6.25)$$

$$\Lambda_{mm'} = \frac{\kappa^{|m|}}{(|m| + |m'|)} \binom{|m| + |m'|/2}{|m|},$$

$$\Delta\nu_m = \nu - m\nu_{x1}, \quad \Delta = m\nu_{x1} - m'\nu_{x2} - n.$$

The eigenvalues of system (4.6.25)

$$\Delta\nu = \frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + 4mm'\xi_1\xi_2\Lambda_{mm'}^2} \quad (4.6.26)$$

indicate that oscillations in sum resonance $mm' < 0$ will be unstable within the stopbands

*These resonances may also occur if colliding bunches move in separate rings with strong differences in betatron tunes.

$$|\Delta| \leq 4 \sqrt{mm' \xi_1 \xi_2} \frac{\kappa^{|m|}}{(|m| + |m'|)} \binom{(|m| + |m'|)/2}{|m|}. \quad (4.6.27)$$

Similar instabilities were detailed in §4.2. We can use the main results of that section to study, for instance, the influence of beam cooling on the spectra of coherent beam-beam oscillations. In formal respect the singular distributions considered here correspond to zero betatron tune spreads in colliding bunches. Therefore, if λ_x is the cooling decrement of a particle, the influence of the beam cooling on collective spectra can be taken into account by the substitution $\nu_x \rightarrow \nu_x - i|m|\lambda_x/2$. As a result, besides the dynamic-type, the dissipative beam-beam instabilities can occur, provided that

$$\frac{\Delta^2}{4} + 4mm' \xi_1 \xi_2 \Lambda_{mm'}^2 + \lambda_1 \lambda_2 < \frac{\Delta^2}{4} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2. \quad (4.6.28)$$

Here, $\lambda_1 = |m|\lambda_x/2$, and $\lambda_2 = |m'|\lambda_x/2$. In particular, for e^-p colliders one has $\lambda_1 \lambda_2 = 0$ and condition (4.6.28) will be satisfied for all sum modes $4mm' \xi_1 \xi_2 \Lambda_{mm'}^2 < 0$.

In $e^-e^-; e^+e^-$ colliders the beam cooling due to the synchrotron radiation can damp the instability of sum modes with high multipole numbers

$$\lambda_x^2 \gg 16 \xi_1 \xi_2 \frac{\kappa^{|m|}}{(|m| + |m'|)} \binom{(|m| + |m'|)/2}{|m|}. \quad (4.6.29)$$

4.6.3 A more realistic solvable model of coherent beam-beam oscillations for flat bunches ($\sigma_x \gg \sigma_z$) can be constructed if we take as the stationary distributions in beams the functions

$$f_0^{(1,2)} = \frac{\theta(I_{0z} - I_z)}{I_0} \frac{1}{I_{0x} + I_x}. \quad (4.6.30)$$

These distributions cannot be normalized and, therefore, are still singular. Nevertheless, Eq. (4.6.30) describes the bunches concentrated within the region

$$|z| \leq \sqrt{\varepsilon_z \beta_z^*}, \quad |x| \leq \sqrt{\varepsilon_x \beta_x^*}, \quad (I_0)_{x,z} = p \varepsilon_{x,z}/2.$$

We shall show now that this circumstance turns out to be very important for the properties of collective spectra. Below we assume the typical ratio $\varepsilon_x \gg \varepsilon_z$ between horizontal and vertical emittances in flat bunches.

For vertical coherent oscillations $\mathbf{m} = \{m, 0, 0\}$ the substitution of distributions (4.6.30) in Eqs. (4.6.9) yields ($N_1 = N_2$)

$$\begin{aligned}
f_{\mathbf{m}}^{(1)} &= -\frac{2Ne^2}{pc\varepsilon_z} \frac{im\delta(I_z - I_0)}{1 - \lambda \exp(2\pi imv_z)} \frac{[U_{1,2}]_{m,0}}{I_{0x} + I_x}, \\
f_{\mathbf{m}}^{(2)} &= -\frac{2Ne^2}{pc\varepsilon_z} \frac{im\delta(I_z - I_0)}{1 - \lambda \exp(2\pi imv_z)} \frac{[U_{2,1}]_{m,0}}{I_{0x} + I_x}.
\end{aligned} \tag{4.6.31}$$

Using here

$$f_{\mathbf{m}}^{(1,2)} = \delta(I_z - I_0) w_m^{(1,2)}(a_x), \quad I_x = \frac{pa_x^2}{2\beta_x^*} \tag{4.6.32}$$

and

$$w^{(1,2)}(k_x) = \int_0^\infty dx x J_0(k_x x) w^{(1,2)}(x), \tag{4.6.33}$$

$$\sigma_{x,z}^2 = (\varepsilon\beta^*)_{x,z}, \quad x = a_x/\sigma_x$$

as well as Eq. (4.6.11), one can transform Eqs. (4.6.31) into the following system of integral equations [$\lambda_m = \lambda \exp(im\mu_z)$]:

$$\begin{aligned}
[1 - \lambda_m] w_m^{(1)}(k) &= -4\pi i \xi_z m \sum_{m'} M_{m,m'} \int_0^\infty dk' \mathcal{K}(k, k') w_{m'}^{(2)}(k'), \\
[1 - \lambda_m] w_m^{(2)}(k) &= -4\pi i \xi_z m \sum_{m'} M_{m,m'} \int_0^\infty dk' \mathcal{K}(k, k') w_{m'}^{(1)}(k'),
\end{aligned} \tag{4.6.34}$$

$$\mathcal{K}(k, k') = \begin{cases} K_0(k) I_0(k'), & k > k' \\ I_0(k) K_0(k'), & k < k', \end{cases} \tag{4.6.35}$$

$$M_{m,m'} = \begin{cases} 0, & |m| + |m'| = 2l + 1, \\ \frac{1}{\pi} \frac{(-1)^{|m| - l + 1}}{[(|m| - l)^2 - 1/4][l^2 - 1/4]}, & |m| + |m'| = 2l, \end{cases} \tag{4.6.36}$$

$$\xi_{1,2} = \frac{N_{2,1} e^2 \beta_z^*}{2\pi p c (\sigma_z)_{1,2} \sigma_x}.$$

Here $I_0(x)$ and $K_0(x)$ are modified Bessel functions.¹⁴ One can construct the functions $w_m^{(1,2)}(k)$ as expansions

$$w_m^{(a)}(k) = \sum_j C_{mj}^{(a)} w_j(k), \quad a=1,2 \tag{4.6.37}$$

in eigenfunctions of the integral equation

$$\Lambda_j w_j(k) = \int_0^\infty dk' \mathcal{K}(k, k') w_j(k').$$

A similar equation has been discussed in §3.5. By double differentiation it can be transformed into the differential equation

$$w'' + w'/k - [1 - 1/(\Lambda k)] w = 0.$$

After the substitutions $z=2k$ and $w=\exp(-z/2)v(z)$ it reads

$$v'' + (1-z)v' + \frac{1-\Lambda}{2\Lambda} v = 0.$$

Solution of this equation will coincide with Laguerre polynomials⁵²

$$v_j = c_j L_j(z),$$

provided that

$$1/\Lambda - 1 = 2j, \quad j=0, 1, \dots$$

The normalization condition

$$\int_0^\infty dk w_j(k) w_{j'}(k) = \delta_{jj'} \frac{|c_j|^2}{2} \int_0^\infty dz \exp(-z) L_j^2(z) = \delta_{jj'} \frac{|c_j|^2}{2} = 1$$

yields the eigenfunctions

$$w_j(k) = \sqrt{2} e^{-k} L_j(2k), \quad \Lambda_j = \frac{1}{1+2j},$$

$$\int_0^\infty dk w_j(k) w_{j'}(k) = \delta_{j,j'}, \quad j=0, 1, \dots \quad (4.6.38)$$

The substitution of Eqs. (4.6.37) in Eqs. (4.6.34) transforms it into the system of algebraic equations for coefficients $C_{mj}^{(1,2)}$:

$$\begin{aligned} [1 - \lambda \exp(im\mu_z)] C_{mj}^{(1)} &= -4\pi i \xi_z m \Lambda \sum_{m'} M_{m,m'} C_{m'j}^{(2)}(k'), \\ [1 - \lambda \exp(im\mu_z)] C_{mj}^{(2)} &= -4\pi i \xi_z m \Lambda \sum_{m'} M_{m,m'} C_{m'j}^{(1)}(k'). \end{aligned} \quad (4.6.39)$$

In typical regions $\xi \ll 1$ the oscillations can be unstable near the resonances $v_z \simeq n/m$ when the dispersion equations for π -modes get the form

$$1 = \frac{4\xi \Delta \Lambda_{mj}}{(v/m)^2 - \Delta^2}, \quad \Lambda_{mj} = \frac{1}{2j+1} \frac{4/\pi}{m^2 - 1/4}.$$

The requirement of real eigenfrequencies

$$\nu_{m,j} = \pm \{\Delta^2 + 4\xi\Delta\Lambda_{m,j}\}^{1/2} \quad (4.6.40)$$

determines the widths of the stopbands

$$-\frac{16\xi/\pi}{m^2 - 1/4} < \Delta < 0, \quad (4.6.41)$$

which again are placed below the resonances. Outside the stopbands ($\Delta > 0$) the spectrum has well defined the ground-state line ($j=0$), whereas the distance between higher-state lines ($j \gg 1$) decreases like $1/j^2$. Qualitatively this behavior of the spectrum coincides with results of measurements. (See, for instance, Ref. 71.) The splitting of the spectrum near the given harmonic $m\nu_z$ into the series of lines describes the dependence of the solution on amplitudes of radial oscillations and generally is specific for smooth distribution functions.

4.6.4 The models that adopt as the stationary distributions the steplike functions in fact describe coherent oscillations in quasimonochromatic beams. In this case, once coherent oscillations propagate along the bunch surface in its phase space

$$f_m(I) \propto \delta(I - I_0),$$

nonlinearities of incoherent betatron oscillations only modify unperturbed tunes, but do not cause Landau damping of coherent modes. Here we shall consider the model that enables one to include in the calculations the dispersion of betatron oscillations in a more or less straightforward way and, thus, to predict the influence of Landau damping on coherent beam-beam oscillations.⁷²

Let us take as the stationary distribution the function

$$f_0^{(1,2)}(I_x, I_z) = \delta(I_z) \frac{\exp[-I_x/I_0^{(1,2)}]}{I_0^{(1,2)}}, \quad (4.6.42)$$

$$I_0^{(1,2)} = p_s \varepsilon_{1,2}.$$

Some important values referring to incoherent beam-beam interaction for bunches with these distributions have been calculated in §4.4. Here we shall try to calculate collective spectra of horizontal coherent oscillations for such colliding bunches. To simplify calculations we shall consider the collisions of identical bunches ($N_1 = N_2$, $\varepsilon_1 = \varepsilon_2$) with $\xi \ll 1$.

Expecting unstable horizontal coherent oscillations near the resonances $m_x \nu_x \simeq n$, we rewrite Eqs. (4.6.9) in the form (the subscript x will be omitted for short)

$$\chi_m^{(1,2)} = (f_m + f_{-m})^{(1,2)}, \quad m > 0,$$

$$\begin{aligned}\chi_m^{(1)} &= -\frac{2m^2\Delta(x)\xi}{v^2-m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x, x') \exp(-x'^2/2) \chi_m^{(2)}, \\ \chi_m^{(2)} &= -\frac{2m^2\Delta(x)\xi}{v^2-m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x, x') \exp(-x'^2/2) \chi_m^{(1)},\end{aligned}\quad (4.6.43)$$

$$M_m(x, x') = 2 \int_0^\infty \frac{dk}{k} J_m(kx) J_m(kx'), \quad (4.6.44)$$

$$\Delta(x) = v_x(x) - n/m = \Delta_0 + \Delta v_x(x), \quad x = \sqrt{J/\varepsilon},$$

ξ coincides with that given by Eq. (4.6.14), and $\Delta v_x(x)$ is given by Eq. (4.3.46). The amplitudes of π -modes

$$\chi_m^- = \chi_m^{(1)} - \chi_m^{(2)},$$

satisfy the integral equation

$$\chi_m^- = \frac{2m^2\Delta(x)\xi}{v^2-m^2\Delta^2(x)} \int_0^\infty dx' x' M_m(x, x') \exp(-x'^2/2) \chi_m^-. \quad (4.6.45)$$

The kernel $M_m(x, x')$ is a discontinuous function of x and x' :

$$M_m(x, x') = \frac{2}{m} \begin{cases} (x/x')^m, & x < x', \\ (x'/x)^m, & x \geq x'. \end{cases} \quad (4.6.46)$$

The substitution

$$\chi_m^- = \frac{4m^2\Delta(x)\xi}{v^2-m^2\Delta^2(x)} X(x) \quad (4.6.47)$$

transforms Eq. (4.6.45) into

$$X(x) = \frac{x^{-m}}{m} \int_0^x dx' x'^{m+1} V_m(x') X(x') + \frac{x^m}{m} \int_x^\infty dx' x'^{1-m} V_m(x') X(x'), \quad (4.6.48)$$

where

$$V_m(x) = \frac{4\Delta(x)\xi}{(v/m)^2 - \Delta^2(x)} e^{-x^2/2}. \quad (4.6.49)$$

One more substitution, $X = w(x)/\sqrt{x}$, and subsequent double differentiation of Eq. (4.6.48) transform it into the differential equation

$$w'' + \left(2V_m(x) - \frac{m^2 - 1/4}{x^2} \right) w(x) = 0. \quad (4.6.50)$$

It is obvious that the choice of a Gaussian function as the stationary distribution is not principal for these calculations. The same equation will describe oscillations of beams with arbitrary distribution of horizontal oscillation amplitudes $f_0(x)$, if one replaces in Eq. (4.6.49)

$$\exp(-x^2/2) \rightarrow -2\partial f_0/\partial x^2.$$

Let us consider first some cases when Eq. (4.6.48) can be solved directly. For instance, if $f_0(x)$ is the step function $f_0 \propto \theta(x^2 - 1)$, one has

$$V_m(x) = \frac{1}{\Lambda_m} \delta(x-1), \quad (1/\Lambda_m) = \frac{4m^2 \Delta(x) \xi}{v^2 - m^2 \Delta^2(x)} \quad (4.6.51)$$

and Eq. (4.6.50) yields

$$w(x) = \begin{cases} x^{1/2+m}, & x \leq 1, \\ x^{1/2-m}, & x \geq 1, \end{cases} \quad \Lambda_m = 1/m, \quad (4.6.52)$$

as well as a dispersion equation coinciding with Eq. (4.6.19):

$$1 = \frac{4\Delta(1)\xi/m}{(v/m)^2 - \Delta^2(1)}, \quad \xi \ll 1.$$

A histogram,

$$f_0(x) = f_j, \quad x_{j-1} \leq x \leq x_j, \quad j = 1, 2, \dots, q, \quad x_0 = 0, \quad (4.6.53)$$

can be used to simulate the spectrum in the beam with a smooth distribution function. In this case we have

$$V_m(x) = \sum_{k=1}^q V_{mj} \delta(x-x_j)/x_j, \quad V_{mj} = \frac{4\Delta_j \xi}{(v/m)^2 - \Delta_j^2} (f_j - f_{j-1}). \quad (4.6.54)$$

Solutions of Eq. (4.6.50) can be written in the form

$$w_j(x) = B_j x^{1/2+m} + C_j x^{1/2-m}, \quad x_{j-1} \leq x \leq x_j. \quad (4.6.55)$$

Using the boundary conditions

$$w_j(x_j) = w_{j+1}(x_j), \quad (w'_{j+1} - w'_j)_{x=x_j} = -\frac{2V_{mj}}{x_j} w_{j+1}(x_j)$$

one can find the recurrence equations for coefficients B_j and C_j :

$$C_j = C_{j+1} \left(1 - \frac{V_{mj}}{m} \right) - \frac{V_{mj}}{m} B_{j+1} x_j^{2m}, \quad (4.6.56)$$

$$B_j = B_{j+1} + (C_{j+1} - C_j)x_j^{-2m}.$$

With obvious initial conditions

$$C_1 = B_{q+1} = 0, \quad C_{q+1} = 1,$$

Eqs. (4.6.56) enable one to find all coefficients B_j, C_j as well as the dispersion equation

$$\sum_{j=1}^q (C_{j+1} - C_j) = 1 = \sum_{j=1}^q \frac{V_{mj}}{m} (C_{j+1} + B_{j+1}x_j^{2m}), \quad (4.6.57)$$

but results in cumbersome expressions that can be useful only for numerical analysis though these expressions indicate a very important general result. If for the sake of simplicity we neglect in Eqs. (4.6.54) through (4.6.57) the tune spread and introduce the values

$$1/\Lambda_m = \frac{4\Delta\xi}{(\nu/m)^2 - \Delta^2},$$

we see that Eq. (4.6.57) is a polynomial of $1/\Lambda_m$ of the power q , which therefore has exactly q roots. In contrast with the single-step distribution function the spectrum of beam with histogram distribution (4.6.53) will consist of q lines.

The direct solution of Eq. (4.6.50) in cases when the distribution f_0 is a smooth function of I still is very difficult. However, many general properties of its eigenfunctions and spectra can be predicted using the analogy of Eq. (4.6.50) and the Schrödinger equation in the quantum mechanic, which is written for a particle with the zeroth energy, moving in the effective potential well:

$$U_{eff}(x) = \frac{m^2 - 1/4}{x^2} - 2V_m(x). \quad (4.6.58)$$

Let us again start by discussing properties of solutions for monochromatic beams $[\Delta(x) \rightarrow \Delta_0]$. Taking again

$$1/\Lambda_m = \frac{2\Delta_0\xi}{(\nu/m)^2 - \Delta_0^2}, \quad (4.6.59)$$

we rewrite Eq. (4.6.50) in the form

$$w'' - U_{eff}(x)w(x) = 0, \quad (4.6.60)$$

$$U_{eff}(x) = \frac{2}{\Lambda_m} e^{-x^2/2} - \frac{m^2 - 1/4}{x^2}. \quad (4.6.61)$$

Equation (4.6.60) may have nontrivial solutions corresponding to the discrete, nondegenerate spectrum if in some region of x the function U_{eff} becomes negative. (See, for instance, Fig. 4.13.) Adopting this for the minimum value of

U_{eff} and replacing the function $2x^2e^{-x^2/2}$ by its maximum value $4/e$, one can estimate Λ_m for the ground state mode at least by

$$\Lambda_m \leq \frac{4/e}{m^2 - 1/4}. \quad (4.6.62)$$

Qualitatively, this behavior can be illustrated by Figs. 4.13(a,b). If the numbers $j=0, 1, \dots$ mark the normal modes of Eq. (4.6.60), one may expect that eigenvalues Λ_{mj} decrease at least like $1/j^2$ when j increases. This circumstance can be verified, for instance, by the direct calculation of the sum

$$\Lambda_\Sigma = \sum_{j=1}^{\infty} \Lambda_{mj}.$$

Using Eq. (4.6.48), one can easily find $\Lambda_\Sigma = 1/m$.

Since solutions of Eq. (4.6.60) have asymptotes

$$w(x) = \begin{cases} x^{1/2+m}, & x \rightarrow 0, \\ x^{1/2-m}, & x \rightarrow \infty, \end{cases}$$

they are mainly concentrated between the roots of $U_{eff}(x)$, which correspond to stop points of the mechanical problem

$$U_{eff}(x_{1,2}) = 0,$$

or

$$2(x^2e^{-x^2/2})_{1,2} = \Lambda_{mj}(m^2 - 1/4). \quad (4.6.63)$$

The behavior of U_{eff} (see, for example, Fig. 4.13) indicates that while the localization of the ground state mode only slightly moves outside the bunch, when m increases, the higher modes can significantly penetrate inside the bunches ($x_1 \rightarrow 0, \Lambda_{mj} \rightarrow 0$). The total amount of modes corresponding to the given value of Λ_m can be estimated by the phase space volume of the potential well:

$$\Delta j \simeq \frac{1}{\pi} \int_{x_1}^{x_2} dx \sqrt{-U_{eff}}. \quad (4.6.64)$$

The sensitivity of eigensolutions to the tune spread due to beam-beam interaction can be studied in a similar way. The direct inspection of the effective potential curves, calculated for $\nu = \pm i\delta$ (see Figs. 4.14 and Fig. 4.15) definitely indicates the existence of eigenmodes within the stopbands which are slightly displaced but have roughly the same widths as the corresponding stopbands for monochromatic bunches. Analogous stopbands can be found out by the inspection of corresponding curves for quadrupole ($m = \pm 2$, see Fig. 4.16), sextupole ($m = \pm 3$, Fig. 4.17), and other oscillations. Such influence of Landau damping

on coherent beam-beam instability could be expected beforehand. (See also §4.2.) For resonant coherent instabilities the nonlinearity of betatron oscillations rather helps to form the buckets in the phase space than to damp the oscillations.

As Eq. (4.6.50) presents the particular case of the more general Sturm-Liouville equation, its spectra can be estimated using the general relationships. For instance, exact solutions $w(x) = w_j(x)$ minimize the integral

$$I[w] = \int_0^\infty dx \left[(w')^2 + \frac{m^2 - 1/4}{x^2} w^2 - 2V_m w^2 \right]. \quad (4.6.65)$$

This yields the normalization condition for eigensolutions

$$\int_0^\infty dx \left((w'_j)^2 + \frac{m^2 - 1/4}{x^2} w_j^2 \right) = 2,$$

and the dispersion equation of the problem

$$\int_0^\infty dx |w_j|^2 \frac{4\Delta(x)\xi}{(\nu/m)^2 - \Delta^2(x)} e^{-x^2/2} = 1, \quad \text{Im}\nu > 0. \quad (4.6.66)$$

If the eigenfunctions w_j are not known, both they and spectra can be found using the minimization routines starting from a more or less suitable set of probe functions $w(x)$.

4.6.5 To conclude this section we briefly discuss some specific features of two-dimensional coherent beam-beam oscillations of short bunches. Neglecting Landau damping and assuming Gaussian stationary distributions

$$f_0^{(1,2)} = \frac{1}{I_{0x}I_{0z}} \exp\left(-\frac{I_x}{I_{0x}} - \frac{I_z}{I_{0z}}\right),$$

as well as the working point of the ring close to the resonance

$$m\nu \simeq n, \quad (4.6.67)$$

we can write [see Eqs. (4.6.9)]:

$$\begin{aligned} f_{\mathbf{m}}^{(1)} &= -2 \frac{Ne^2}{2\pi pc} \left(\frac{m_x}{\varepsilon_x} + \frac{m_z}{\varepsilon_z} \right) f_0 \frac{[U_{1,2}]_m}{\nu - \Delta_m}, \\ f_{\mathbf{m}}^{(2)} &= -2 \frac{Ne^2}{2\pi pc} \left(\frac{m_x}{\varepsilon_x} + \frac{m_z}{\varepsilon_z} \right) f_0 \frac{[U_{2,1}]_m}{\nu - \Delta_m}, \end{aligned} \quad (4.6.68)$$

$$\Delta_m = m\nu - n.$$

Among others these equations show that in the round beams $\varepsilon_x = \varepsilon_z$ coherent modes cannot be excited on the line of the main coupling resonance $\nu_x + \nu_z = n$. For other modes it is convenient to transform Eqs. (4.6.68) into equations for the moments

$$f_m^{(1,2)}(\mathbf{k}) = \int d\Gamma \{ \exp(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp) \}_m f_m^{(1,2)}(\mathbf{I}).$$

Near the resonances (4.6.67) and for π -modes

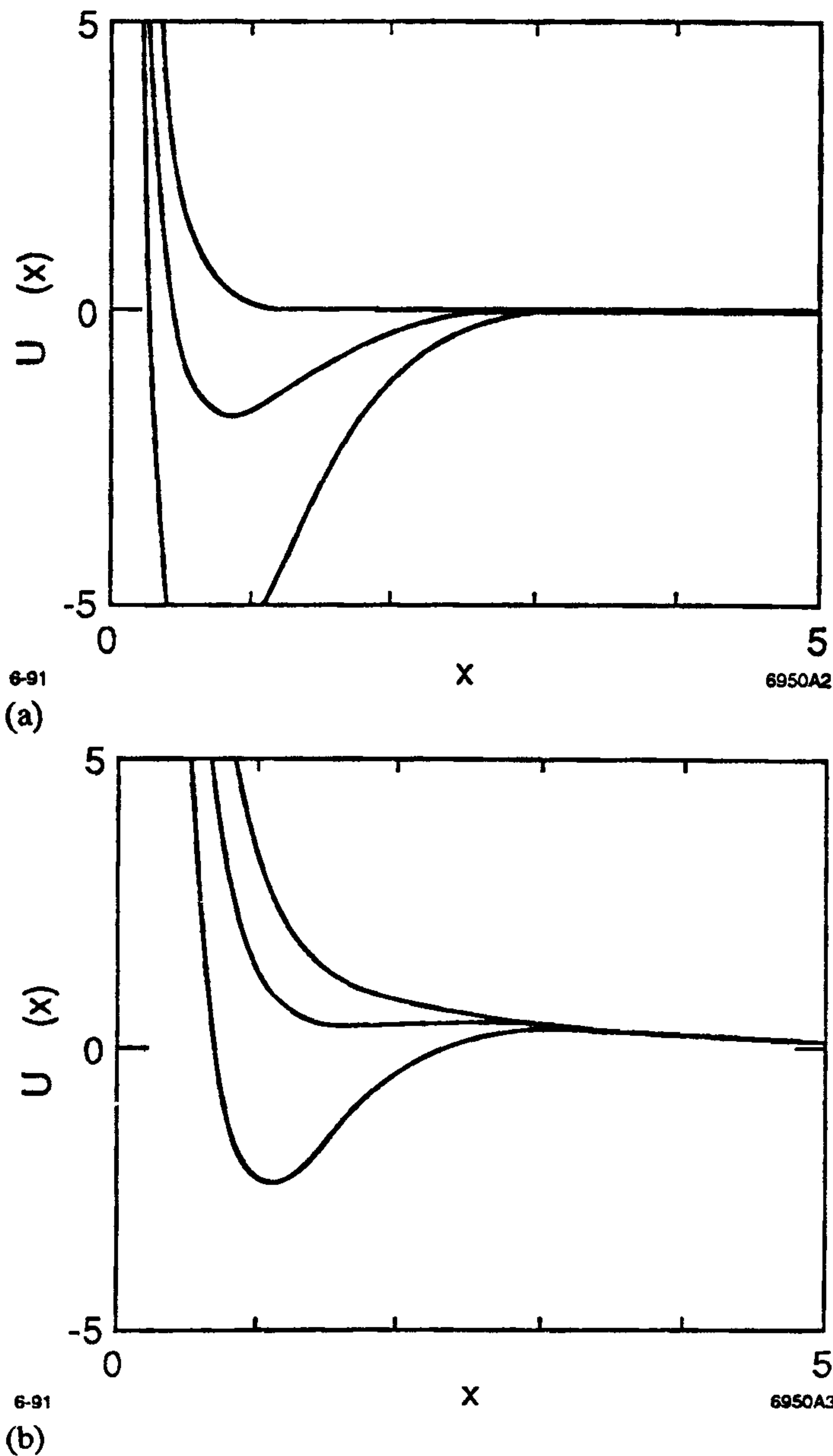


FIG. 4.13. Effective potential from Eq. (4.6.61) for dipole (a) and quadrupole (b) coherent oscillations; from top to bottom $\Lambda_m=2, 0.5, 0.2$.

$$\chi_m^- = f_m^{(1)}(\mathbf{k}) - f_m^{(2)}(\mathbf{k})$$

it reads

$$\Lambda_m \chi_m^- = \frac{2}{\pi} \int \frac{dx' dz'}{x'^2 r + z'^2 / r} g_m(x, x') g_m(z, z') \chi_m^-(x', z'), \quad (4.6.69)$$

with the dispersion equation

$$1 = \frac{Ne^2 \Lambda_m}{2\pi pc} \left(\frac{m_x}{\epsilon_x} + \frac{m_z}{\epsilon_z} \right) \frac{2\Delta_m}{v^2 - \Delta_m^2}. \quad (4.6.70)$$

Here

$$g(x, x') = \exp\left(-\frac{x^2 + x'^2}{2}\right) I_m(xx'), \quad (4.6.71)$$

and $r = \sigma_z / \sigma_x$ is the beam aspect ratio. So, to calculate the widths of the stop-bands we need the eigenvalues Λ_m . As the direct solution of Eq. (4.6.69) is very complicated, we can try to estimate Λ_m using the extrapolation described in §3.5 of eigensolutions from the short-wavelength region. Once asymptotically

$$g(x, x') \simeq \frac{\exp[-(x - x')^2]}{\sqrt{2\pi |xx'|}}, \quad |x| \gg |m|,$$

we can write

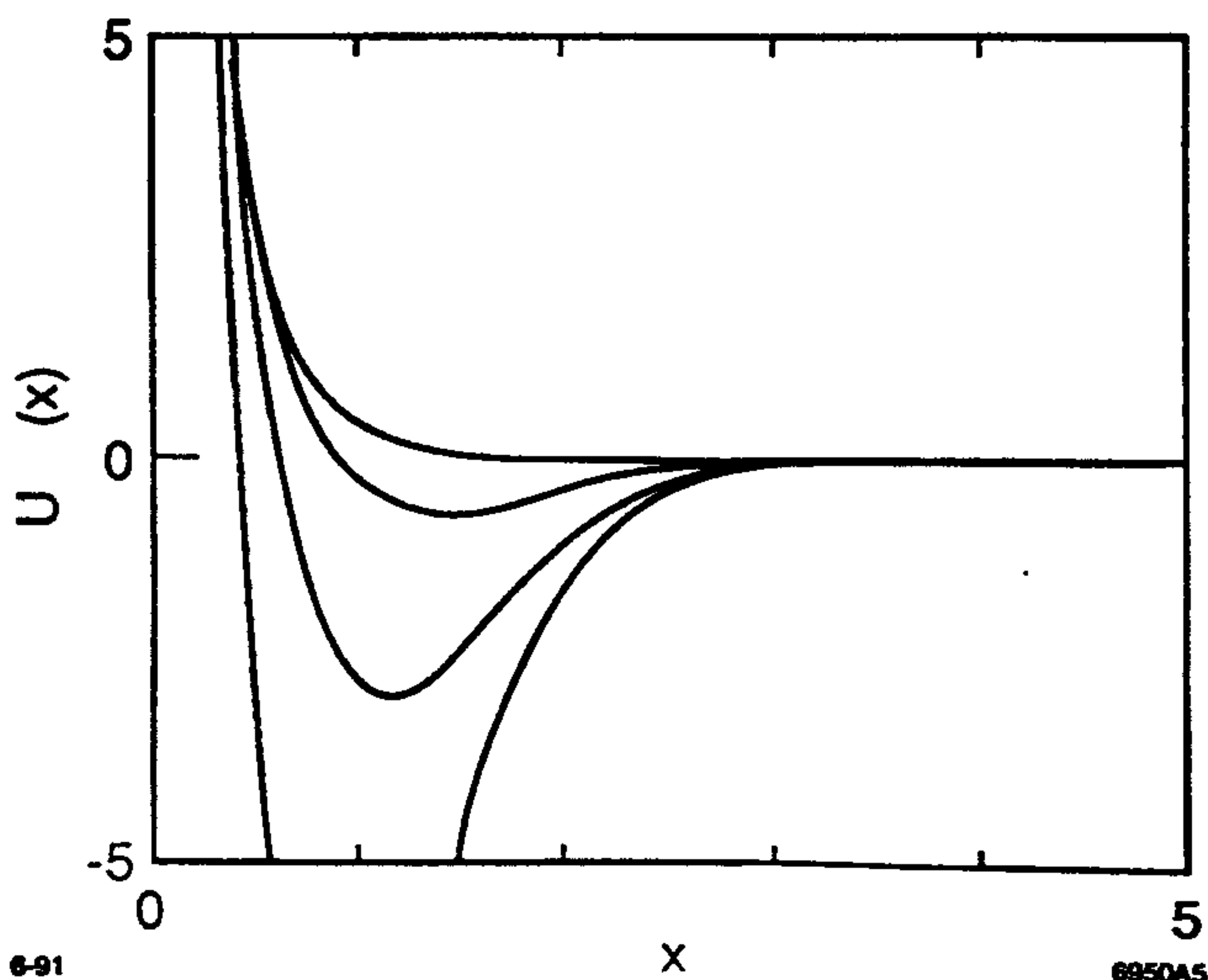


FIG. 4.14. Effective potential for dipole ($m = \pm 1$) oscillations; $\Delta_0 = -0.05$, $\xi = 0.05$, from top to bottom— $iv/m = 0.1, 0.075, 0.05, 0.025$.

$$\Lambda_m \simeq \frac{1}{|m_x m_z|} \frac{1}{m_x^2 r + m_z^2 / r}, \quad |m_x|, |m_z| > 1. \quad (4.6.72)$$

After the substitution of this expression in Eq. (4.6.70) we obtain a dispersion equation in the form:

$$1 = \frac{2\Delta_m \Xi_m}{v^2 - \Delta_m^2}, \quad (4.6.73)$$

$$\Xi_m = \frac{m_x \xi_x (1+r) + m_z \xi_z (1+1/r)}{|m_x m_z| (m_x^2 r + m_z^2 / r)}, \quad (4.6.74)$$

where the beam-beam parameters ξ_x and ξ_z are defined by Eqs. (4.3.23) and (4.3.24). This indicates that in two-dimension cases, positions of the stopbands are determined by unequations

$$\Delta_m \Xi_m < 0, \quad (4.6.75)$$

$$|\Delta_m| < 2|\Xi_m|. \quad (4.6.76)$$

According to these unequations (see also Fig. 4.18), the relative importance of various two-dimensional resonances depends both on the symmetry of the mode m_x, m_z and on the aspect ratio of the beam r . For flat bunches $r \ll 1$ the widths of stopbands is mainly related to excitation of vertical oscillations:

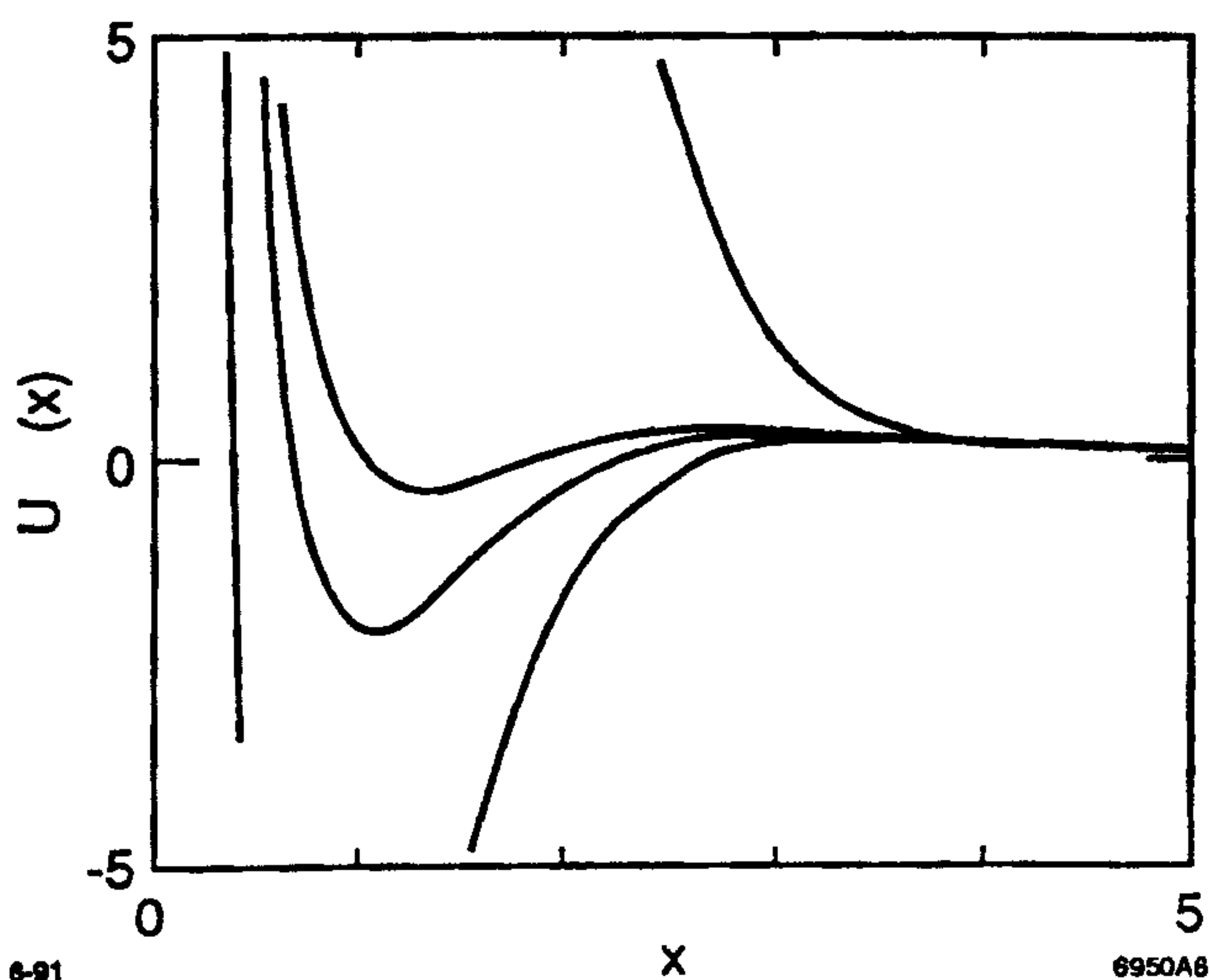


FIG. 4.15. Effective potential for dipole ($m = \pm 1$) oscillations; $\xi = 0.05$, $v = 0.1$, for the top line $\Delta_0 = 0$, then from the bottom $\Delta_0 = -0.125, -0.25, -0.375$.

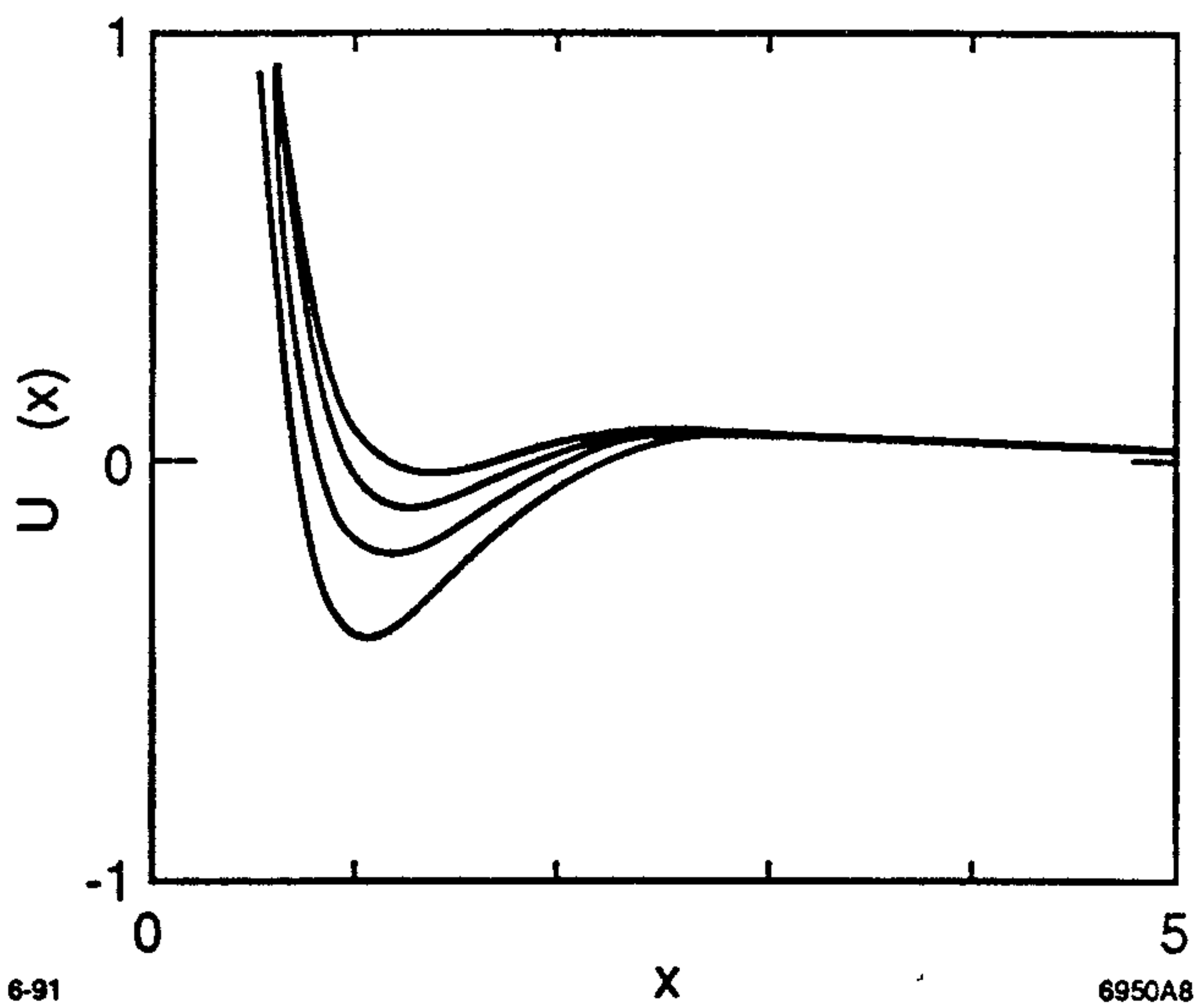


FIG. 4.16. Effective potential for quadrupole ($m = \pm 2$) oscillations; $\xi = 0.05$, $\nu = 0.1$, from top to bottom $\Delta_0 = -0.4, -0.35, -0.3, -0.25$.

$$\Delta_0 \simeq \frac{2\xi_z}{m_z^2 |m_x|}.$$

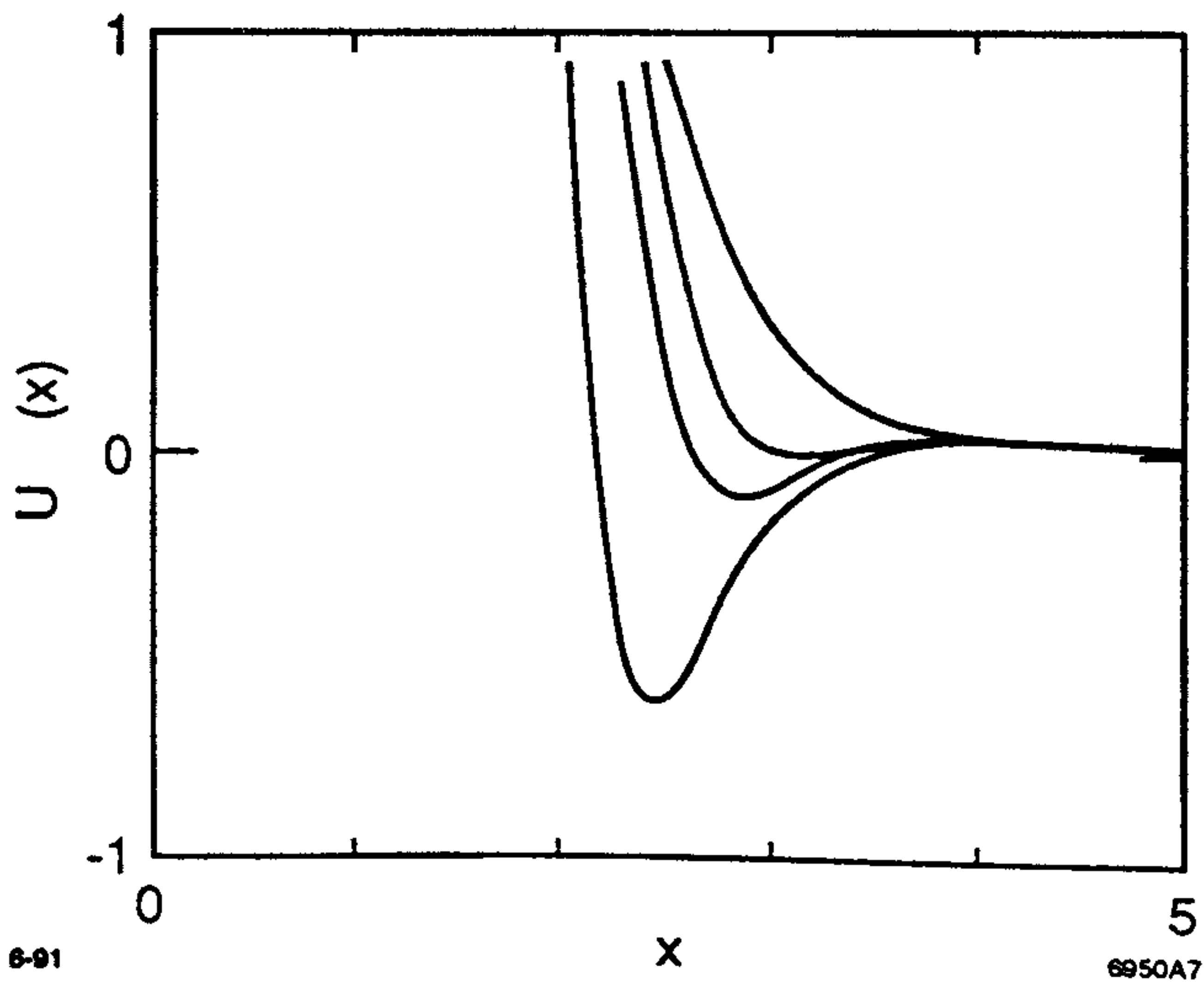


FIG. 4.17. Effective potential for sextupole ($m = \pm 3$) oscillations; $\xi = 0.05$, $\nu = 0.1$, from top to bottom $\Delta_0 = 0, -0.0125, -0.015, -0.02$.

The most broad are sum-type resonance $\nu_x + \nu_z = 1$ and the main coupling resonance $\nu_x = \nu_z$.^{*}

For round beams ($\varepsilon_x = \varepsilon_z$), as we mentioned, there is no stopband along the main coupling resonance, while the most powerful two-dimensional is the linear sum-type resonance. As with the increase of the beam current the working point in this case moves along the line $\nu_x = \nu_z$, its vicinity may become more preferable for the position of the ring working point.

4.7. COHERENT BEAM-BEAM OSCILLATIONS OF LONG BUNCHES

If the bunch length σ_c becomes comparable with β^* , one may expect strong suppression of coherent beam-beam oscillations due to the phase averaging effect discussed in §4.4. This important possibility was already mentioned in §4.5 and will be now discussed in a more straightforward way.⁷³ As we mentioned before, the most complete description of coherent oscillations of colliding bunches is given by the system of equations (4.6.1). For the contributing Lagrangians $L_{1,2}$ or $L_{2,1}$ we can use either Eq. (4.6.2) or equivalent equation

$$L_{1,2} = 2 \frac{Ne^2}{c} U_{1,2},$$

$$U_{1,2} = \int \frac{dk_x dk_z}{\pi k^2} \exp(i\mathbf{k}\mathbf{r}_1^{(1)}) \rho^{(2)}(\mathbf{k}, \vartheta), \quad k^2 = k_x^2 + k_z^2, \quad (4.7.1)$$

$$\rho^{(2)}(\mathbf{k}, \vartheta) = \int d^2r_1 d^3p \exp(-i\mathbf{k}\mathbf{r}_1) f^{(2)}(\mathbf{r}_1, \vartheta + \vartheta_s, \mathbf{p}, \vartheta_s). \quad (4.7.2)$$

Substituting these equations in Eqs. (4.6.1) and using the Fourier expansion for its solutions,

$$f^{(1,2)}(\mathbf{I}, \psi, \vartheta_s) = \sum_{\mathbf{m}} f_{\mathbf{m}}^{(1,2)}(\mathbf{I}, \vartheta_s) \exp(i\mathbf{m}\psi), \quad (4.7.3)$$

we can rewrite Eqs. (4.6.1) in the form

$$\begin{aligned} \frac{\partial f_{\mathbf{m}}^{(1)}}{\partial \vartheta_s} + i\mathbf{m}\nu_1 f_{\mathbf{m}}^{(1)} &= -i\mathbf{m} \frac{\partial f_0^{(1)}}{\partial \mathbf{I}} \frac{2Ne^2}{c} [U_{1,2}]_{\mathbf{m}}, \\ \frac{\partial f_{\mathbf{m}}^{(2)}}{\partial \vartheta_s} + i\mathbf{m}\nu_2 f_{\mathbf{m}}^{(2)} &= -i\mathbf{m} \frac{\partial f_0^{(2)}}{\partial \mathbf{I}} \frac{2Ne^2}{c} [U_{2,1}]_{\mathbf{m}}. \end{aligned} \quad (4.7.4)$$

Since the phase of synchrotron oscillations ψ_c is a cyclic variable, Eqs. (4.7.3) and (4.7.4) generally contain harmonics of the distribution function in synchro-

^{*}Here we certainly mean the fractional parts of tunes.

tron phases f_{m_x, m_z, m_c} , describing the coherent synchrotron (or, respectively, synchrobetatron) oscillations of the bunches. The possibility to select such modes as the normal modes of coherent motion and thus to classify solutions using definite multipolarity of synchrotron oscillations m_c is generally determined by the ratio between the rate of time variation of amplitudes of coherent oscillations ($\text{Im}\nu$) and ν_c . For the beam-beam interaction this rate is specified by the beam-beam parameter ξ . To simplify the calculations we initially assume

$$\text{Im}\nu \gg \nu_c, \quad (4.7.5)$$

when synchrotron oscillations of particles can cause only adiabatically slow variations of the parameters of coherent oscillations, but do not classify the collective modes. For these fast coherent oscillations the coupling of synchrotron modes is so strong that it becomes more reliable to describe the collective motion assuming the variables Δp and φ as the integrals of the motion. In this case the distribution functions can be presented by the expansions

$$f^{(1,2)} = \sum_{\mathbf{m}_\perp} f_{\mathbf{m}_\perp}^{(1,2)}(\mathbf{I}_\perp, \Delta p, \varphi, \vartheta_s) \exp(im_\perp \psi_\perp), \quad (4.7.6)$$

describing the bunch with a rigid longitudinal distribution.

Assuming that the unperturbed distribution functions are:

$$f_0^{(1,2)} = F_0(\mathbf{J}_\perp) \delta(\Delta p) \rho_0(\varphi), \quad (4.7.7)$$

let us write solutions of Eqs. (4.7.4) in the form:

$$f_m^{(1,2)} = \rho_0(\vartheta \mp \vartheta_s) \delta(\Delta p) \exp(-im_\perp \psi_\perp \mp \vartheta_s) A_m^{(1,2)}(\mathbf{J}_\perp, \varphi, \vartheta_s). \quad (4.7.8)$$

The substitution of these expressions into Eqs. (4.7.4) yields the system of equations for amplitudes $A_m^{(1,2)}$:

$$\begin{aligned} \frac{\partial A_m^{(1)}}{\partial \vartheta_s} = & -im_\perp \frac{\partial F_0}{\partial \mathbf{J}_\perp} \left(\frac{4Ne^2}{pc} \right) \rho_0(\varphi) \rho_0(2\vartheta_s + \varphi) \int \frac{d^2k}{\pi k^2} Q_m(\mathbf{J}, \vartheta_s) \\ & \times \sum_{m'_\perp} \exp[i(\mathbf{m}'_\perp + \mathbf{m}_\perp) \phi_\perp(\vartheta_s)] \\ & \times \int d\Gamma'_\perp d\varphi' Q_{m'}^*(\mathbf{J}', \vartheta_s) A_{m'}^{(2)}(\mathbf{J}', \varphi', \vartheta_s), \end{aligned} \quad (4.7.9)$$

$$\begin{aligned} \frac{\partial A_m^{(2)}}{\partial \vartheta_s} = & -im_\perp \frac{\partial F_0}{\partial \mathbf{J}_\perp} \left(\frac{4Ne^2}{pc} \right) \rho_0(\varphi) \rho_0(2\vartheta_s + \varphi) \int \frac{d^2k}{\pi k^2} Q_m(\mathbf{J}, \vartheta_s) \\ & \times \exp[i\mathbf{m}_\perp \phi_\perp(\vartheta_s)] \sum_{m'_\perp} \exp[i\mathbf{m}'_\perp \phi_\perp(\vartheta_s)] \end{aligned}$$

$$\times \int d\Gamma'_1 d\varphi' Q_m^*(J',\vartheta_s) A_{m'}^{(1)}(J',\varphi',\vartheta_s),$$

where

$$Q_m(J,\vartheta_s)=J_{m_x}[k_x\sqrt{J_x\beta_x(\vartheta_s)}]J_{m_z}[k_z\sqrt{J_z\beta_z(\vartheta_s)}]. \tag{4.7.10}$$

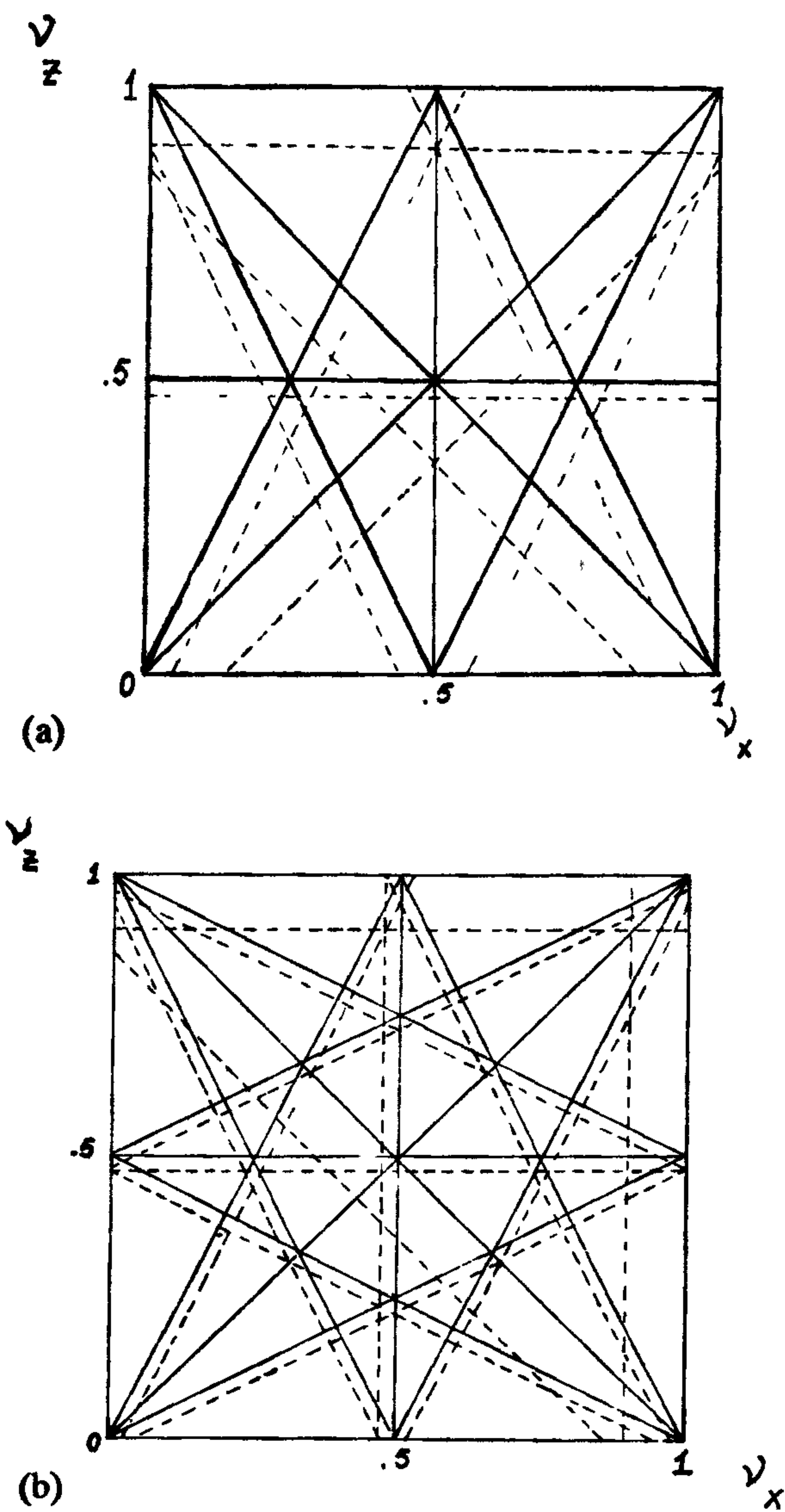


FIG. 4.18. Positions of coherent betatron stopbands for modes with multipole numbers below 3 in the working plane for similar flat ($\xi_z \gg \xi_x, \sigma_x \gg \sigma_z$) (a) and similar round (b) colliding beams; $F=0.05$.

Finally, introducing the momenta:

$$X_m^{(a)}(J, \vartheta_s) = \int_0^{2\pi} d\varphi A_m^{(a)}(J, \varphi, \vartheta_s), \quad a=1,2 \quad (4.7.11)$$

and assuming the Gaussian linear density distributions in bunches,

$$\rho_0(\varphi) = \frac{1}{\sqrt{2\pi}\sigma_\phi} \exp\left(-\frac{\varphi^2}{2\sigma_\phi^2}\right), \quad \sigma_c = R_0\sigma_\phi, \quad (4.7.12)$$

one can get for $X_m^{(1,2)}$

$$\begin{aligned} \frac{\partial X_m^{(1)}}{\partial \vartheta_s} &= (1/\sqrt{\pi}\sigma_\phi) \exp\left(-\frac{\vartheta_s^2}{\sigma_\phi^2} + im_\perp \phi_\perp(\vartheta_s)\right) \sum_{m'_\perp} \exp[im'_\perp \phi_\perp(\vartheta_s)] \\ &\times \int d\Gamma'_\perp K_{m,m'}(J, J', \vartheta_s) X_{m'}^{(2)}(J'), \end{aligned} \quad (4.7.13)$$

$$\begin{aligned} \frac{\partial X_m^{(2)}}{\partial \vartheta_s} &= (1/\sqrt{\pi}\sigma_\phi) \exp\left(-\frac{\vartheta_s^2}{\sigma_\phi^2} + im_\perp \phi_\perp(\vartheta_s)\right) \sum_{m'_\perp} \exp[im'_\perp \phi_\perp(\vartheta_s)] \\ &\times \int d\Gamma'_\perp K_{m,m'}(J, J', \vartheta_s) X_{m'}^{(1)}(J'), \end{aligned}$$

where

$$K_{m,m'}(J, J', \vartheta_s) = -im_\perp \frac{\partial F_0}{\partial J_\perp} \left(\frac{2Ne^2}{pc} \right) \int Q_m(J, \vartheta_s) Q_{m'}^*(J', \vartheta_s) \frac{d^2k}{\pi k^2}. \quad (4.7.14)$$

These equations again describe the perturbation of colliding bunches caused by the countermoving partner by a sequence of rather short periodic kicks. Hence, like in §4.6 we may expect eigensolutions in the form of Eq. (4.6.8):

$$f_m(0+) = \lambda e^{2\pi im\nu} f_m(0-). \quad (4.7.15)$$

The calculation of the variations of $X_m^{(1,2)}$ after the crossing the IP $\delta X_m^{(1,2)}$ can be significantly simplified if a particular geometry of colliding bunches is assumed. Therefore below we shall consider separately two important cases: collisions of bunches with round cross sections ($\sigma_x = \sigma_z$; $\beta_x = \beta_z$) and collisions of flat bunches ($\sigma_x \gg \sigma_z$). Since we have assumed that the lengths of bunches σ_c and β^* can have comparable values, in these calculations we must take into account the variation of the β -function in the close vicinity of the interaction point:

$$\beta_\alpha(s) = \beta_\alpha^* + \frac{s^2}{\beta_\alpha^*}, \quad (4.7.16)$$

and thus, that the corresponding phase advance of betatron oscillations can be large enough:

$$\phi_\alpha(s) = \arctan(s/\beta_\alpha^*). \quad (4.7.17)$$

Let us start with the case when bunches have round cross sections. Taking into account that the nontrivial behavior of the amplitudes $X_m^{(1,2)}$ is associated with the tuning of the working point toward the nonlinear resonances $m\nu = n$, and assuming $\xi \ll 1$, one can rewrite, for instance, the first equation from Eqs. (4.7.13) in the form

$$\frac{\partial X_m^{(1)}}{\partial \vartheta_s} = (1/\sqrt{\pi}\sigma_\phi) \exp\left(-\frac{\vartheta_s^2}{\sigma_\phi^2} + 2im_\perp \phi_\perp(\vartheta_s)\right) \int d\Gamma'_1 K_{m,m'}(\mathbf{J}, \mathbf{J}', \vartheta_s) X_{m'}^{(2)}(\mathbf{J}'). \quad (4.7.18)$$

If $\beta_x(s) = \beta_z(s) = \beta(s)$, after the substitutions $k_x \rightarrow k_x \sqrt{\beta(s)}$ and $k_z \rightarrow k_z \sqrt{\beta(s)}$ we can see from Eqs. (4.7.10) and (4.7.14) that the kernel $K_{m,m}$ in Eq. (4.7.18) does not depend on θ_s . Then, the integration of Eq. (4.7.18) over θ_s yields

$$\delta X_m^{(1)} = Y_m(\sigma_c/\beta^*) \int d\Gamma'_1 K_{m,m}(\mathbf{J}, \mathbf{J}') X_m^{(2)}(\mathbf{J}'), \quad (4.7.19)$$

where

$$Y_m(\xi) = \int_{-\infty}^{\infty} du \sqrt{\pi} \exp[-u^2 + 2i(m_x + m_z) \arctan(u/\xi)], \quad \xi = \beta^*/\sigma_c.$$

Except for the factor $Y_m(\xi)$, Eq. (4.7.19) coincides with a corresponding equation describing the variation of amplitudes of a short bunch. [See, for instance, Eq. (4.6.7).] Thus, we may state that for round beams the influence of the expansion of the interaction region due to finite lengths of colliding bunches as well as the influence of Floquet modulation is equivalent to the renormalization of the beam-beam parameter,

$$\xi_{eff} = Y_m(\xi) \xi, \quad (4.7.20)$$

and, provided the factor $Y_m(\xi)$ is small, describes the suppression of coherent beam-beam resonances. As one more interesting feature of this modification of ξ , we have to mention the following. Since the position of the stopbands relative to particular resonance $\nu = n/m$ is determined by the sign of the production (see §4.6),

$$\xi_{eff} \Delta = \xi Y(\xi) \Delta \leq 0, \quad \Delta = \nu - n/m,$$

in the regions of ξ , where $Y(\xi) < 0$, the stopbands occur above the resonant values of ν . If their widths are smaller than that of incoherent oscillations, the stability of bunches will be determined by the stability of o-modes.

The asymptotes of $Y_m(\xi)$ can be easily calculated for very short or very long bunches. For short bunches $\xi \gg 1$ one can use the expansion

$$\arctan(u) \simeq u, \quad u \ll 1,$$

when the calculation of the integral in Eq. (4.7.20) yields

$$Y_m(\xi) = \exp(-q^2/\xi^2) \simeq 1 - (q/\xi)^2, \quad q = m_x + m_z, \quad (4.7.21)$$

It indicates weak suppression of coherent beam-beam resonances when the interaction region is only slightly extended due to finite bunch length.

The inverse asymptote of $Y_m(\xi)$ can be obtained by rewriting this function in the form

$$Y_m(\xi) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} e^{-u^2} T_{2q}\left(\frac{\xi}{\sqrt{\xi^2 + u^2}}\right),$$

where $T_q(x)$ are the Chebyshev polynomials.¹⁴ Then, using the relationship $T_{2q}(0) = (-1)^q$, one can easily find that for round, long bunches

$$Y_q(\xi) \rightarrow (-1)^q, \quad \xi \ll 1, \quad (4.7.22)$$

the infinite extension of the interaction region of the bunches does not suppress the coherent beam-beam instability. These properties generally agree with predictions obtained in §4.4 for incoherent beam-beam oscillations.

The behavior of $Y_m(\xi)$ for a wider region cannot be predicted analytically. The inspection of the results of numerical calculations (see Fig. 4.19) indicates that coherent resonances, especially of higher orders, can be significantly suppressed for round bunches with lengths $\sigma_{\parallel} \sim \beta^*$. On the other hand, as condition (4.7.5) must be valid, the value of the suppression factor cannot be too small. Using for estimations, for instance, Eq. (4.6.72), for a given combination (m_x, m_z) we can rewrite (4.7.5) as

$$|Y(\xi)| \gg \frac{\nu_c}{4\xi} (m_x^2 + m_z^2).$$

Although this condition holds better for smaller frequency of synchrotron oscillations [$\xi \gg \nu_c$], it still can be violated for modes with multipole numbers

$$|m_{x,z}| \sim 2 \sqrt{\xi/\nu_c}.$$

In some cases the calculation of Y_m can be simplified representing this function in terms of the confluent hypergeometric function $\Psi(a, b, z)$ ¹⁴

$$\Psi(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty dt t^{a-1}(1+t)^{b-a-1}e^{-zt}.$$

Direct calculations result in

$$Y_m(\zeta)=\frac{(2m)!}{4^m}\sum_{k=0}^m(-1)^{m-k}\frac{(4\zeta^2)^k}{(2k)!(m-k)!}\Psi(m,k+1/2,\zeta^2). \tag{4.7.25}$$

Let us now consider in more detail the case, when colliding bunches are flat ($\sigma_x \gg \sigma_z$). If also $\beta_x^* = \beta_z^*$, coherent oscillations of flat and round bunches are described by similar equations with the same suppression factor $Y_m(\zeta)$. Due to $\xi_z \gg \xi_x$ we may expect that resonances of vertical oscillations $m_x \nu_z = n$ will be the most dangerous for the collective stability of bunches.

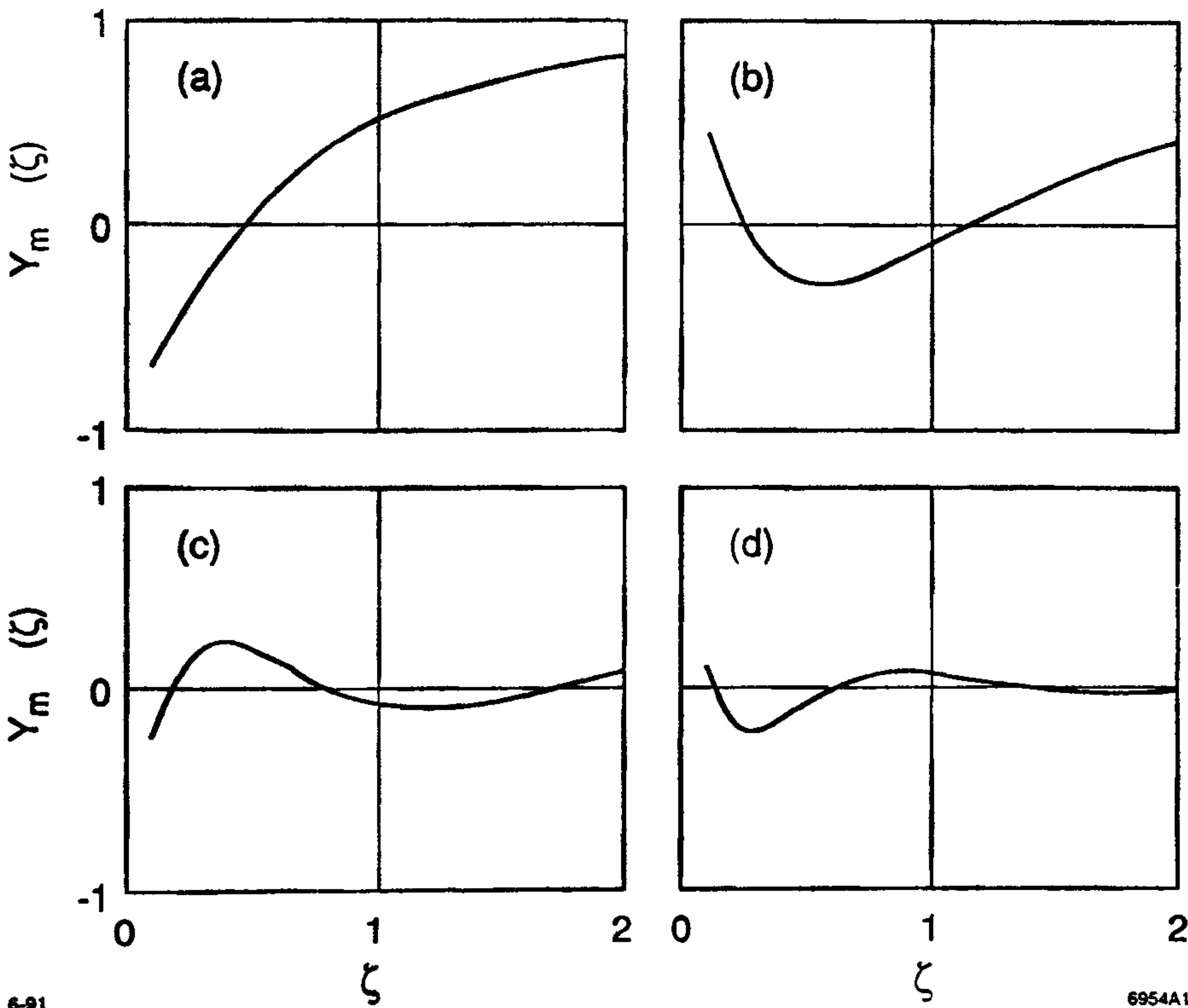


FIG. 4.19. Dependence of the suppression factor of coherent beam-beam resonances on the bunch length ($\zeta = \beta^* / \sigma_z$) for round bunches; a: $q=1$, b: $q=2$, c: $q=3$, d: $q=4$.

Below, we focus on the case, when $\beta_x^* \gg \beta_z^*$. If $\beta_x^* \gg \beta_z^*$, the kernel $K_{m,m}$ in Eqs. (4.7.13), (4.7.18) reads

$$K_{m,m'}(\mathbf{J}, \mathbf{J}', \vartheta) = -im \frac{\partial F_0}{\partial \mathbf{J}} \left(\frac{2Ne^2}{pc} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dw dv$$

$$\times \frac{J_{m_x}(w \sqrt{J_x}) J_{m_z}(v \sqrt{J_z}) J_{m'_x}(w \sqrt{J'_x}) J_{m'_z}(v \sqrt{J'_z})}{\pi \left(w^2 \sqrt{\frac{\beta_z(\vartheta)}{\beta_x(\vartheta)}} + v^2 \sqrt{\frac{\beta_x(\vartheta)}{\beta_z(\vartheta)}} \right)}.$$

(4.7.26)

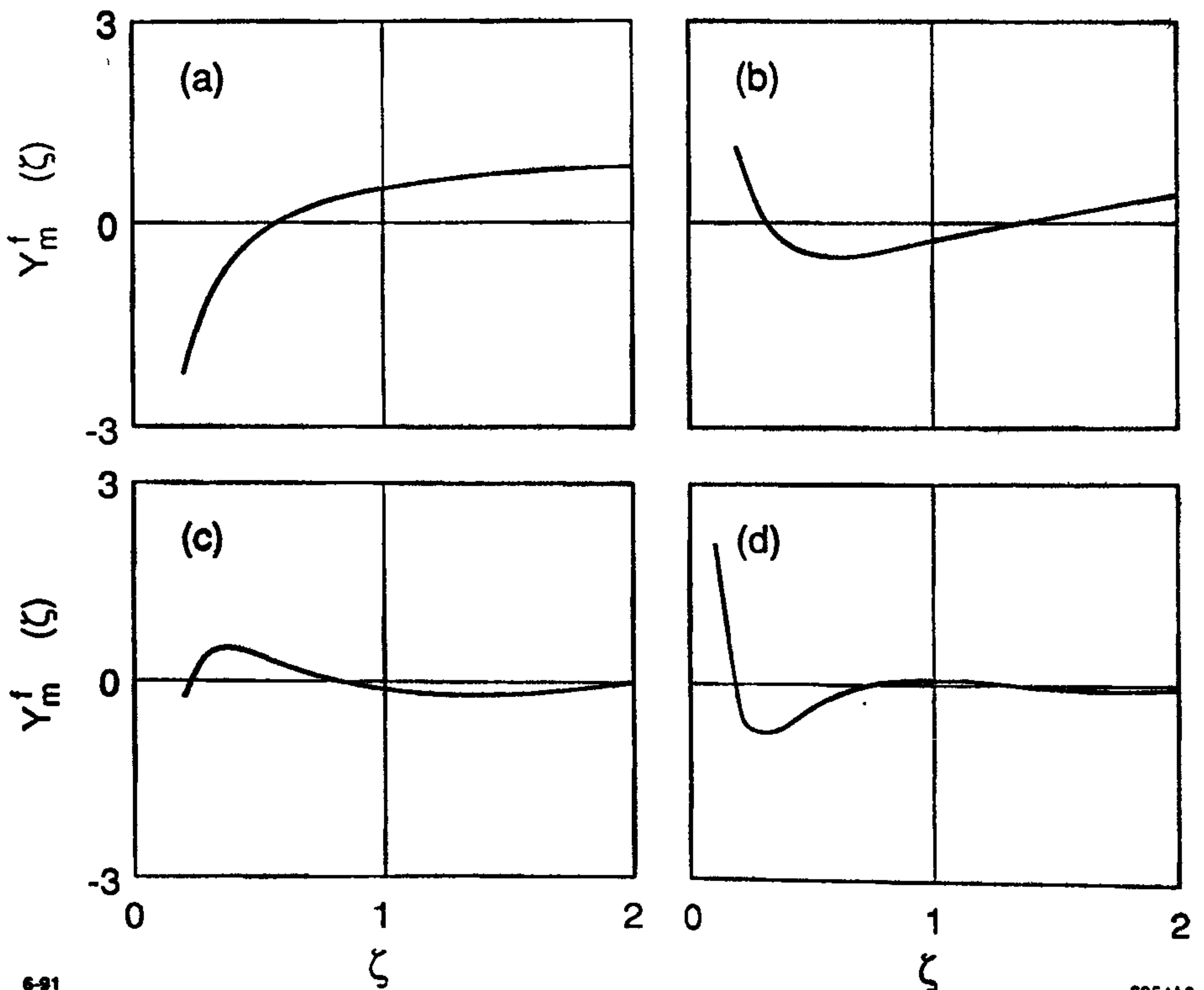


FIG. 4.20. The same as in Fig. 4.19, but for flat beams.

Since the leading contributions to the right-hand side of Eq. (4.7.19) makes the region $J_x \leq \varepsilon_x$ and $J_z \leq \varepsilon_z$, kernel $K_{m,m}(J_x, J_z, \vartheta)$ can be factorized asymptotically for modes with $m_z \neq 0$. According to Eq. (4.7.26), the leading term of the expansion of $K_{m,m}(J_x, J_z, \vartheta)$ in the powers of $\sqrt{\beta_z/\beta_x}$ is

$$K_{m,m'}(\mathbf{J}, \mathbf{J}', \vartheta) = -im \frac{\partial F_0}{\partial \mathbf{J}} \left(\frac{2Ne^2}{pc} \right) \sqrt{\beta_z(\vartheta)/\beta_x(\vartheta)} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dw dv}{\pi v^2} J_{m_x}(w \sqrt{J_x}) J_{m_z}(v \sqrt{J_z}) J_{m'_x}(w \sqrt{J'_x}) J_{m'_z}(v \sqrt{J'_z}). \quad (4.7.27)$$

Neglecting here the modulation of β_x along the (IR) and substituting this kernel in Eq. (4.7.18), we obtain for the variations of $\delta X_m^{(1,2)}$ the expressions, similar to that in Eq. (4.7.19), but with a different factor $Y_m(\xi)$:

$$Y_m^f(\xi) = \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} \exp[-u^2 + 2im_z \arctan(u/\xi)] \sqrt{1 + \frac{u^2}{\xi^2}}. \\ = \frac{(2m)! \xi}{4\pi} \sum_{k=0}^m (-1)^{m-k} \frac{(4\xi^2)^k}{(2k)!(m-k)!} \Psi(m-1/2, k, \xi^2). \quad (4.7.28)$$

For short bunches ($\xi \gg 1$), this function has the asymptote similar to that of $Y_m(\xi)$ in Eq. (4.7.20). The behavior of $Y_m(\xi)$ and $Y_m^f(\xi)$ strongly differs for long bunches [$\xi \rightarrow 0$], when Y_m^f diverges proportional to $1/\xi$. This behavior of Y_m^f describes the mismatching between the beam-beam kick and the β -function, which takes place for flat colliding bunches. As can be seen from Fig. 4.20, within the intermediate region of parameters ($\sigma_{\parallel} \sim \beta^*$), the extension of the interaction region generally suppresses coherent resonances, especially of higher orders.

In the region, where

$$\sigma_{\parallel} \simeq \beta_z^* \ll \beta_x^*, \quad (4.7.29)$$

the radial beam-beam resonances are not suppressed. This can be proved, for instance, repeating the calculations for radial oscillations [$m_z = 0$] and assuming the model distribution function, discussed in § 4.6:

$$F_0(I_x, I_z) = \delta(I_z) \frac{\exp(-I_x/I_0)}{(2\pi)^2 I_0}, \quad (4.7.30)$$

when the kernel of Eq. (4.7.19) gets the form:

$$im_x \delta(I_z) \left(\frac{2Ne^2}{pc\varepsilon_0} \right) \exp(-J_x/2\varepsilon_0) \int_{-\infty}^{\infty} J_{m_x}(u \sqrt{J_x}) J_{m_x}(u \sqrt{J'_x}) \frac{du}{u}, \quad (4.7.31)$$

$$I_0 = p\varepsilon_0, \quad \sigma_{\parallel} \ll \beta_x^*.$$

which has exactly the same form as the kernel, calculated for a short bunch. This means that in the region (4.7.29) the widths of stopbands as well as increments of unstable radial coherent beam-beam oscillations will be determined by the nominal value of ξ_x . Hence, the limitations on the ring performance due to coherent beam-beam effects in this case will be weak, provided that

$$\xi_x \ll (\xi_z)_{\text{eff}} \ll 1. \quad (4.7.32)$$

The suppression of the instability of coherent beam-beam oscillations in the region $\sigma_{\parallel} \simeq \beta^*$ generally eliminates the limitations on the ring performance due to this instability. For instance, this can open new possibilities for the designing the schemes like 4-beam compensated colliding beams, or asymmetric electron-positron factories with different perimeters of the low and high energy rings.

The phase-averaging of the collective beam-beam resonances can essentially enhance the effect of Landau damping on the stability of the collective beam-beam modes. In this case, with the increase of the multipole numbers (m_z and m_x) the strength Landau damping generally increases, while the values of Y_m (or, Y_m^f) decrease. Even for short bunches ($\sigma_{\parallel} \ll \beta^*$) this fact can result in substantial decrease of increments of unstable modes, provided that $m_{x,z}$ are high enough.

CHAPTER 5

Single-Turn Instabilities of Bunched Beams

Another wide class of collective effects that can limit the current in a storage ring is caused by perturbations from induced fields decaying faster than the revolution period of particles in the ring. As was mentioned in Chapter 3, by the accepted terminology such effects are called the single-turn ones. As the specific feature of single-turn phenomena, very weak dependence of decrements (increments) of coherent oscillations on the position of the working point of the ring can be pointed out.

The main features of single-turn coherent oscillations of bunched beams significantly depend on the ratio of coherent frequency shift Ω_m to the frequency of synchrotron oscillations of particles. If $|\Omega_m| \ll \omega_c$, harmonics of ω_c are well pronounced in coherent oscillations, which, therefore, can be specified by a definite multipole number of synchrotron motion m_c . In this region the betatron ($m_c=0$), synchrotron ($m_1^2=0$), and synchrobetatron coherent oscillations can be selected.

In the inverse region $|\Omega_m| \gg \omega_c$ the interaction mixes synchrotron modes and this selection becomes impossible. These fast oscillations have many specific features. In particular, they have no normal modes and, thus, the formally written dispersion equations have no solutions. The physical reason for such behavior is simple. As the rise time for these effects is estimated by $|\Omega_m|$, during the instability the particles do not change their positions in the bunch. In these conditions the instability of coherent oscillations is caused by the redistribution of energy between, say, head-on and tail-on parts of the bunch.

5.1. EFFECT OF THE FAST DAMPING

As was found in early experiments in the storage ring VEPP-2 (INP, Novosibirsk),⁷⁵ an increase in the current of a short bunch results in the proportional increase in the damping rates of its vertical coherent oscillations. This phenomenon, now the so-called effect of the fast damping, was understood in the interaction of the bunch coherent oscillations with the fields, which these oscillations radiated in the matched deflecting plates, placed in the vacuum chamber of the VEPP-2.

The single-turn interaction of a bunch with the matched plates will be detailed in another section. Here we describe basic properties of the single-turn betatron

and synchrotron coherent oscillations spectra, which can be found out within very general assumptions about the bunch environment and the properties of its Green functions. For this case, the general integral equation (3.1.24) reads

$$\Delta\omega_m\chi_m = \rho(\varphi_c) \sum_{n=-\infty}^{\infty} \Omega_{m,n} J_{m_c}(n_1\varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1\varphi'_c) \chi_m(\varphi'_c),$$

$$m_{\perp}^2 \neq 0. \quad (5.1.1)$$

As previously, we use $\Delta\omega_m = \omega - \mathbf{m}_{\perp} \omega_{\perp} - m_c \omega_c$ for the coherent frequency shift, $\rho(\varphi_c)$ is the distribution function over the amplitudes of synchrotron oscillations, $n_1 = n + \mathbf{m}_{\perp} d\omega_{\perp}/d\omega_s$, and $\Omega_{m,n}$ is the coherent frequency shift of the coasting beam. By Eq. (3.1.19) $\Omega_{m,n}$ is related to harmonics of the Green function. Using Eq. (3.1.26)

$$\sum_{n=-\infty}^{\infty} b_n = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dn b(n) \exp(2\pi i l n), \quad (5.1.2)$$

and neglecting in Eq. (5.1.1) the multi-turn terms, we replace in Eq. (5.1.1) the summation over n by the integration to transform it into

$$\Delta\omega_m\chi_m = \rho(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1\varphi_c) \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1\varphi'_c) \chi_m(\varphi'_c). \quad (5.1.3)$$

According to Eq. (3.1.35) we can write the decrements of eigensolutions of Eq. (5.1.3) $[\delta = -\text{Im}\Delta\omega_m]$ in the following form:

$$\delta_{m,\alpha} = - \int_{-\infty}^{\infty} dn \text{Im}\Omega_{m,n} \left| \int_0^{\infty} d\varphi_c \sqrt{\varphi_c \rho(\varphi_c)} J_{m_c}(n_1\varphi_c) w_{m,\alpha}(\varphi_c) \right|^2,$$

where α marks the modes for given m and $\chi_{m,\alpha} = w_{m,\alpha} \sqrt{\rho(\varphi_c)/\varphi_c}$. Although this expression enables very general study of the stability of betatron and synchrotron coherent oscillations, in order to focus the discussion on the basic properties of the decrements due to single-turn interaction and to avoid mathematical embarrassments, we consider a special case when

$$\rho(\varphi_c) = \delta(\varphi_c^2 - \varphi_0^2). \quad (5.1.4)$$

Besides, we assume $m_x = 0$. The substitution of Eq. (5.1.4) in Eq. (5.1.3) yields $\chi_m \propto \delta(\varphi_c^2 - \varphi_0^2)$ and the decrements of oscillations $\delta_m = -\text{Im}\omega$ in the form

$$\delta_m = - \int_{-\infty}^{\infty} dn J_{m_c}^2(n_1\varphi_0) \text{Im}(\Omega_{m,n}) \quad (5.1.5)$$

or, using Eq. (3.1.19)

$$\delta_m = \frac{Ne^2 R_0}{p_s v_z} \mathcal{A}_m \operatorname{sgn}(m_z) I_m(\varphi_0), \quad (5.1.6)$$

$$I_m(\varphi_0) = \int_{-\infty}^{\infty} dn J_{m_c}^2(n\varphi_0) g''_{mn}(n\omega_0 + m_z \omega_z), \quad (5.1.7)$$

$$\mathcal{A}_m = \frac{1}{\Gamma^2(|m_z|)} \int_0^{\infty} da_z^2 \left(\frac{a_z}{2}\right)^{2|m_z|-2} F_0(a_z^2). \quad (5.1.8)$$

The remaining calculations in this section will be done for pure vertical oscillations ($m_c=0$). Besides, we neglect for a moment the chromaticity of betatron oscillations. Then, instead of Eq. (5.1.7) we write

$$I_m(\varphi_0) = \int_{-\infty}^{\infty} dn J_0^2(n\varphi_0) g''_{mn}(m_z \omega_z + n\omega_0). \quad (5.1.9)$$

Since $g''_{mn}(\omega)$ is an odd function of ω and that, due to the symmetry of the Green functions $G(1,2,\omega) = G(2,1,\omega)$, $g''_{mn}(\omega)$ is an even function of m and n , we can rewrite Eq. (5.1.9) as

$$I_m(\varphi_0) = \int_0^{\infty} dn J_0^2(n\varphi_0) [g''_{mn}(n\omega_0 + m_z \omega_z) - g''_{mn}(n\omega_0 - m_z \omega_z)]. \quad (5.1.10)$$

The product $m_z I_m(\varphi_0)$ this expression will be positive and, hence, the oscillations will decay if

$$\int_0^{\infty} dn J_0^2(n\varphi_0) g''_{mn}(n\omega_0 + m_z \omega_z) > \int_0^{\infty} dn J_0^2(n\varphi_0) g''_{mn}(n\omega_0 - m_z \omega_z).$$

In general agreement with the results of §4.2 this means that an interaction of the bunch with its surroundings will damp the coherent oscillations, if the contributions of the differential-type resonances exceed the contributions of the sum-type resonances. Eq. (5.1.10) also shows that I_m vanishes if we neglect the dependence of g'' on the subscript n . The last possibility corresponds to the case when the bunch interacts with electrodes of zero length. For this reason, the effect of the damping becomes stronger if the length of the electrodes increases.

To evaluate the integral in Eq. (5.1.9) we assume that g''_{mn} contains as a parameter some ultimate harmonic number $W \gg 1$ (and $|m_z| v_z \ll W$), after which $g''_{mn}(\omega)$ decreases at least like $1/\omega$. In this case, the integral in Eq. (5.1.9) can be calculated expanding $g''_{mn}(n + m_z v_z)$ in the powers series of $m_z v_z$. The main contribution in $I_m(\varphi_0)$ yields

$$I_m = m_z \omega_z \int_{-\infty}^{\infty} dn J_0^2(n\varphi_0) \left(\frac{\partial g_{mn}}{\partial \omega} \Big|_{\omega = n\omega_s} \right) \quad (5.1.11)$$

or

$$I_m \simeq 2m_z \omega_z W \begin{cases} \overline{\left(\frac{\partial g_{mn}}{\partial \omega} \right)_{\omega=n\omega_s}}, & W\varphi_0 \ll 1, \\ \frac{\ln(W\varphi_0)}{W\varphi_0} \left(\frac{\partial g_{mn}(\omega=n\omega_s)}{\partial \omega} \right)_{n\varphi_0 \sim 1}, & W\varphi_0 \gg 1, \end{cases} \quad (5.1.12)$$

where

$$\overline{(\dots)} = \frac{1}{W} \int_0^W dn(\dots).$$

After the substitution of Eq. (5.1.11) [or (5.1.12)] in Eq. (5.1.6) we find out that decrements of vertical oscillations of the short bunch depend on its total current (N) and do not depend on its length φ_0 as well as on betatron tunes.

$$\delta_m = -\frac{Ne^2 \mathcal{A}_m}{p_s} 2|m_z| v_s W \overline{\left(\frac{\partial g_{mn}(\omega=n\omega_s)}{\partial \omega} \right)}, \quad W\varphi_0 \ll 1. \quad (5.1.13)$$

In the inverse case ($W\varphi_0 \gg 1$) decrements also do not depend on ν , but are proportional to the bunch linear density N/φ_0 :

$$\delta_m = -\frac{Ne^2 \mathcal{A}_m}{p_s \varphi_0} 2|m_z| v_s \ln(W\varphi_0) \left(\frac{\partial g_{mn}(\omega=n\omega_s)}{\partial \omega} \right)_{n\varphi_0 \sim 1}.$$

The enhancement of the damping of coherent oscillations with the increase of the bunch current was primarily observed in the storage ring VEPP-2^{28,75} and was called an effect of the fast damping. Instabilities with increments proportional to the bunch linear density were observed in different rings starting with experiments on ACO and ADONE.⁷⁶ But, except for the linear density, the increments of these instabilities were found to depend on the ring chromaticity.

As was mentioned in §3.2, if we can neglect the excitation of wake-fields by the transverse beam current, the interaction of dipole coherent oscillations with surrounding electrodes can be described in terms of the coupling impedances. In this case, from Eqs. (3.2.45) and (3.1.19) we have

$$g_{mn}(\omega_{mn}) = \frac{\omega_0}{2\Pi} Z_n^\perp(\omega_{mn}), \quad \omega_{mn} = m_z \omega_z + n\omega_0. \quad (5.1.14)$$

Then for instance, Eq. (5.1.6) gets the form ($m_z = \pm 1$).

$$\delta_m = m_z \frac{Ne^2 \omega_0}{4\pi p \nu_z} \int_{-\infty}^{\infty} dn J_{m_c}^2(n_1 \varphi_0) \text{Im}[Z_n^\perp(\omega_{mn})]. \quad (5.1.15)$$

For the vacuum chamber with a round cross section [of the radius l_1], using the relationship between the transverse of longitudinal coupling impedances, [see Eq. (3.2.48)] we can rewrite Eq. (5.1.15) like

$$\delta_m = m_z \frac{Ne^2 \omega_0 R_0^2}{4\pi p v_z J_1^2} \int_{-\infty}^{\infty} dn J_{m_c}^2(n\varphi_0) \frac{\text{Re}\{Z_n^{\parallel}(n+m_z v_z)\}}{n+m_z v_z}. \quad (5.1.16)$$

Note, that according to Eqs. (3.2.9) and (3.3.18) the ratio $Z_n(\omega)/\omega$ is the nonsingular function of ω at $\omega=0$.

If we neglect the lattice chromaticity, the substitution of Eq. (5.1.14) in Eq. (5.1.10) results in the following stability criterion for vertical dipole oscillations

$$\int_0^{\infty} dn J_0^2(n\varphi_0) \text{Im}[Z_n^{\perp}(n\omega_0 + \omega_z) - Z_n^{\perp}(n\omega_0 - \omega_z)] > 0. \quad (5.1.17)$$

5.2. HEAD-TAIL EFFECT

5.2.1 Let us now estimate the decrements of synchrotron oscillations $m_c \neq 0$. To take into account in Eq. (5.1.7) the leading terms we assume that the azimuthal size of the system ϑ_0 is significantly smaller than φ_0 and $1/W$. This means that for the estimation I_m one may neglect the dependence of $g_{mn}''(\omega)$ on the subscript n :

$$I_m(\varphi_0) = \int_0^{\infty} dn J_{m_c}^2(n\varphi_0) g_m''\left(n - m_z \frac{dv_z}{d\ln\omega_0}\right). \quad (5.2.1)$$

If the ring chromaticity is small $|\varphi_0 m_z dv_z/d\ln\omega_0| \ll |m_c|$, the integral in Eq. (5.2.1) can be expanded in the powers of $m_z dv_z/d\ln\omega_0$. The main term of this expansion is

$$I_m \simeq -2m_z \frac{dv_z}{d\ln\omega_0} \int_0^{\infty} dn \frac{\partial g_{mn}''}{\partial n} J_{m_c}^2(n\varphi_0). \quad (5.2.2)$$

Taking into account the asymptotes of the Bessel function

$$J_m(x) \simeq \begin{cases} (x/2)^m/m!, & |x| \ll |m|, \\ \sqrt{\frac{2}{\pi x}} \cos\left[x - \left(m - \frac{1}{2}\right)\frac{\pi}{2}\right], & |x| \gg |m|, \end{cases}$$

we can find out that for short bunches $\varphi_0 W \ll |m_c|$ the main contributions to I_m make harmonics from the region $W \ll |n| \leq |m_c|/\varphi_0$, where we replace $\partial g''/\partial n$ by $-g''(W)/n^2$. Hence, for short bunches Eq. (5.2.2) can be rewritten as

$$I_m \simeq 2m_z \frac{dv_z}{d\ln\omega_0} g''(W) \int_0^\infty dn J_{m_c}^2(n\varphi_c)/n^2$$

or

$$I_m \simeq \frac{2g''(W)}{\pi} m_z \frac{dv_z}{d\ln\omega_0} \frac{\varphi_0}{m_c^2 - 1/4}. \quad (5.2.3)$$

The substitution of Eq. (5.2.3) in Eq. (5.1.6) gives the following formula for decrements of synchrobetatron coherent oscillations

$$\delta_m = \frac{Ne^2 \mathcal{A}_m}{p_s} |m_z| \frac{dv_z}{d\ln\omega_0} \frac{\varphi_0}{m_c^2 - 1/4} \frac{2R_0}{\pi} g''(W). \quad (5.2.4)$$

The contribution from the ring chromaticity into decrements of betatron oscillations can be certainly estimated starting from Eq. (5.2.1) directly. However, these calculations can be done more simply using the following steps. Let us note that Eq. (5.2.1) with given m_z yields

$$\sum_{m_c=-\infty}^{\infty} \delta_{m_z, m_c} = 0$$

(Due to our simplifying assumption, the contribution of the fast damping is omitted.) Therefore, the decrement of the betatron mode reads

$$\delta_{m_z}(m_c=0) = -2 \sum_{m_c=1}^{\infty} \delta_{m_z, m_c}. \quad (5.2.5)$$

Now, since

$$\cot \pi x = \frac{1}{\pi x} + \frac{1}{\pi x} 2 \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2},$$

we find

$$\sum_{m_c=1}^{\infty} \frac{1}{m_c^2 - 1/4} = 2$$

and therefore,

$$\delta_{m_c=0} = -\frac{Ne^2 \mathcal{A}_m}{p_s} |m_z| \frac{dv_z}{d\ln\omega_0} 4\varphi_0 \frac{2R_0}{\pi} g''(W). \quad (5.2.6)$$

This expression shows that Eq. (5.2.4) yields the decrement of the betatron mode, if we make $m_c=0$. To take into account in Eq. (5.2.1) the effect of fast damping one has to retain in Eq. (5.2.1) the dependence of $g''(\omega)$ on the

subscript n .^{*} Then, for instance, for the short bunch $\varphi_0 W \ll |m_c|$ Eq. (5.2.1) shows that the contribution of the fast damping into decrements of synchrobetatron oscillations contains the factor

$$J_{m_c}^2 \left(\varphi_0 m_z \frac{dv_z}{d \ln \omega_s} \right).$$

Thus, this contribution to decrements of synchrobetatron modes is negligible. For the same reasons, the decrement of the betatron mode is given by the combination of Eq. (5.1.13) and (5.2.4):

$$\delta_m = \frac{2Ne^2 |m_z| v \mathcal{A}_m}{p_s} \left\{ - \int_0^W dn \left(\frac{\partial g_{mn}(\omega)}{\partial \omega} \right)_{\omega=n\omega_s} - \frac{4\varphi_0}{\pi} \frac{dv_z}{d\omega_s} g''(W) \right\},$$

$W\varphi_0 \ll 1.$

(5.2.7)

Eqs (5.2.4) and (5.2.7) indicate that the ring chromaticity makes the contributions of opposite sign into decrements of betatron and synchrobetatron oscillations. The importance of this effect was pointed out by C. Pellegrini³⁸ and M. Sands,³⁹ who called it the head–tail effect. The physical sense of this instability is clarified by Eq. (5.2.5), which indicates that the modulation of betatron frequencies by synchrotron oscillations redistributes decrements between betatron and synchrobetatron modes of the bunch.

This statement can be illustrated by the following simple calculations. Let us consider the bunch containing only two particles that have the charges $Ne/2$, execute betatron and synchrotron oscillations with the same amplitudes (a and φ_0 , respectively), but are shifted by π in the phase of synchrotron oscillations. (See Fig. 5.1.) We shall describe the interaction of those particles using the force

$$\mathcal{F}_{1,2} = - \frac{Ne^2 g}{2\pi l_1^2} \theta(\varphi_1 - \varphi_2) z_1,$$

$$\theta(\varphi) = \begin{cases} 1, & \varphi > 0, \\ 0, & \varphi < 0. \end{cases} \quad (5.2.8)$$

Transverse coherent oscillations in such a bunch are described by the equation

^{*}As we mentioned, the effect of the fast damping is caused by the interaction of the bunch with the wake on the whole length of the system. On the contrary, the main contributions to the head–tail effect are due to localized wake fields.

$$z_1 = a \cos \left(\psi_z + \varphi_1 \frac{d\omega_z}{d\omega_s} \right), \quad z_2 = a \cos \left(\psi_z + \varphi_2 \frac{d\omega_z}{d\omega_s} \right),$$

$$\langle z \rangle = \frac{z_1 + z_2}{2}, \quad \varphi_1 = -\varphi_2 = \varphi_0 \cos \psi_c$$

and

$$\langle z \rangle'' + \nu_z^2 \langle z \rangle = -\frac{Ne^2 g}{2\Pi l_1^2} \theta(\varphi_1 - \varphi_2) z_1. \quad (5.2.9)$$

Here, the subscript 1 marks the head-on particle. Due to modulation of z by synchrotron oscillations the solution of Eq. (5.2.9) must have the form

$$\langle z \rangle = \sum_{m_z = \pm 1} \sum_{m_c = -\infty}^{\infty} \langle z \rangle_{m_z, m_c}(\vartheta) \exp[i(m_z \nu_z + m_c \nu_c) \vartheta], \quad (5.2.10)$$

where the systematic behavior of $\langle z \rangle_{m_z, m_c}$ on ϑ is determined by the dispersion equation of the problem. Instead of calculating of this equation let us obtain the rate of the variation of the energy of coherent oscillations in such a bunch. Multiplying Eq. (5.2.9) by $\langle z \rangle$ and defining

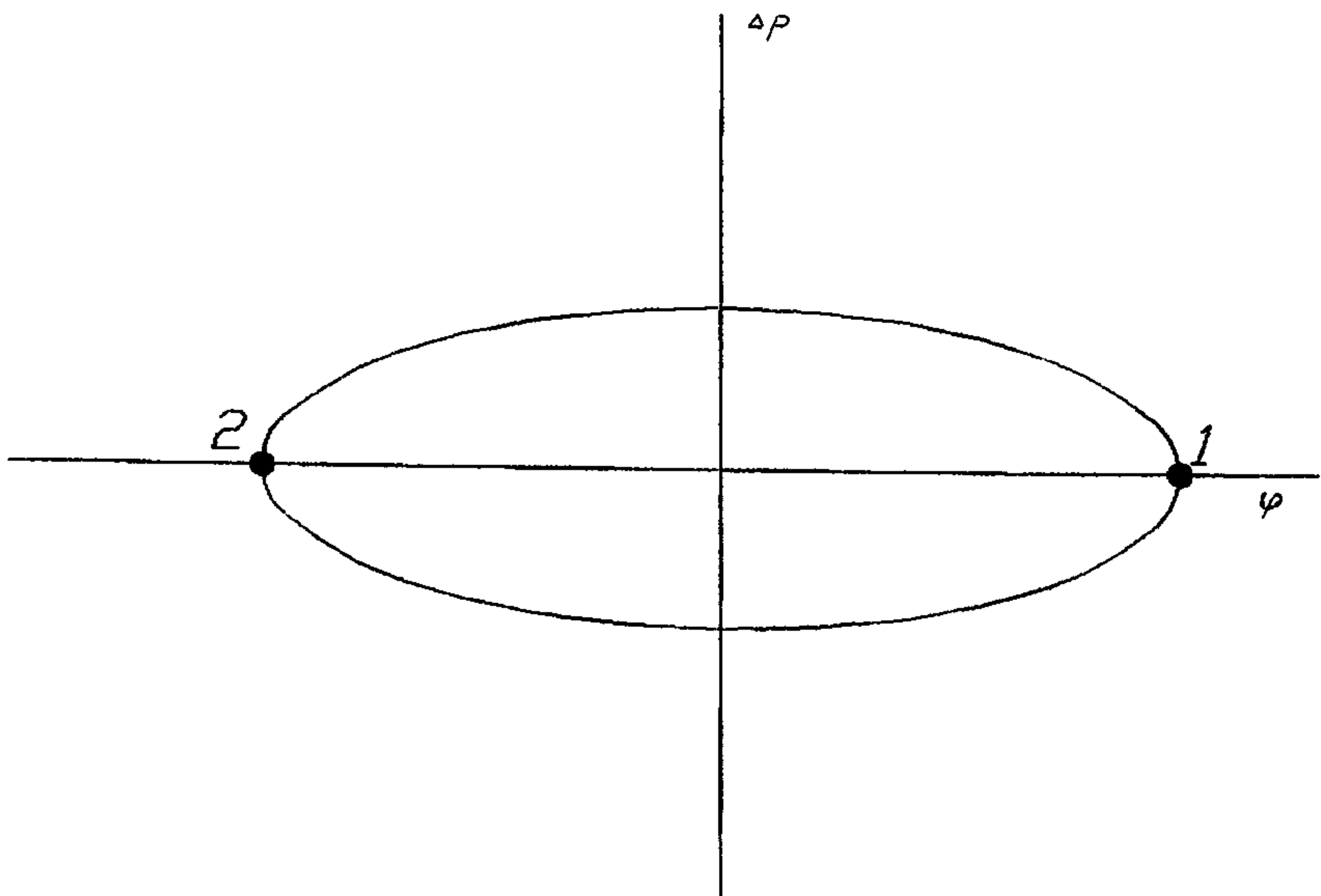


FIG. 5.1. To the head-tail instability mechanism.

$$\begin{aligned}\mathcal{E} &= \sum_{m_z=\pm 1} \sum_{m_c=-\infty}^{\infty} \mathcal{E}_{m_z, m_c} \\ &= \frac{\langle \dot{z} \rangle^2}{2} + \frac{v^2 \langle z \rangle^2}{2} = \sum_{m_z=\pm 1} \sum_{m_c=-\infty}^{\infty} \frac{v_z^2 |\langle z \rangle_{m_z, m_c}|^2}{2},\end{aligned}\quad (5.2.11)$$

we can write

$$\begin{aligned}\frac{d\mathcal{E}}{d\vartheta} &= \frac{Ne^2g}{2\Pi l_1^2} \theta(\varphi_1 - \varphi_2) v_z a^2 \cos\left(\psi_{z1} + \varphi_1 \frac{d\omega_z}{d\omega_s}\right) \sin\left(\psi_{z2} + \varphi_2 \frac{d\omega_z}{d\omega_s}\right) \\ &\approx \frac{Ne^2g}{2\Pi l_1^2} \theta(\varphi_1 - \varphi_2) v_z a^2 \left\{ \cos\psi_z \sin\psi_z - \varphi_1 \frac{d\omega_z}{d\omega_s} \cos^2\psi_z + \varphi_2 \frac{d\omega_z}{d\omega_s} \sin^2\psi_z \right\}.\end{aligned}$$

The averaging of this equation over time gives

$$\begin{aligned}\overline{\frac{d\mathcal{E}}{d\vartheta}} &= \sum_{m_z=\pm 1} \sum_{m_c=-\infty}^{\infty} \overline{\frac{d\mathcal{E}_{m_z, m_c}}{d\vartheta}}, \quad \mathcal{E}_{in} = \frac{v_z^2 a^2}{2}, \\ \delta_{1,0} &= -\frac{1}{\mathcal{E}_{in}} \overline{\frac{d\mathcal{E}_{1,0}}{d\vartheta}} = 4 \frac{Ne^2g}{2\Pi l_1^2} \varphi_0 \left(1 + \frac{d\ln v_z}{d\ln \omega_s} \right), \\ \delta_{1,m_c} &= -\frac{1}{\mathcal{E}_{in}} \overline{\frac{d\ln \mathcal{E}_{1,m_c}}{d\vartheta}} = -\frac{Ne^2g}{2\Pi l_1^2} \varphi_0 \left(1 + \frac{d\ln v_z}{d\ln \omega_s} \right) \frac{1}{m_c^2 - 1/4}.\end{aligned}\quad (5.2.12)$$

In these calculations the values $\mathcal{E}_{m_z,0}$ and $\mathcal{E}_{m_z,m_c}$ represent respectively the energies of the betatron and synchrobetatron modes of the beam. The summation of the right-hand sides in Eq. (5.2.12) yields Eq. (5.2.5) directly.

The discussed redistribution of decrements due to ring chromaticity is only one from the series of the phenomena, due to the coupling between transverse and longitudinal degrees of freedom, which is generally specific for the motion of particles in storage rings. Without the mentioned effect of the fast damping, the head-tail effect manifests itself as a global coherent instability when either betatron or synchrobetatron oscillations of the bunch become unstable.

5.2.2 Another global coherent instability can be associated with the coupling of the synchrobetatron modes due to interaction of the bunch with its environment. It is clear that such a coupling will be essential only in cases when the coherent frequency shift of the bunch becomes comparable or higher than the frequency of synchrotron oscillations of particles. As has been shown in § 3.1, under these conditions the integral equation (5.1.3) must be replaced by the

following infinite system of integral equations:

$$\begin{aligned}
 (\Delta\omega_m - m_c\omega_c)\chi_m = & \rho(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_m(n) J_{m_c}(n\varphi_c) \\
 & \times \sum_{m'_c=-\infty}^{\infty} \int_0^{\infty} d\varphi'_c \varphi'_c J_{m'_c}(n\varphi'_c) \chi_m(\varphi'_c)
 \end{aligned}
 \quad (5.2.13)$$

Since the solution of these equations in an analytic form is typically very complicated, next we consider a simplified model when it can be solved exactly. Namely, we take

$$\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2), \quad \Omega_m(n) = \frac{i\Omega_m}{\pi(n + i\Delta)}, \quad (5.2.14)$$

and neglect the ring chromaticity ($dv_z/d\omega_s = 0$). We remind the reader that in the region $|\Omega_m| \ll \omega_c$ the quantity Ω_m gives the coherent frequency shift of the betatron mode ($m_c = 0$). Since in this model

$$\chi_m(\varphi_c) = C_m \delta(\varphi_0^2 - \varphi_c^2), \quad (5.2.15)$$

we replace Eqs. (5.2.13) by an equivalent system of algebraic equations:

$$(\Delta\omega_m - m_c\omega_c)C_m = \frac{i\Omega_m}{\pi} \sum_{m'_c=-\infty}^{\infty} C_{m'_c} \int_{-\infty}^{\infty} \frac{dn}{n + i\Delta} J_{m_c}(n\varphi_0) J_{m'_c}(n\varphi_0), \quad (5.2.16)$$

or defining

$$\begin{aligned}
 x &= \frac{\Delta\omega_m}{\omega_c}, \quad w = \frac{\Omega_m}{\omega_c}, \\
 (x - m_c)C_m &= \frac{iw}{\pi} \sum_{m'_c=-\infty}^{\infty} C_{m'_c} \int_{-\infty}^{\infty} \frac{dn}{n + i\Delta} J_{m_c}(n) J_{m'_c}(n).
 \end{aligned}
 \quad (5.2.17)$$

Now, using

$$\begin{aligned}
 J_m(0) &= \delta_{m,0}, \quad J_m(-n) = (-1)^m J_m(n), \\
 J_{-m}(n) &= (-1)^m J_m(n)
 \end{aligned}$$

and ⁵²

$$\frac{\pi}{2}(m^2 - m'^2) \int_0^{\infty} \frac{dn}{n} J_m(n) J_{m'}(n) = \sin\left[\frac{\pi}{2}(m - m')\right],$$

we transform the system of Eqs. (5.2.17) into:

$$(x-w)C_0 = iW \sum_{k=0}^{\infty} \frac{C_{2k+1}^-}{(2k+1)^2}, \quad (5.2.18)$$

$$C_{2k+1}^- = iW \frac{2x}{x^2 - (2k+1)^2} \left[\frac{C_0}{(2k+1)^2} + \sum_{p=1}^{\infty} \frac{C_{2p}^+}{(2k+1)^2 - 4p^2} \right], \quad (5.2.19)$$

$$C_{2p}^+ = iW \frac{2x}{x^2 - 4p^2} \sum_{k=0}^{\infty} \frac{C_{2k+1}^-}{(2k+1)^2 - 4p^2}. \quad (5.2.20)$$

Here $W = (4w/\pi^2)$ and

$$C_{2k}^+ = C_{2k} + C_{-2k}, \quad C_{2k+1}^- = C_{2k+1} - C_{-(2k+1)}.$$

Substituting Eq. (5.2.20) into Eq. (5.2.19), we rewrite Eq. (5.2.19) in the form

$$\begin{aligned} C_{2k+1}^- = iW \frac{2x}{x^2 - (2k+1)^2} & \left[\frac{C_0}{(2k+1)^2} + \sum_{k'=0}^{\infty} C_{2k'+1}^- \right. \\ & \times \sum_{p=1}^{\infty} \frac{2iWx}{[(2k+1)^2 - 4p^2][x^2 - 4p^2][(2k'+1)^2 - 4p^2]} \left. \right]. \end{aligned} \quad (5.2.21)$$

The sum over p in this equation can be easily calculated using the identity

$$\sum_{p=1}^{\infty} \frac{1}{\alpha^2 - 4p^2} = \frac{\pi}{4\alpha} \left[\cot \frac{\pi\alpha}{2} - \frac{2}{\pi\alpha} \right],$$

which results in

$$\begin{aligned} & \sum_{p=1}^{\infty} \frac{1}{[(2k+1)^2 - 4p^2][x^2 - 4p^2][(2k'+1)^2 - 4p^2]} \\ & = -\frac{1}{2x^2(2k+1)^2(2k'+1)^2} + \frac{\pi}{4x} \frac{\cot(\pi x/2)}{[x^2 - (2k+1)^2][x^2 - (2k'+1)^2]}. \end{aligned}$$

Using this expression instead of Eq. (5.2.21) we obtain

$$\begin{aligned} C_{2k+1}^- = & -\frac{2W^2}{(2k+1)^2[x^2 - (2k+1)^2]} \frac{w}{x-w} \sum_{k'=0}^{\infty} \frac{C_{2k'+1}^-}{(2k'+1)^2} \\ & - W^2 \frac{\pi x \cot(\pi x/2)}{[x^2 - (2k+1)^2]^2} \sum_{k'=0}^{\infty} \frac{C_{2k'+1}^-}{x^2 - (2k'+1)^2}. \end{aligned} \quad (5.2.22)$$

Solutions of these equations obviously read

$$C_{2k+1}^- = \frac{A(x)}{(2k+1)^2[x^2 - (2k+1)^2]} + \frac{B(x)}{[x^2 - (2k+1)^2]^2}. \quad (5.2.23)$$

The substitution of Eq. (5.2.23) in Eq. (5.2.22) yields the dispersion equation

$$x = w \left\{ 1 - 2W^2 \left(F_1(x) - \frac{W^2 \pi x \cot(\pi x/2)}{1 + W^2 \pi x \cot(\pi x/2) F_3(x)} F_2^2(x) \right) \right\}, \quad (5.2.24)$$

where

$$\begin{aligned} F_1(x) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4 [x^2 - (2k+1)^2]}, \\ F_2(x) &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 [x^2 - (2k+1)^2]^2}, \\ F_3(x) &= \sum_{k=0}^{\infty} \frac{1}{[x^2 - (2k+1)^2]^3}. \end{aligned} \quad (5.2.25)$$

The functions F_1 , F_2 , and F_3 can be easily calculated using the identity

$$\tan \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - x^2},$$

which yields

$$\begin{aligned} F_1(x) &= \frac{\pi^4}{96x^2} + \frac{\pi^2}{8x^4} - \frac{\pi}{4x^5} \tan \frac{\pi x}{2}, \\ F_2(x) &= \frac{\pi^2}{8x^4} \left(1 + \frac{1}{2\cos^2(\pi x/2)} - \frac{3}{\pi x} \tan \frac{\pi x}{2} \right), \\ F_3(x) &= -\frac{\pi}{32x^3} \left\{ \left(\frac{3}{x^2} + \frac{\pi^2}{2\cos^2(\pi x/2)} \right) \tan \frac{\pi x}{2} - \frac{3\pi}{2x\cos^2(\pi x/2)} \right\}. \end{aligned}$$

In the region $|x| \ll 1$ (and, correspondingly, $|w| \ll 1$) these functions have the asymptotes:

$$F_3 \simeq F_1 \simeq -F_2 \simeq -\frac{\pi^6}{960}.$$

Substituting these values into Eq. (5.2.24) we can estimate the first-order correction to the coherent frequency shift in the case of the weak mode coupling:

$$x \simeq w \left(1 + \frac{\pi^2}{30} w^2 \right).$$

In the region of the strong synchrotron mode coupling $|w| \geq 1$ the dispersion equation must be solved numerically. In this region it definitely has roots with $\text{Im} \omega > 0$ and therefore describes unstable solutions. As can be seen from Eqs.

(5.2.18) and (5.2.19) the lowest threshold ($|\omega_{th}| \approx 0.56$) appears due to the coupling of the baricentric ($m_c=0$) and one of the first synchrotron sidebands ($m_c = \pm 1$). With the increase in the bunch current the interaction will overlap more sidebands, which will result in the appearance of more unstable modes. We have to emphasize that, in contrast with the previously discussed effect of the fast damping and the head-tail instability, this instability due to the coupling of synchrotron modes can be caused by pure reactive interaction of particles. This specific feature will be discussed in more detail in §5.6.

Returning to the head-tail instability of uncoupled synchrotron modes, we note that relevant contributions into decrements of coherent oscillations are determined by fields, which are induced by the longitudinal motion of the bunch.^{31,32} Apart from the example considered, this can be clearly demonstrated by calculating the decrements of coherent oscillations for the bunch, interacting with matched plates.^{31,32} Since this problem can also be important from other points of view, below we shall use it as the basis to study single-turn effects in bunched beams.

5.3. THE FIELDS IN CYLINDRICAL WAVEGUIDES WITH PERFECTLY CONDUCTING WALLS

From a general point of view, various elements of the vacuum chamber of a storage ring present the segments of waveguides. Therefore, generally the motion of a bunch can be accompanied by the excitation and propagation of electromagnetic waves along the vacuum chamber. As shown in § 5.1, if the energy of these waves dissipates, such processes can cause the damping of bunch coherent oscillations. On the other hand, previously for beam-beam calculations we assumed that the fields of the countermoving bunch are quasi-stationary in the bunch rest frame. Such a possibility occurs due to some specific features of the excitation of waveguide modes by the beam.

Here we briefly describe the simplest properties of waveguide modes, which can be excited by a beam in some typical elements of the vacuum chamber. (A more extended review of this subject can be found, for instance, in the textbook Ref. 9.) The most simple case corresponds to the case, when the beam moves inside the vacuum chamber, which has smooth and perfectly conducting walls. The eigenfunctions of waveguide modes

$$\begin{aligned} \mathbf{E}(\mathbf{r}_\perp, s, t) &= \mathbf{E}_k(\mathbf{r}_\perp) \exp(iks - i\omega t), \\ \mathbf{H}(\mathbf{r}_\perp, s, t) &= \mathbf{H}_k(\mathbf{r}_\perp) \exp(iks - i\omega t), \end{aligned} \quad (5.3.1)$$

can be calculated using the sourceless Maxwell equations

$$\begin{aligned} \operatorname{rot} \mathbf{E} &= \frac{i\omega}{c} \mathbf{H}, \quad \operatorname{div} \mathbf{E} = 0, \\ \operatorname{rot} \mathbf{H} &= -\frac{i\omega}{c} \mathbf{E}, \quad \operatorname{div} \mathbf{H} = 0. \end{aligned} \quad (5.3.2)$$

We assume here that we can neglect the curvature of the orbit inside the element of interest. Combining Eq. (5.3.1) and (5.3.2), we write

$$(\Delta_{\perp} + k_{\perp}^2) \{\mathbf{E}, \mathbf{H}\} = 0, \quad k_{\perp}^2 = \frac{\omega^2}{c^2} - k^2, \quad (5.3.3)$$

where $\Delta_{\perp} = \Delta - k^2$ is a two-dimensional Laplace operator. Selecting in Eqs. (5.3.2) the longitudinal and transverse to the orbit components of $\mathbf{E}$ and $\mathbf{H}$

$$\begin{aligned} E_{\parallel} &= \boldsymbol{\tau} \mathbf{E}, \quad H_{\parallel} = \boldsymbol{\tau} \mathbf{H}, \\ \mathbf{E}_{\perp} &= \mathbf{E} - \boldsymbol{\tau} E_{\parallel}, \quad \mathbf{H}_{\perp} = \mathbf{H} - \boldsymbol{\tau} H_{\parallel}, \end{aligned}$$

from rotor equations in Eq. (5.3.2) one can easily express $\mathbf{E}_{\perp}$ and $\mathbf{H}_{\perp}$ through the longitudinal components of these fields

$$\begin{aligned} \mathbf{E}_{\perp} &= \frac{1}{(\omega/c)^2 - k^2} \left[ik \nabla_{\perp} E_{\parallel} - i \frac{\omega}{c} \boldsymbol{\tau} \times \nabla_{\perp} H_{\parallel} \right], \\ \mathbf{H}_{\perp} &= \frac{1}{(\omega/c)^2 - k^2} \left[ik \nabla_{\perp} H_{\parallel} + i \frac{\omega}{c} \boldsymbol{\tau} \times \nabla_{\perp} E_{\parallel} \right]. \end{aligned} \quad (5.3.4)$$

These equations show that the fields $\mathbf{E}$ and $\mathbf{H}$ can be calculated using solutions of two-dimensional Eqs. (5.3.3) for $E_{\parallel}$ and $H_{\parallel}$. Due to assumed perfect conductivity of the chamber walls, the fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the boundary conditions

$$\mathbf{n} \mathbf{H}|_{\Sigma} = 0, \quad \mathbf{n} \times \mathbf{E}|_{\Sigma} = 0, \quad (5.3.5)$$

where $\mathbf{n}$ is the unit vector normal to the surface of electrodes. Taking into account Eq. (5.3.4), we replace these conditions by that for components $E_{\parallel}$ and $H_{\parallel}$:

$$E_{\parallel}|_{\Sigma} = 0, \quad \left. \frac{\partial H_{\parallel}}{\partial n} \right|_{\Sigma} = 0. \quad (5.3.6)$$

Eqs. (5.3.3) together with boundary conditions (5.3.6) form the eigenvalue problem. They have nontrivial solutions only for the set of eigenvalues $k_{\perp}^2 = (\omega/c)^2 - k^2$. As a rule, the boundary conditions cannot be satisfied simultaneously. That is why the modes in waveguides are split onto the modes with transverse electric field (*TE*-modes)

$$E_{\parallel} = 0, \quad H_{\parallel} = \Psi_{TE}, \quad \partial \Psi_{TE} / \partial n|_{\Sigma} = 0 \quad (5.3.7)$$

and the modes with transverse magnetic field (*TM*-modes)

$$H_{\parallel} = 0, \quad E_{\parallel} = \Psi_{TM}, \quad \Psi_{TM}|_{\Sigma} = 0. \quad (5.3.8)$$

This mode separation has been used in § 4.2. As an illustration, let us calculate Ψ_{TE} and Ψ_{TM} for the vacuum pipe with circular cross section. In cylindrical coordinates Eq. (5.3.3) reads

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \varphi^2} + k_1^2 \Psi = 0, \quad (5.3.9)$$

r and φ are the polar coordinates in the chamber cross section. Solutions of Eq. (5.3.9) have the form

$$\Psi_{k_1}(r, \varphi) = J_l(k_1 r) e^{il\varphi}, \quad l = 0, \pm 1, \dots,$$

where $J_l(x)$ is the Bessel function. The eigenvalues k_1 must be calculated from boundary conditions. For *TE*-waves Eq. (5.3.7) yields

$$\frac{d}{dr} J_l(k_1 l_1) = 0,$$

where l_1 is the pipe radius. Hence

$$k_{1,\alpha} = \kappa_{\alpha} / l_1, \quad \alpha = 1, 2, \dots, \quad (5.3.10)$$

where κ_{α} are the roots of the first derivation of the Bessel function. Analogous calculations express k_1 for *TM*-modes in terms of roots of the Bessel function. By definition

$$k_{\alpha}^2(\omega) = \frac{\omega^2}{c^2} - k_{1,\alpha}^2 = \frac{\omega^2}{c^2} - \left(\frac{\kappa_{\alpha}}{l_1} \right)^2.$$

This means that the propagation of *TE* and *TM* is possible if ω exceeds the cutoff frequency of the waveguide

$$\omega > \omega^c = c \kappa_{\min} / l_1. \quad (5.3.11)$$

Otherwise the fields exponentially decay within the longitudinal distance $\simeq 1/l_1$. By the substitution in Eq. (5.3.11) of the combination frequencies ω_{mn} one can verify that for most of harmonics $|n| \gg |\mathbf{m}\mathbf{v}|$ and $|n| < R_0/\beta l_1$ the vacuum chamber presents the waveguide below the cutoff. Thus, the beam-induced fields are quasistatic in its comoving system.

The behavior of induced fields can completely change if the piece of the chamber contains the conducting plate, which is terminated by the characteristic impedance. Such a piece of the vacuum chamber presents the segment of the double-connected waveguide, where the fundamental TEM-wave can propagate

with $E_{\parallel} = H_{\parallel} = 0$. Equations (5.3.3) and (5.3.4) indicate that in an infinite double-connected waveguide the spectrum of TEM-oscillations is continuous,

$$\omega = \pm kc, \quad (5.3.12)$$

whereas amplitude functions satisfy the Laplace equation

$$\begin{aligned} \mathbf{H}_{\perp} &= -\frac{ic}{\omega} \nabla_{\perp} \times \mathbf{E}_{\perp}(\mathbf{r}_{\perp}), \\ \Delta_{\perp} \mathbf{E}_{\perp}(\mathbf{r}_{\perp}) &= 0. \end{aligned} \quad (5.3.13)$$

Nontrivial solutions in the case of the double-connected geometry of the waveguide correspond to the unit voltage between the chamber and the plate. In the high-frequency region ($\omega > c/l_{\perp}$) the propagation of TE- and TM-waves becomes possible.

The vector potential $\mathbf{A}(\mathbf{r}, t)$ describing the field of TEM-waves in the infinite double-connected waveguide reads

$$\mathbf{A}(\mathbf{r}, t) = c \mathbf{A}_0(\mathbf{r}_{\perp}) \int_{-\infty}^{\infty} \frac{dk}{2\pi} Q_k(t) e^{iks}, \quad (5.3.14)$$

where $\mathbf{A}_0(\mathbf{r}_{\perp})$ describes transverse distribution of the field. It is proportional to the electric field, which is created by voltage U_0 between electrodes

$$\mathbf{A}_0(\mathbf{r}_{\perp}) = \sqrt{cZ_0} \frac{\mathbf{E}(\mathbf{r}_{\perp})}{U_0}, \quad \Delta_{\perp} \mathbf{E}_{\perp}(\mathbf{r}_{\perp}) = 0.$$

Here Z_0 is the characteristic impedance of the waveguide.

In fact, the beam interacts with the piece of the waveguide terminated by the characteristic impedance. In the low-frequency region $\omega l_{\perp} \ll c$ one may use Eq. (5.3.14) as the potential of such a double-connected waveguide if $\mathbf{A}_0$ is replaced by the real electrostatic field $\mathbf{A}_0(\mathbf{r})$, which exponentially decays within the distance $\Delta s \simeq l_{\perp}$ outside the waveguide. It is important that near the edges of the waveguide the electric field of the fundamental wave has the longitudinal component and, therefore, can be excited by the longitudinal motion of particles.

Assuming that $\mathbf{A}_0(\mathbf{r})$ is normalized by the condition

$$\int d^3r |\mathbf{A}_0(\mathbf{r})|^2 = 4\pi,$$

we can write the following equation for field oscillators $Q_k(t)$:

$$\ddot{Q}_k + c^2 k^2 Q_k = e \int d\Gamma \mathbf{v} \mathbf{A}_0(\mathbf{r}) \exp(-iks) f_m \exp(im\psi - i\omega t). \quad (5.3.15)$$

Its solution has the form

$$Q_k(t) = \sum_{n=-\infty}^{\infty} Q_{kn} e^{-i(\omega+n\omega_s)t}, \quad V_{mn}(k) = [\nabla A_0(\mathbf{r}) e^{ikR_0\vartheta}]_{\mathbf{m},n},$$

$$Q_{kn} = \frac{e \int d\Gamma f_{\mathbf{m}\omega} V_{mn}^*(k)}{k^2 c^2 - (\omega + n\omega_s + i0)^2}. \quad (5.3.16)$$

For the averaged harmonics of the Lagrangian describing the interaction of a particle with the TEM-wave this yields

$$\bar{L}_{\mathbf{m},\omega} = \frac{e}{c} \int_0^T \frac{dt}{T_s} (\nabla A(\mathbf{r},t))_{\mathbf{m},\omega} = e^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{V_{mn}(k) j_{k,m}}{k^2 c^2 - (\omega + n\omega_s + i0)^2}, \quad (5.3.17)$$

where

$$j_{k,m} = \int d\Gamma V_{mn}^*(k) f_{\mathbf{m},\omega}(\mathbf{I}).$$

The calculation of harmonics $[\nabla A_0(\mathbf{r}) \exp(ikR_0\vartheta)]_{\mathbf{m},n}$ in Eq. (5.3.17) can be simplified using the fact that the field $\mathbf{E}$ has a potential:

$$\mathbf{E}(\mathbf{r}) = -\frac{U_0}{\sqrt{cZ_0}} \nabla U(\mathbf{r}). \quad (5.3.18)$$

By definition we can write

$$V_{mn}(k) = \int_0^T \frac{dt}{T_s} \frac{dU}{dt} \exp[ikR_0\vartheta(t) - im\psi(t) - in\omega_s t],$$

or, after the integration by parts,

$$V_{mn}(k) = i(kv - n\omega_s - m\omega) U_{\mathbf{m},n}. \quad (5.3.19)$$

Here, the term proportional to $kv - n\omega_s$ describes the excitation of the TEM-wave by longitudinal motion of a particle, while the term proportional $m\omega$ describes the excitation of the TEM-wave by transverse oscillations.

For low-frequency field oscillations $kl_{\perp} \ll 1$ the azimuthal dependence of $U(\mathbf{r}_{\perp}, \vartheta)$ can be approximated by the step function

$$U(\mathbf{r}_{\perp}, \vartheta) = \begin{cases} U(\mathbf{r}_{\perp}), & |\vartheta| \leq \vartheta_0 = l/2R_0, \\ 0, & |\vartheta| > \vartheta_0, \end{cases} \quad (5.3.20)$$

where l is the length of the plate. Subsequent multipole expansion of $U(\mathbf{r}_{\perp})$

$$U(\mathbf{r}_{\perp}) = \sum_l (x + iz)^l U_l + \text{c.c.}$$

enables the calculation of the particular harmonics ($U_{m,n}$).

5.4. STABILITY OF VERTICAL SYNCHROBETATRON OSCILLATIONS OF A BEAM INTERACTING WITH MATCHED PLATES

5.4.1. The radiation by the beam of TEM-waves in the low-frequency region $0 < \omega < c/l_1$ enables one to use matched plates as dampers for coherent oscillations in storage rings. Primarily this possibility was suggested in Ref. 77 and was studied in more detail in Ref. 27–35. Here we shall mainly follow Ref. 32.

Let us start by discussing effects of the interaction of a bunch with the matched plate placed in one of the straight sections of a ring. The spectrum of coherent oscillations can be calculated solving the integral equation

$$\Delta\omega_m f_{m,\omega} = \left(\mathbf{m}_\perp \frac{\partial f_0}{\partial \mathbf{I}_\perp} \right) \overline{L_{m,\omega}} \quad (5.4.1)$$

or, substituting Eq. (5.3.17), and (5.3.18) and assuming the single-turn interaction ($\pi > \vartheta_0 + \varphi_b$)

$$\begin{aligned} \Delta\omega_m f_{m,\omega} = & Ne^2 \rho(\varphi_c) m_z \frac{\partial F_0}{\partial I_z} \int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(kv - n\omega_s - m_z \omega_z)^2}{k^2 c^2 - (\omega + n\omega_s + i0)^2} \\ & \times U_{m,n} \int d\Gamma U_{m,n}^* f_{m,\omega}(\mathbf{I}). \end{aligned} \quad (5.4.2)$$

For the sake of simplicity below we assume the smoothed focusing approximation. Then, taking into account Eq. (5.3.19) we write

$$U_{\mathbf{m}}(\mathbf{I}_\perp, \varphi_c) = U_{\mathbf{m}_1}(\mathbf{I}_\perp) J_{m_c}(n_1 \varphi_c) b_{kn}, \quad (5.4.3)$$

$$U_{\mathbf{m}_1}(\mathbf{I}_\perp) = \int \frac{d^2 \psi_\perp}{(2\pi)^2} U(\mathbf{r}_\perp) \exp(-i \mathbf{m}_1 \cdot \psi_\perp), \quad (5.4.4)$$

$$b_{kn} = \int_{-\vartheta_0}^{\vartheta_0} \frac{d\vartheta}{2\pi} e^{i(kR_0 - n)\vartheta} = \frac{\sin \vartheta_0 (kR_0 - n)}{\pi (kR_0 - n)}. \quad (5.4.5)$$

Expecting the solution of Eq. (5.4.2) of the form

$$f_{m,\omega}(\mathbf{I}) = (\mathbf{m} \partial F_0 / \partial \mathbf{I}) U_{\mathbf{m}_1}(\mathbf{I}_\perp) \chi_m(\varphi_c), \quad (5.4.6)$$

we rewrite this equation in the form of Eq. (5.1.3),

$$\Delta\omega_m \chi_m = \rho(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1 \varphi_c) \int_0^\infty d\varphi'_c \varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c), \quad (5.4.7)$$

where, as previously,

$$\Omega_{m,n} = -Ne^2 \left\langle m_z \frac{\partial}{\partial I^2} |U_{m_1}|^2 \right\rangle \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(kv - \omega_{mn})^2 |b_{kn}|^2}{k^2 c^2 - (\omega + n\omega_s + i0)^2} \quad (5.4.8)$$

is the coherent frequency shift of the coasting beam. The contributing integral here can be calculated exactly. Using

$$|b_{kn}|^2 = 2 \left(\frac{\vartheta_0}{2\pi} \right)^2 \int_0^1 dz (1-z) \cos(2\vartheta_0 z [kR_0 - n]), \quad (5.4.9)$$

we rewrite it in the form

$$I_n = \left(\frac{\vartheta_0}{2\pi} \right)^2 \int_0^1 dz (1-z) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(kv - \omega_{mn})^2}{k^2 c^2 - (\omega + n\omega_s + i0)^2} \\ \times [\exp(2i\vartheta_0 z [kR_0 - n]) + \text{c.c.}].$$

The integral over k is determined by residuals in the simple poles

$$kc = \pm (\omega_{mn} + i0), \quad (5.4.10)$$

where the sign $+$ corresponds to forward radiation and $-$ to backward radiation. The remaining integration over z does not present any difficulties and yields

$$I_n = - \left(\frac{\vartheta_0}{\pi} \right)^2 \frac{\omega_s (m_z v_z + n)}{4c} \left\{ \frac{(1-\beta)^2}{2v_- \vartheta_0} \left(1 - \frac{\sin 2\vartheta_0 v_-}{2v_- \vartheta_0} \right) \right. \\ \left. + \frac{(1+\beta)^2}{2v_+ \vartheta_0} \left(1 - \frac{\sin 2\vartheta_0 v_+}{2v_+ \vartheta_0} \right) - i \left[(1-\beta)^2 \left(\frac{\sin \vartheta_0 v_-}{v_- \vartheta_0} \right)^2 \right. \right. \\ \left. \left. + (1+\beta)^2 \left(\frac{\sin \vartheta_0 v_+}{v_+ \vartheta_0} \right)^2 \right] \right\}, \\ v_{\pm} = \beta (m_z v_z + n) \pm n, \quad \beta = v/c. \quad (5.4.11)$$

Substituting this formula in Eq. (5.4.8) and taking into account that in the smoothed focusing approximation $I_z = p_s v_z a_z^2 / 2R_0$ we rewrite $\Omega_{m,n}$ in the form

$$\Omega_{m,n} = \frac{Nr_0 c}{4\gamma v_z} \left\langle m_z \frac{\partial}{\partial a_z^2} |U_{m_1}|^2 \right\rangle (m_z v_z + n) \left(\frac{\vartheta_0}{\pi} \right)^2 \\ \times \left\{ \frac{(1-\beta)^2}{2v_- \vartheta_0} \left(1 - \frac{\sin 2\vartheta_0 v_-}{2v_- \vartheta_0} \right) + \frac{(1+\beta)^2}{2v_+ \vartheta_0} \left(1 - \frac{\sin 2\vartheta_0 v_+}{2v_+ \vartheta_0} \right) \right. \\ \left. - i \left[(1-\beta)^2 \left(\frac{\sin \vartheta_0 v_-}{v_- \vartheta_0} \right)^2 + (1+\beta)^2 \left(\frac{\sin \vartheta_0 v_+}{v_+ \vartheta_0} \right)^2 \right] \right\}. \quad (5.4.12)$$

Equation (5.4.12) indicates that the contribution into $\Omega_{m,n}$ from forward radiation significantly depends on the particle's energy. In relativistic region $\gamma \gg 1$ it is suppressed in $1/\gamma^2$ times as compared with the contribution from backward radiation.

5.4.2. Let us now proceed to the calculation of decrements as well as stability criteria for particular coherent modes. To avoid the known mathematical difficulties we start with the simple case when the bunch distribution over amplitudes of synchrotron oscillations is given by Eq. (5.1.4). This yields $\chi_m(\varphi_c) \propto \delta(\varphi_c^2 - \varphi_0^2)$ and respectively

$$\omega - m_z \omega_z = \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}^2 \left(\varphi_0 \left[n + m_z \frac{d\omega_z}{d\omega_0} \right] \right) \quad (5.4.13)$$

or

$$\delta_m = -\text{Im} \omega = - \int_{-\infty}^{\infty} dn \text{Im}(\Omega_{m,n}) J_{m_c}^2 \left(\varphi_0 \left[n + m_z \frac{d\omega_z}{d\omega_0} \right] \right). \quad (5.4.14)$$

The direct calculation of integral in Eq. (5.4.14) still leads to very complicated expressions. To simplify these calculations below we consider the most important for the applications case, when $\gamma \gg 1$. Then,

$$\text{Im}(\Omega_{m,n}) = \delta_0 (m_z v_z + n) \left(\frac{\sin \vartheta_0 (m_z v_z + 2n)}{m_z v_z + 2n} \right)^2 \quad (5.4.15)$$

and

$$\delta_m = \delta_0 I_m = \delta_0 \int_{-\infty}^{\infty} dn (m_z v_z + n) J_{m_c}^2(n_1 \varphi_c) \left(\frac{\sin \vartheta_0 (m_z v_z + 2n)}{m_z v_z + 2n} \right)^2, \quad (5.4.16)$$

where

$$\delta_0 = \frac{Nr_0 c}{4\pi^2 \gamma v_z} \left\langle m_z \frac{\partial}{\partial a_z^2} \left| U_{m_1} \right|^2 \right\rangle \quad (5.4.17)$$

Using new variable $u = n_1 \varphi_0$ as well as Eq. (5.4.9) we can transform the integral in Eq. (5.4.16)

$$I_m = q^2 \int_0^1 dz \left\{ v(1-z) \cos(2qz\bar{\xi}) - \frac{\sin(2qz\bar{\xi})}{q} \right\} \int_{-\infty}^{\infty} du J_{m_c}^2(u) \cos(2qzu). \quad (5.4.18)$$

Here,

$$\xi = \varphi_0 m_z \frac{dv_z}{d \ln \omega_0}, \quad v = m_z v_z \varphi_0, \quad \bar{\xi} = \xi + v/2, \quad q = 2\vartheta_0/\varphi_0.$$

These expressions indicate that I_m generally depends on the ratio of the bunch and the plate lengths as well as on the phase advance of betatron oscillations on the bunch length $\bar{\xi}$.

Let us consider first the region $l_b \ll 2l$. Using the formula⁵²

$$J_{m_c}^2(u) = \int_0^\pi \frac{d\alpha}{\pi} J_0(2u \sin \alpha) \cos(2m_c \alpha),$$

we transform Eq. (5.4.18) into

$$I_m = \frac{\pi q v}{2} J_{m_c}^2(\bar{\xi}) - \frac{\pi}{2} \left(1 + \frac{v}{2} \frac{d}{d\bar{\xi}} \right) I_m^{(1)}(\bar{\xi}), \quad (5.4.19)$$

$$I_m^{(1)}(\bar{\xi}) = \int_0^\pi \frac{d\alpha}{\pi} \mathbf{H}_0(2\bar{\xi} \sin \alpha) \cos(2m_c \alpha). \quad (5.4.20)$$

Here, $\mathbf{H}_0(x)$ is the Struve's function.⁵² Since below we shall calculate in fact the asymptotic values of $I_m^{(1)}(\bar{\xi})$, the representation of $\mathbf{H}_0(x)$ as the Melline integral will be useful:

$$\mathbf{H}_0(x) = \frac{\pi}{2} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \left(\frac{x}{2} \right)^{-s} \frac{1}{\cos(\pi s/2) \Gamma^2(1-s/2)}, \quad |c| < 1.$$

Then the integration in Eq. (5.4.20) over α yields

$$I_m^{(1)} = \frac{1}{2\sqrt{\pi}} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} (\bar{\xi})^{-s} \frac{\tan^2(\pi s/2) \Gamma(s/2) \Gamma(|m_c| + s/2)}{\Gamma[(s+1)/2] \Gamma(|m_c| + 1 - s/2)}. \quad (5.4.21)$$

In the region $|\bar{\xi}| \leq (|m_c| + 1)$ we calculate this integral looping the integration contour in the left half-plane of s . This results in the series in powers of $\bar{\xi}$

$$I_m^{(1)}(\bar{\xi}) = \sum_{l=0}^{\infty} \frac{(-1)^{l+|m_c|} (\bar{\xi})^{2l+1}}{\sqrt{\pi} [(l+1/2)^2 - m_c^2]} \times \frac{1}{\Gamma(l+3/2) \Gamma(1/2+l+|m_c|) \Gamma(1/2+l-|m_c|)}. \quad (5.4.22)$$

In the inverse region $|\bar{\xi}| \geq (|m_c| + 1)$ the integration contour in Eq. (5.4.21) can be looped in the right half-plane of s , which presents the integral $I_m^{(1)}$ as the series in powers of $1/(\bar{\xi})$. Its leading term is

$$I_m^{(1)}(\bar{\xi}) = \frac{4}{\pi^{3/2} \bar{\xi}} \begin{cases} \ln(8|\bar{\xi}|) + C, & m_c = 0, \\ \ln(2|\bar{\xi}|/|m_c|) + 1/|m_c| - \psi(|m_c| + 1/2), & m_c \neq 0. \end{cases} \quad (5.4.23)$$

Here, $\psi(x)$ is the logarithmic derivative of the Γ -function, and $C \simeq 0.577$ is the Euler's constant.

Substituting these expressions in Eq. (5.4.19) and then in Eq. (5.4.16) we can calculate the decrements of coherent oscillations in the wide region of parameters. For instance, for short bunch $|\bar{\xi}| \ll (|m_c| + 1)$, we find from Eq. (5.4.22)

$$\delta_m = \frac{Nr_0c}{4\gamma} \left\langle m_z^2 \frac{\partial |U_m|^2}{\partial a_z^2} \right\rangle \left[\frac{l}{\Pi} J_{|m_c|}^2(\bar{\xi}) + \frac{\varphi_0}{\pi^3} \frac{1 + \frac{d\ln v_z}{d\ln \omega_0}}{m_c^2 - 1/4} \right]. \quad (5.4.24)$$

Using the multipole expansion of $U(\mathbf{r}_1)$ and assuming the Gaussian distribution in amplitudes of betatron oscillations

$$F_0(a_z) = \frac{1}{a_0^2} \exp\left(-\frac{a_z^2}{a_0^2}\right),$$

one can write

$$\left\langle \frac{\partial}{\partial a_z^2} |U_m|^2 \right\rangle = \frac{(a_0/2)^{2|m_z|-2}}{4(|m_z|!)} \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \Big|_{\mathbf{r}_1=0} \right)^2. \quad (5.4.25)$$

Since for relativistic energies $\gamma\alpha \gg 1$, one also has

$$\frac{d\ln \omega_0}{d\ln p} = -\alpha, \quad \frac{d\ln v_z}{d\ln \omega_0} \simeq -\frac{1}{\alpha} \frac{d\ln v_z}{d\ln p} = -\frac{d\ln v_z}{d\ln R_0}, \quad (5.4.26)$$

Eq. (5.4.24) can be rewritten in the form

$$\delta_m = \frac{Nr_0c}{8\gamma} \left[\frac{l}{\Pi} J_{m_c}^2(\bar{\xi}) + \frac{\varphi_0}{\pi^3} \frac{1 + \frac{d\ln v_z}{d\ln \omega_0}}{m_c^2 - 1/4} \right] \times m_z^2 \frac{(a_0/2)^{2|m_z|-2}}{4(|m_z|!)} \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \Big|_{\mathbf{r}_1=0} \right)^2. \quad (5.4.27)$$

In agreement with the general expectations discussed in § 5.1 and § 5.2, Eq. (5.4.27) presents δ_m as the sum of two terms. The first, which is proportional to the length of the plate (l), describes the damping of coherent oscillations of the beam due to the excitation of the TEM-wave by the transverse oscillations. This term describes the previously mentioned effect of fast damping and it always is positive. The second term, which does not depend on the plate length, describes the mutual influence of the beam particles (head-tail effect). The sign of this term is determined by the synchrotron mode number m_c as well as by the sign of the ring chromaticity $d v_z / d \ln R_0$. Physically it is caused by the excitation of the TEM-waves in edges regions by the beam longitudinal motion. Most clearly this can be demonstrated by the formula for oscillation decrements, which takes into account the modulation of the ring β -functions.^{31,32} It can be obtained from Eq. (5.4.27) if one replaces $a_0^2 \rightarrow \varepsilon_0$ (ε is the beam emittance) and the expression in square brackets by

$$\frac{l}{\Pi} J_{|m_c|}^2(\bar{\xi}) \Phi_1 + \frac{(\varphi_0/\pi^3)}{4m_c^2-1} \left(\Phi_2 - \frac{d\ln v_z}{d\ln R_0} \Phi_3 \right),$$

$$\Phi_1 = \int_{-\vartheta_0}^{\vartheta_0} \frac{d\vartheta}{2\vartheta_0} |\beta_z(\vartheta)|^{|m_z|-1}, \quad (5.4.28)$$

$$\Phi_2 = \begin{cases} \frac{3|m_z|-2}{2|m_z|-2} \beta_z^{|m_z|-1}(\vartheta_0) + \frac{|m_z|-2}{2|m_z|-2} \beta_z^{|m_z|-1}(-\vartheta_0), & |m_z| \neq 1, \\ 2 + \ln[\beta_z(\vartheta_0)/\beta_z(-\vartheta_0)], & |m_z| = 1, \end{cases}$$

$$\Phi_3 = \frac{1}{R_0} \{ \beta_z^{|m_z|}(\vartheta_0) + \beta_z^{|m_z|}(-\vartheta_0) \}.$$

In these expressions the term proportional to Φ_1 describes the interaction of particles with TEM-wave along the plate. This term results in the damping of coherent oscillations. Remaining terms describe the head-tail instability due to interaction of particles with TEM-wave in the regions near the edges of the plate. It is remarkable that the edges terms in decrements are proportional to the $|m_z| - th$ power of the β -function, which can be important for the choice of the plates locations in rings with strong modulation of β -functions.

Equations (5.4.27) and (5.4.28) indicate that the contribution of the fast damping in decrements of synchrotron oscillations of short bunches is proportional to the power of small value

$$\bar{\xi} = \frac{1}{2} \left(m_z l_b \frac{dv_z}{dR_0} \right).$$

Therefore, the stability of synchrotron oscillations is mainly determined by the sign of the ring chromaticity $dv_z/d\ln R_0$. For instance, for a weak-focusing ring with linear lattice one has:

$$\frac{dv_z}{d\ln R_0} = \frac{n(1+n)}{2v_z}, \quad v_z = \sqrt{n}, \quad 0 < n < 1.$$

(Here n is the field index.) Thus, the stability criterion gets the form

$$\frac{4}{\pi^2} \frac{l}{l_b} \frac{1}{v_z} \frac{d\omega_z}{d\omega_0} = \frac{2l_b}{\pi^2 l} (1-n) < 1, \quad m_c = 0$$

$$\frac{d\omega_z}{d\omega_0} = \frac{1-n}{2} > 0, \quad m_c \neq 0, \quad (5.4.29)$$

which for the short bunch is always valid. Stability criteria for a bunch in a strong focusing ring can be obtained directly from Eqs (5.4.27) and (5.4.28). As

seen from Eq. (5.4.28), one can guarantee the stability of betatron modes provided that the plate is long enough and is placed at positions with low β -functions. This enables one to use head-tail redistribution for simultaneous damping of both betatron and synchrobetatron collective modes.

For dipole oscillations ($|m_z|=1$) a combination of Eq. (5.4.27) and Eq. (5.4.28) results in

$$\delta_m = \frac{Nr_0 c}{4\gamma} \left(\frac{\partial U}{\partial z} \Big|_{r_1=0} \right)^2 \left\{ \frac{l}{\Pi} J_{m_c}^2 \left(\frac{l_b}{2} \frac{dv_z}{dR_0} \right) + \frac{2l_b/R_0}{\pi^3(4m_c^2-1)} \left[2 + \ln \frac{\beta_z(\vartheta_0)}{\beta_z(-\vartheta_0)} - \frac{dv_z}{dR_0} [\beta_z(\vartheta_0) + \beta_z(-\vartheta_0)] \right] \right\}. \quad (5.4.30)$$

In the ring with high chromaticity betatron phase advance on the bunch length can be large:

$$\frac{l_b}{R_0} \left| m_z \frac{dv_z}{d \ln p} \right| \gg \alpha (|m_c| + 1).$$

In this case, as can be seen from Eqs (5.4.19) through (5.4.23), the decrements will mainly be determined by interaction of particles via the plate, while the contribution of the fast damping will be small. For the most important case, $|m_z|=1$, the substitution of Eq. (5.4.23) in Eq. (5.4.16) yields:

$$\delta_m \simeq \frac{Nr_0 c}{\pi^{5/2} \gamma v_z} \frac{Z_0 c}{l_1} \frac{L(m_c)}{l_b \frac{dv_z}{d \ln R_0}},$$

$$L(m_c) = \begin{cases} \ln(8|\bar{\xi}|) + C, & m_c = 0, \\ \ln\left(\frac{2|\bar{\xi}|}{|m_c|}\right) + \frac{1}{|m_c|} - \psi\left(|m_c| + \frac{1}{2}\right), & m_c \neq 0, \end{cases} \quad (5.4.31)$$

l_1 is some specific transverse size of the vacuum chamber at the plate location. Note that in this case decrements of synchrobetatron modes change the sign when multipole numbers are sufficiently high:

$$|m_c| \sim \sqrt{|\bar{\xi}|} > 1.$$

Generally, Eq. (5.4.31) indicates that decrements δ_m decrease with an increase of the ring chromaticity, which in some sense can be considered a stabilizing factor.

For long bunches $l_b \gg l(|m_c| + 1)$ decrements can be estimated directly from Eq. (5.4.18). Since in the case of interest q is small, one may replace the logarithmic accuracy I_m by the integral

$$I_m \simeq -\frac{q}{m_c} \int_0^{2q|m_z|} dz \left\{ v \left(1 + \frac{z}{2q|m_c|} \right) \cos \left(\frac{q\bar{\xi}}{|m_c|} \right) - \frac{\sin \frac{z\bar{\xi}}{|m_c|}}{q} \right\} \ln z. \quad (5.4.32)$$

The result of the integration depends on the betatron phase advance on the plate length $q\bar{\xi}$. In the region $|q\bar{\xi}| \ll 1$ the trigonometric functions in the integrand slowly vary inside the integration region, so one may replace

$$\cos \left(\frac{q\bar{\xi}}{|m_c|} \right) \simeq 1, \quad \sin \left(\frac{q\bar{\xi}}{|m_c|} \right) \simeq \frac{q\bar{\xi}}{|m_c|}.$$

The integration of the remaining expression yields

$$I_m \simeq -2q^2(2\nu + \xi) \ln \left(\frac{1}{2q|m_c|} \right) + q^2\nu. \quad (5.4.33)$$

Hence, the decrements for instances of dipole oscillations $|m_z| = 1$ read

$$\delta_m = \frac{Nr_0 c l^2}{4\pi\gamma \Pi l_b} \frac{Z_0 c}{l_1} \left(1 + 2 \frac{d \ln \nu_z}{d \ln R_0} \ln \frac{l_b}{2l(|m_c| + 1)} \right). \quad (5.4.34)$$

One can see that the negative sign of the chromaticity of the ring ($d \ln \nu_z / d \ln R_0 < 0$) can cause the instability. Note also that δ_m in Eq. (5.4.34) is inversely proportional to the bunch length l_b . Here are the reasons for that dependence. With $l_b \gg |m_c|l$ the main contribution into the integral in Eq. (5.4.16) gives harmonics from interval $\Delta n \sim R_0/l_b$; the contribution of each harmonic is of the order of $\sim (l/2R_0)^2$, thus $\delta \propto l^2/l_b R_0$.

In the region

$$q|\bar{\xi}| \simeq l \left| m_z \frac{d\nu_z}{dR_0} \right| \gg 1,$$

I_m can be presented by the series in powers of $1/\bar{\xi}$. The leading term of this expansion is

$$I_m = \frac{\ln \left(\frac{1}{2q|m_c|} \right)}{\bar{\xi}} [(1 + \nu q) \sin(2q\bar{\xi}) - 2\sin^2(q\bar{\xi})]. \quad (5.4.35)$$

The substitution of this expression in Eq. (5.4.16) yields

$$\delta_m = \frac{\delta_0 \ln(2q|m_c|)}{\bar{\xi}} [2\sin^2(q\bar{\xi}) - (1 + q\nu) \sin(2q\bar{\xi})]. \quad (5.4.36)$$

In particular, for dipole oscillations we can write

$$\delta_m \simeq \frac{Nr_0 c}{4\pi^2 \gamma \left(l_b \frac{dv_z}{dR_0} \right)} \frac{Z_0 c}{v_z l_{\perp}^2} \ln \frac{l_b}{2l(|m_c| + 1)} \times \left[2\sin^2 \left(l \frac{dv_z}{dR_0} \right) + \left(1 + \frac{v_z l}{\Pi} \right) \sin \left(2l \frac{dv_z}{dR_0} \right) \right]. \quad (5.4.37)$$

We have to note that the logarithmic divergence of the calculated expressions for the decrements is specific for the distribution (5.1.4). For smooth distributions $\rho(\varphi_c)$ obtained formulas determine the corresponding sums of decrements. These logarithmic factors indicate that at high mode numbers λ the decrements slowly decay with λ (presumably like $\delta_{m,\lambda} \propto 1/\lambda$).

5.4.3 Let us now consider some limiting cases when Eq. (5.4.2) can be transformed into an integral equation with real and symmetrical kernels. The general properties of eigenvalues for such equations are well known.⁴² Thus we can study the stability conditions for a reasonably wide class of smooth distributions $\rho(\varphi_c)$ without a direct solution of Eq. (5.4.2).

For the sake of simplicity below we shall assume the smoothed lattice approximation as well as that $\rho(\varphi_c)$ has only one parameter—the bunch length φ_0 :

$$\rho(\varphi_c) = q(\varphi_c/\varphi_0)/\varphi_0^2, \quad \int_0^\infty dy y q(y) = 1, \quad \varphi_c = \varphi_0 y. \quad (5.4.38)$$

By definition, $q(y) > 0$. With these notations we can rewrite Eq. (5.4.2) in the form

$$\Lambda \chi(y) = \int_0^\infty dy' y' q(y') \mathcal{K}_1(y, y') \chi(y'), \quad (5.4.39)$$

where $\omega = m\omega - \Lambda\delta_1/\varphi_0$, and

$$\delta_1 = \frac{Nr_0 c}{2\pi^2 \gamma} \frac{1}{\Gamma^2(|m_z|)\Gamma^2(|m_x|)} \left(\frac{\partial^{|m_x|+|m_z|} U}{\partial x^{|m_x|} \partial z^{|m_z|}} \right)^2 \times \left\langle \left(\frac{a_x}{2} \right)^{2|m_x|-2} \left(\frac{a_z}{2} \right)^{2|m_z|-2} \left[\frac{\text{sgn}(m_x) a_z^2}{v_x m_z^2} + \frac{\text{sgn}(m_z) a_x^2}{v_z m_x^2} \right] \right\rangle,$$

$$\mathcal{K}_1(y, y') = \int_{-\infty}^\infty dx \Omega \left(\frac{x}{\varphi_0} - \mathbf{m}_1 \frac{d\omega_1}{d\omega_0} \right) J_{m_c}(xy) J_{m_c}(xy'), \quad (5.4.40)$$

while

$$\Omega(x) = \frac{m v \vartheta_0}{2x + m v} \frac{x \sin 2\vartheta_0 (2x + m v)}{(2x + m v)^2} + 2i(x + m v) \frac{\sin^2 \vartheta_0 (2x + m v)}{(2x + m v)^2}. \quad (5.4.41)$$

The kernel $\mathcal{K}_1$ obviously is symmetrical:

$$\mathcal{K}_1(y, y') = \mathcal{K}_1(y', y).$$

Let us consider first one-dimensional, say, vertical coherent oscillations of short bunch $l_b \ll l$. Neglecting in Eq. (5.4.40) the values of the order of

$$\frac{l_b}{l} \left| \frac{dv_z}{d \ln R_0} \right| \ll 1, \quad l_b v_z \ll l,$$

we can rewrite $\mathcal{K}_1$ in the form

$$\begin{aligned} \mathcal{K}_1(y, y') &= i \frac{\varphi_0 m_z v_z l}{R_0} K(y, y'), \\ K(y, y') &= \int_0^\infty \frac{dx}{x^2} \sin^2 x J_0\left(\frac{xy l_b}{2l}\right) J_0\left(\frac{xy' l_b}{2l}\right). \end{aligned} \quad (5.4.42)$$

This transforms Eq. (5.4.39) into

$$\Lambda_1 \chi_m = \int dy' y' q(y') K(y, y') \chi_m(y'), \quad (5.4.43)$$

$$\Lambda_1 = -i \frac{\Lambda R_0}{m_z v_z \varphi_0}.$$

By the substitution

$$\Phi_m(y) = \sqrt{y q(y)} \chi_m(y)$$

one can transform Eq. (5.4.43) into an integral equation with symmetrical kernel:

$$\Lambda_1 \Phi_m = \int_0^\infty dy' \sqrt{y y' q(y) q(y')} K(y, y') \Phi_m(y'). \quad (5.4.44)$$

It is important that the kernel of Eq. (5.4.44) is a positively defined function. This means that the integration of Eq. (5.4.44) with any function satisfying

$$\int_0^\infty \Phi_m^2(y) dy = 1 \quad dy < \infty$$

yields

$$\int_0^\infty \frac{dx}{x^2} \sin^2 x \left[\int_0^\infty dy [y q(y)]^{1/2} J_0\left(\frac{xy l_b}{2l}\right) \Phi_m(y) \right]^2 > 0. \quad (5.4.45)$$

Since for any eigenvalue Λ_α we can write

$$\Lambda_\alpha = \int_0^\infty dy dy' \Phi_{m,\alpha} \sqrt{yy' q(y) q(y')} K(y, y') \Phi_{m,\alpha} > 0,$$

where $\Phi_{m,\alpha}(y)$ is the eigensolution of Eq. (5.4.44) corresponding to the eigenvalue Λ_α and

$$\int_0^\infty \Phi_{m,\alpha}^2 = 1,$$

all eigenvalues of integral equations with symmetrical and positively defined kernels are real and positive numbers. (See, for instance, Ref. 42.) Applying this to Eq. (5.4.43) we can find that decrements of all its modes have the form

$$\delta_{m,\alpha} = \Lambda_{m,\alpha} \frac{Nr_0 c |m_z|}{\pi \gamma \Gamma^2(|m_z|)} \frac{l}{\Pi} \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \right\rangle \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \Big|_{r_1=0} \right)^2, \quad (5.4.46)$$

and are positive values. The upper limit for any Λ_α is obviously determined by their sum:

$$\Lambda_\alpha < \sum_\alpha \Lambda_\alpha.$$

For Eq. (5.4.43) this yields

$$\Lambda_1 < \int_0^\infty dy y q(y) K(y, y')$$

or

$$\Lambda_1 < \int_0^\infty dy y q(y) \int_{-\infty}^\infty \frac{dx}{x^2} \sin^2 x J_0^2 \left(xy \frac{l_b}{2l} \right) < \int_0^\infty \frac{dx}{x^2} \sin^2 x = \frac{\pi}{2}.$$

This gives an upper limit on the decrements of modes

$$\delta_{m,\alpha} = \frac{Nr_0 c |m_z|}{2\gamma \Gamma^2(|m_z|)} \frac{l}{\Pi} \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \right\rangle \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \Big|_{r_1=0} \right)^2, \quad (5.4.47)$$

which coincides with the decrement of the bunch with zero length. When the value $|d\omega_z/d\omega_0| l_b/l$ approaches 1, the numerical solution of Eq. (5.4.39) must be employed.

In case of two-dimensional betatron modes coherent oscillations may become unstable. The decrements of these modes read

$$\delta_m = \Lambda' \delta_1 2\vartheta_0 (m_z v_z + m_x v_x), \quad \Lambda' > 0.$$

This gives the stability condition in the form

$$(m_z v_z + m_x v_x) \left\langle \left(\frac{a_x}{2} \right)^{2|m_x|-2} \left(\frac{a_z}{2} \right)^{2|m_z|-2} \left[\frac{\text{sgn}(m_z)}{v_z} \left(\frac{a_x}{m_x} \right)^2 + \frac{\text{sgn}(m_x)}{m_x} \left(\frac{a_z}{m_z} \right)^2 \right] \right\rangle > 0. \quad (5.4.48)$$

Similar transformations for synchrobetatron vertical oscillations give different results depending on the ratio between the bunch and the plate lengths. Let us consider two limiting cases: a short bunch and a long bunch.

Case 1: A Short Bunch. For a short bunch $l_b \ll |m_c| l$ the main contributions of Taylor expansion of the kernel $\mathcal{K}_1$ over $l_b |dv - z/dR_0| \ll 1$ yields

$$\mathcal{K}_1 \simeq \left[4\vartheta_0 \left(\frac{dv_z}{d \ln R_0} - \frac{v_z}{2} \right) + i \frac{d\omega_z}{d\omega_0} \right] m_z \varphi_0^2 \int_0^\infty \frac{dx'}{x^2} J_{m_c}(xy) J_{m_c}(xy').$$

Therefore, both decrements (or increments) of these modes

$$\delta_m = \Lambda \frac{N r_0 c |m_z|}{4\pi^2 \Gamma^2(|m_z|)} \varphi_0 \frac{d\omega_z}{d\omega_0} \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \right\rangle \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \right)^2, \quad (5.4.49)$$

and coherent frequency shifts are proportional to the eigenvalues of the integral equation

$$\Lambda \chi = \int_0^\infty \frac{dx}{x^2} J_{m_c}(xy) \int_0^\infty dy' y' q(y') J_{m_c}(xy') \chi(y'). \quad (5.4.50)$$

Due to the positive definition of the kernel of this equation, all Λ are positive numbers. Hence, the stability condition

$$\frac{d\omega_z}{d\omega_0} > 0 \quad (5.4.51)$$

coincides with that obtained above for δ -function distribution (5.1.4).

Although solutions of Eq. (5.4.50) cannot be found in a general way, the maximum eigenvalue ($\Lambda_{\max}$) can be calculated using the fact that a functional

$$\Lambda[\chi] = \int_0^\infty \frac{dx}{x^2} \left| \int_0^\infty dy \sqrt{yq(y)} J_{|m_c|}(xy) \chi(y) \right|^2 \quad (5.4.52)$$

reaches its maximum value, when a probe function (χ) coincides with one of the eigenfunctions (χ_α) of Eq. (5.4.50). In the last case, if the probe function is normalized by the condition

$$\int_0^\infty dy \chi^2(y) = 1, \quad (5.4.53)$$

the value of the functional ($\Lambda[\chi]$) coincides with the eigenvalue Λ_α . This fact enables an evaluation of the $\Lambda_{\max}$ using the inequality

$$\Lambda[\chi] < \Lambda_{\max} < \Lambda_\Sigma. \quad (5.4.54)$$

Here,

$$\Lambda_\Sigma = \int_0^\infty \frac{dx}{x^2} \int_0^\infty dy y q(y) J_{m_c}^2(xy) \quad (5.4.55)$$

is the sum of the eigenvalues of Eq. (5.4.50). The accuracy of this estimation depends on the deviation of the chosen probe function from the exact eigenfunction. Taking for instance

$$q(y) = 2e^{-y^2}$$

and ($m_c > 0$)

$$\chi(y) = C_m y^{m_c+1/2} e^{-\beta y^2}, \quad C_m^2 = \frac{2}{m_c!} (2\beta)^{m_c+1}, \quad (5.4.56)$$

we find, from Eq. (5.4.52)

$$\begin{aligned} \Lambda(\beta) &= 2C_m^2 \int_0^\infty \frac{dx}{x^2} \left[\int_0^\infty dy y^{m_c+1} \exp\left(-\frac{y^2(1+2\beta)}{2}\right) J_{m_c}(xy) \right]^2 \\ &= 2C_m^2 \int_0^\infty dx \frac{x^{2m_c-2}}{(1+2\beta)^{2m_c+2}} \exp\left(-\frac{x^2}{1+2\beta}\right), \end{aligned}$$

or

$$\Lambda(\beta) = \frac{2\Gamma(m_c - 1/2)(2\beta)^{m_c+1}}{m_c!(1+2\beta)^{m_c+5/2}}. \quad (5.4.57)$$

The value of β is determined by the equation $d\Lambda/d\beta = 0$, which results in $\beta = (m_c + 1)/3$ and $[\Lambda(\beta) = \Lambda_m]$:

$$\Lambda_m = \frac{12\sqrt{3}\Gamma(m_c - 1/2)}{m_c!} 2^{m_c} \frac{(m_c + 1)^{m_c+1}}{(2m_c + 5)^{m_c+5/2}}. \quad (5.4.58)$$

Hence, we can rewrite Eq. (5.4.56) in the form

$$\Lambda_m \leq \Lambda_{\max} \leq \frac{1}{2} \frac{4}{\sqrt{\pi} (4m_c^2 - 1)}. \quad (5.4.59)$$

For the first synchrotron sidebands ($|m_c| = 1$) this yields

$$0.3247 \leq \Lambda_{\max} \leq 0.3761.$$

For high multipole numbers of synchrotron oscillations ($|m_c| \gg 1$) the eigenvalues Λ_m decrease with m_c like

$$\Lambda_m \approx \frac{3}{m_c^3} \sqrt{\frac{3}{2e}}, \quad m_c \gg 1. \quad (5.4.60)$$

Such an asymptote of Λ_m is in general agreement with that predicted using the extrapolation from the microwave region. (See in § 3.5.)

Case 2: A Long Bunch. For long bunches $l_b \gg |m_c|l$ the main contribution into interaction of the bunch with the plate gives the harmonics interval $|n| < R_0/l$, when the kernel $\mathcal{K}_1(y, y')$ can be presented in the form

$$\mathcal{K}_1(y, y') \approx \frac{i}{2} \left(\frac{l}{R_0} \right)^2 \frac{dv}{d \ln R_0} \int_0^{x_m} dx J_{|m_c|}(xy) J_{|m_c|}(xy'), \quad x_m \approx l_b/l \gg |m_c|. \quad (5.4.61)$$

Again the decrements

$$\delta_m = \Lambda \frac{Nr_0 c}{\pi \Gamma^2(|m_z|)} \frac{l}{\Pi} \frac{|m_z|l}{l_b} \frac{dv_z}{d \ln R_0} \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \right\rangle \left(\frac{\partial^{|m_z|} U}{\partial z^{|m_z|}} \right)^2 \quad (5.4.62)$$

are expressed in term of eigenvalues of the integral equation

$$\Lambda \chi = \int_0^{x_m} dx J_{|m_c|}(xy) \int_0^\infty dy' y' q(y') J_{|m_c|}(xy') \chi(y'),$$

while their signs are determined only by the sign of the ring chromaticity $dv_z/d \ln R_0$. The maximum increments for this case will be estimated later

As we said in § 3.5 one more estimation of eigenfrequencies in Eq. (5.4.39) can be obtained using the extrapolation of its spectrum from the high-frequency region $|n|\varphi_0 \gg (|m_c| + 1)$ where one can write

$$\Lambda(x) = \begin{cases} \frac{\bar{\Phi}(x)}{x}, & l_b \ll l, \\ \frac{\Phi(x)}{x}, & l_b \gg l, \end{cases} \quad (5.4.63)$$

$$\Phi(x) = \frac{\Omega(x) + \Omega(-x)}{2}$$

The physical sense of the decrease in the correlation of harmonics in the region $|n|\varphi_0 \gg |m_c| + 1$ means that modes that are mainly contributed by the interaction of the bunch with the high-frequency part of the wake are close to synchrotron phase harmonics of “plane waves”:

$$f_m(\mathbf{I}_1, \mathbf{I}_c) \propto \{\exp(in\varphi)\}_{m_c},$$

with the distance between the neighbor mode of the order of $\Delta n \sim 1/\varphi_0$.

The estimations can be simplified in two limiting cases.

Case 1: $l_b \ll l$. The function $\bar{\Phi}(x)$ reads

$$\bar{\Phi}(x) = \begin{cases} \frac{i\varphi_0^2}{2x^2} m_z \frac{d\omega_z}{d\omega_0}, & x \gg |m_z| v_z \varphi_0, \varphi_0 |m_z d\omega_z/d\omega_0|, \\ -\frac{i}{2} \left(m_z \frac{d\omega_z}{d\omega_0} \right)^{-1}, & x \ll \varphi_0 |m_z d\omega_z/d\omega_0|. \end{cases} \quad (5.4.64)$$

Substituting $\bar{\Phi}$ in Eq.(5.4.51) we obtain the decrements of high-frequency modes:

$$\delta = \begin{cases} \frac{\delta_1 \varphi_0}{2x^3} m_z \frac{d\omega_z}{d\omega_0}, & x \gg |m_c| \gg \varphi_0 |m_z d\omega_z/d\omega_0|, \\ -\frac{\delta_1}{2x} \left(m_z \frac{d\omega_z}{d\omega_0} \right)^{-1}, & \varphi_0 |m_z d\omega_z/d\omega_0| \gg x \gg |m_c|. \end{cases} \quad (5.4.65)$$

By integrating δ over x one can easily find the corresponding sums of the decrements ($m_c \neq 0$):

$$\sum_{x \simeq |m_c|}^{\infty} \delta \simeq \begin{cases} \delta_1 \frac{\varphi_0 m_z}{4m_c^2} \frac{d\omega_z}{d\omega_0}, & |m_c| \gg \varphi_0 |m_z d\omega_z/d\omega_0|, \\ -\frac{\delta_1 \ln \left(\frac{\varphi_0}{|m_c|} \left| m_z \frac{d\omega_z}{d\omega_0} \right| \right)}{2\varphi_0 m_z \frac{d\omega_z}{d\omega_0}}, & \varphi_0 |m_z d\omega_z/d\omega_0| \gg |m_c|. \end{cases} \quad (5.4.66)$$

Good coincidence of these formulas with Eqs (5.4.27) and (5.4.31) respectively is not so surprising. The main contribution in synchrobetatron modes of the short bunch yields the interaction with the edge fields, which are sufficiently nonuniform in ϑ . For this reason, one may expect that the spectrum of high-frequency modes fits well the low-frequency one. Using this fact one may conclude that Eqs. (5.4.65) extrapolated to the low-frequency region $x \simeq |m_c|$, can give (at least by the order of magnitude) the maximum increment of synchrobetatron modes of the short bunch

$$\delta_{\max} \simeq \begin{cases} \frac{\delta_1 \varphi_0}{2|m_c|^3} m_z \frac{d\omega_z}{d\omega_0}, & |m_c| \gg \varphi_0 |m_z d\omega_z/d\omega_0|, \\ -\frac{\delta_1}{2|m_c|} \left(m_z \frac{d\omega_z}{d\omega_0} \right)^{-1}, & \varphi_0 |m_z d\omega_z/d\omega_0| \gg |m_c|. \end{cases} \quad (5.4.67)$$

Such an extrapolation is not valid for betatron modes $m_c=0$ since in the last case the main contribution in low-frequency modes yields harmonics $|n| < R_0/l < 1/\varphi_0$.

Case 2: $l_b \gg |m_c|l$. In this asymptote the leading contribution in the maximum increment makes the region $|m_c| \ll x \ll l_b/l$. Using Eq. (5.4.41) we can write here $\Phi(x)$ in the form

$$\Phi(x) = i \left(\frac{l}{R_0} \right)^2 m_z \frac{dv_z}{d \ln R_0}, \quad l_b/l \gg x \gg |m_c|. \quad (5.4.68)$$

The substitution of this expression in Eq. (5.4.63) estimates the maximum decrement by the following expression:

$$\delta_{\max} \simeq \delta_1 \frac{l}{l_b} \frac{l}{R_0} m_z \frac{dv_z}{d \ln R_0} \frac{1}{|m_c| + 1}. \quad (5.4.69)$$

In § 3.5 we mentioned the differential equation, that gives the spectra of collective oscillations for the bunch with the Lorentz distribution over the amplitudes of synchrotron oscillations. If $\Phi(x)$ is defined by Eq. (5.4.68), this equation can be solved directly ($C_m = w_m/\sqrt{x}$):

$$w_m''(x) + \left\{ \frac{\Phi_m}{\Lambda_m x} - 1 - \frac{m_c^2 - 1/4}{x^2} \right\} w_m(x) = 0,$$

$$w_{m,\alpha}(x) = \left(\frac{x}{2} \right)^{|m_c| + 1/2} e^{-x/4} F(-\alpha, 2|m_c| + 1, x/2),$$

$$\Lambda_{m,\alpha} = \frac{1}{2} \frac{\Phi_m}{|m_c| + \frac{1}{2} + \alpha}. \quad (5.4.70)$$

Here $\alpha=0,1,2,\dots$ is the number of the solution; $F(\alpha, \gamma, x)$ is the confluent hypergeometric function. This yields for the maximum decrement ($\alpha=0$) the expression

$$\delta_{\max} \simeq \delta_1 \frac{l}{l_b} \frac{l}{2R_0} m_z \frac{dv_z}{d \ln R_0} \frac{1}{|m_c| + 1}, \quad (5.4.71)$$

which only by factor 2 is smaller then the extrapolated value given by Eq. (5.4.69).

5.5 ON THE DAMPING OF SYNCHROTRON COHERENT OSCILLATIONS BY MATCHED PLATES

Let us now briefly discuss the interaction of synchrotron oscillations with the segment of the vacuum chamber containing the double-connected waveguide. Using definitions introduced in § 5.4 we can write the basic integral equation for this case in the form

$$\Lambda\chi = \int_{-\infty}^{\infty} dx \Omega(x/\varphi_0) \left[U_0^2 + \frac{R_0 \varphi_0 \mu_{\parallel}}{px} \frac{m_c \omega_c}{\bar{\eta}} \frac{\partial U^2}{\partial r} \right] \times J_{m_c}(xy) \int_0^{\infty} dy' \frac{dq}{dy'} J_{m_c}(xy') \chi(y'), \quad (5.5.1)$$

with

$$\omega - m_c \omega_c = \Lambda \frac{2Ne^2 m_c}{\mu_{\parallel} \omega_c \Pi^2 \varphi_0^3}.$$

We assume that the bunch moves along the axis of the waveguide. Since the frequency of synchrotron oscillations is typically small ($\omega_c \ll \omega_s$), we can present the function $\Omega(x)$ in the form

$$\Omega(x) = \frac{\varphi_0}{4} \frac{\sin(4xl/l_b)}{x} + \frac{i\varphi_0}{2} \frac{\sin^2(2xl/l_b)}{x}. \quad (5.5.2)$$

To avoid the cumbersome calculations let us take as $q(y)$ the step-function:

$$q(y) = \begin{cases} 2, & 0 \leq y \leq 1, \\ 0, & y > 1. \end{cases} \quad (5.5.3)$$

This immediately yields the dispersion equation

$$\Lambda = -2 \int_{-\infty}^{\infty} dx \Omega(x/\varphi_0) \left[U_0^2 + \frac{R_0 \varphi_0 \mu_{\parallel}}{px} \frac{m_c \omega_c}{\bar{\eta}} \frac{\partial U^2}{\partial r} \right] J_{m_c}^2(x). \quad (5.5.4)$$

Since $\text{Im}\Omega(x)$ in Eq. (5.5.2) is an odd function of x , the first term in Eq. (5.5.4), which is particularly caused by the total loss of the energy, contributed only in the frequency shift. The second term, which is proportional to the radial gradient of those losses, yields the decrement of oscillations:

$$\delta_m = \frac{2Nr_0 c m_c^2}{\pi^2 \gamma l_b} \bar{\eta} \frac{\partial U^2}{\partial r} \int_0^{\infty} \frac{dx}{x^2} J_{m_c}^2(x) \sin^2(2xl/l_b). \quad (5.5.5)$$

This expression can be simplified in two limiting cases.

Case 1: $l_b \ll |m_c|l$ (Long Plate). In this region the square of the sine in the integrand is the fast oscillating function, which therefore can be replaced by its average value. This results in

$$\delta_m \approx \frac{Nr_0 c}{\pi^3 \gamma l_b} \bar{\eta} \frac{\partial U^2}{\partial r}. \quad (5.5.6)$$

Coherent oscillations will decay provided

$$\bar{\eta} \frac{\partial U^2}{\partial r} > 0. \quad (5.5.7)$$

This condition means that with $\bar{\eta} > 0$, to damp the synchrotron coherent oscillations the plates should be placed on the outer side of the orbit. Physically this makes obvious sense: for damping the modulation of the coherent energy losses should increase the losses when the energy of the oscillations increases. Note another important feature of this particular damper: for all modes with $|m_c| < l_b/l$, the values of decrements do not depend on the mode number m_c . (For higher m_c the excitation of higher modes of the waveguide should be taken into account.) This is because of the fact that the radiation length for this device coincides with the length of the edge field:

$$l_{rad} \sim l_1 \ll l_b / |m_c|.$$

Case 2: $l_b \gg |m_c|l$ (Short Plate). In this case the main contribution in the integral in Eq. (5.5.5) comes from the region $|m_c| < x < l_b/l$. Calculating the integral with the logarithmic accuracy we get

$$\delta_m = \frac{8Nr_0 c}{\pi^3 \gamma l_b} \left(\frac{m_c l}{l_b} \right)^2 \bar{\eta} \frac{\partial U^2}{\partial r} \ln \left(\frac{l_b}{|m_c|l} \right). \quad (5.5.8)$$

The stability condition for this case coincides with that given by Eq. (5.5.7).

Equation (5.5.5) describes the contributions to decrements of synchrotron modes due to the coupling of radial and longitudinal oscillations. Generally (§3.4), such a coupling results in a redistribution of decrements between the coupled oscillations, so that an increase in decrements of synchrotron modes is accompanied by a decrease in decrements of radial ones. Therefore, to make a conclusion on the possibility to use matched plates for damping of synchrotron coherent oscillations of bunches, we must calculate a relevant correction to decrements of radial modes. This can be done using Eq. (3.1.10), if we take into account the contribution made to $\Delta \vartheta(\mathbf{I}, \psi, t)$ by the “betatron phase”:

$$\vartheta_b = [\eta(\vartheta) x'_b - x_b(\vartheta) \eta'(\vartheta)] / R_0^2. \quad (5.5.9)$$

Assuming the smoothed focusing approximation and assuming that along the plate $\eta' = 0$, we can rewrite ϑ_b like

$$\vartheta_b = \eta v_x \frac{dx_b}{d\psi_x}. \quad (5.5.10)$$

Assuming also the distribution over synchrotron amplitudes given by Eq. (5.1.4), we can write decrements of radial synchrobetatron oscillations in the form

$$\delta_m = \delta_m^{(0)} - \delta_m^{(1)}, \quad (5.5.11)$$

where $\delta_m^{(0)}$ is given by Eq. (5.4.16) with the substitution of subscripts x instead of z , while $\delta_m^{(1)}$ yields the correction due to radial-longitudinal coupling. For dipole radial oscillations ($|m_x| = 1$) this reads

$$\delta_m^{(1)} = \frac{Nr_0 c}{2\pi^2 \gamma} \eta \frac{\partial U^2}{\partial r} \int_{-\infty}^{\infty} dn n(n + m_x v_x) J_{m_c}^2(n_1 \varphi_0) \left(\frac{2 \sin \vartheta_0 (2n + m_x v_x)}{(2n + m_x v_x)} \right)^2. \quad (5.5.12)$$

The integral in Eq. (5.5.12) diverges logarithmically. In the model used for this calculation, this is caused by too sharp variation of $U(r_\perp, \vartheta)$ in the edge region. [See Eq. (5.3.20).] In real cases U decays on the distance of the order of $l_\perp$, which cuts the integrand at n of the order of $n_{\max} \sim R_0/l_\perp$. Since the divergency is only a logarithmic one, for the estimation we can simply replace the infinite limits of integration by finite ones with $|n| < n_{\max}$. Let us again give the decrements in two limiting cases.

Case 1: $l_b \ll |m_c| l$. Here

$$\delta_m^{(1)} = \frac{Nr_0 c}{2\pi^3 \gamma l_b} \eta \frac{\partial U^2}{\partial r} \ln \left(\frac{l_b}{l_\perp (|m_c| + 1)} \right). \quad (5.5.13)$$

Note that this correction does not depend on $d\omega_x/d\omega_0$, and with

$$\eta \frac{\partial U^2}{\partial r} > 0 \quad (5.5.14)$$

decreases the decrement of radial oscillations.

Provided that $\delta_m^{(0)} > 0$ the comparison of Eqs. (5.5.13) and (5.4.27) shows that radial betatron oscillations will be unstable when

$$\eta \frac{l_\perp R_0}{ll_b} > 1,$$

and radial synchrobetatron, when

$$\eta \frac{l_\perp R_0}{l_b^2} > 1.$$

In both cases decrements vary inversely proportionally to the bunch length l_b .

Case 2: $l_b \gg |m_c|l$. In this region the addition to the decrement due to radial-longitudinal coupling is estimated by the formula:

$$\delta_m^{(1)} = \frac{Nr_0c}{2\pi^3\gamma} \frac{\eta}{l_b} \ln\left(\frac{l}{l_\perp}\right) \frac{\partial U^2}{\partial r}. \quad (5.5.15)$$

Generally, the decrease in decrements of radial oscillations is not a fatal disadvantage of the described method to damp synchrotron coherent oscillations. Additional damping of radial oscillations can be provided by the special system of plates, which does not contribute in decrements of synchrotron oscillations (i.e., by the system with $\langle \partial \mathcal{F}_\parallel / \partial R \rangle = 0$). Thus, the use of appropriate systems of matched plates enables one to damp both betatron and synchrotron coherent oscillations of a bunched beam.

5.6 FAST SINGLE-TURN INSTABILITIES IN STORAGE RINGS

At the beginning of this chapter we mentioned that the use of synchrotron multipole number m_c as the number specifying a particular mode of collective oscillations makes sense only provided the oscillations are slow enough and are developed during many periods of synchrotron oscillations. In this section we shall consider the opposite limiting case, when coherent:

$$|\Omega_m| \gg \omega_c. \quad (5.6.1)$$

In this region the interaction strongly couples harmonics with different m_c and therefore the separation of modes into betatron and synchrobetatron becomes impossible. Physically this corresponds to rise times of coherent oscillations that are shorter than it takes for particles to vary noticeably their phase position φ . Originally, such instabilities were found in relativistic electron linacs,⁸² where the longitudinal mobility is suppressed by the relativistic saturation of particle velocities, and was called as the beam breakup effect. These instabilities can be described using the standard technique and assuming Δp and $\varphi = \vartheta - \omega_s t$ are the integrals of longitudinal motion. These integrals are similar to that used to describe collective oscillations in the coasting beams. On the other hand, the azimuthal inhomogeneity of the bunch linear density should be taken into account. Both points correspond to the description of the oscillations within a so-called model with rigid longitudinal distribution, which was frequently used to calculate the spectra of bunched beams. (See, for instance Ref. 27, 80, and 81.) For cases when the beam interacts with resonant elements, this approach can give quite reliable estimations. However, it must be applied with some care for calculations related to the single-turn instabilities, when wakes decay during one revolution period.

Let us now show that in the last case the existence of the spectrum of eigenfrequencies is not so obvious and generally should be studied for particular problems. In other words, we expect that in some cases the linearized, homogeneous Vlasov equation has no solutions except the trivial. Indeed, to have the eigenmodes the interaction of the beam with surroundings must be arranged in such a way, when the perturbation of any part of the beam is affected by the response of other particles of that beam. In other words, the interaction must loop the feedback coupling of particles through the beam and thus provide the complete self-consistency of the interaction. Without such a feedback some particles in the beam are driven by wakes and some are not, which results in absence of the beam eigenmodes. Single-turn instabilities by definition are caused by the interaction with the retarding and rapidly decaying induced fields. That results in such an asymmetry of the interaction, when wake-fields of the tail-on part of the bunch do not perturb head-on particles. Since in the region described by Eq. (5.6.1) head-on and tail-on particles cannot change their azimuthal positions during "coherent" times, such an asymmetric interaction does not loop the feedback through the beam and hence does not provide conditions for the development of the beam eigenmodes.

From the other side, the particles in the beam have very close frequencies of oscillations. Hence, the driving by the wakes of the head-on particles can excite the oscillations of tail-on particles very effectively. That is why the study of collective stability in these conditions remains of practical interest. Since the possible instabilities are associated with the resonant excitation, one may expect nonexponential dependence of the amplitudes on time. This feature makes the difference between the fast single-turn instabilities and the synchrotron ones.

The straightforward theory of these fast instabilities was developed in Ref. 83 and 84. Before making particular calculations, let us note that their application indeed can be very wide. Except for the description of coherent oscillations in synchrotrons we may mention the problems on the single-bunch dynamics in linear colliders^{85,86} as well as coherent oscillations in compensated electron beams in electron coolers.⁸⁷

5.6.1. The main features of fast instabilities (or beam breakup effect) in storage rings discussed in the previous section can be demonstrated using the problem of the interaction of a single bunch with the matched plates. Assuming for the sake of simplicity the excitation of vertical oscillations $\omega \simeq m_z \omega_z$, we can rewrite Eq. (3.1) in the form

$$f_{m,\omega} = \frac{if_m^{(0)}}{\omega - \omega_m} + \bar{L}_{m,\omega} \frac{m_z \partial f_0 / \partial I_z}{\omega - \omega_m}, \quad \omega_m = m_z \omega_z \quad (5.6.2)$$

In this equation we omitted the term describing the longitudinal mobility of particles:

$$\omega'_0 \Delta p \frac{\partial f}{\partial \varphi} = \omega_c \varphi_c \frac{\partial f}{\partial \varphi}.$$

The following calculations will indicate that with the true condition (5.6.1) and due to lattice focusing of particles this term is not significant. Since we expect no solution of the homogeneous part of Eq. (5.6.2), we kept the initial conditions $f_m^{(0)} = f_m(I_z, \Delta p, \varphi, t=0)$.

We shall need also the particular expression for the wake $\bar{L}_{m,\omega}$. For the interaction of the bunch with the matched plate it reads:

$$\begin{aligned} \bar{L}_{m,\omega} = & Ne^2 \int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(kv - \omega_{mn})^2 U_m(I_z)}{k^2 c^2 - \omega_{mn}^2} \\ & \times |b_{kn}|^2 \int d\Gamma' U_m^*(I'_z) e^{in(\varphi - \varphi')} f_{m,\omega}, \\ & \omega_{mn} = \omega_m + n\omega_0. \end{aligned} \quad (5.6.3)$$

This equation was written assuming the single-turn interaction of particles. From the other side, this poses the bunch length φ_b from the range

$$\varphi_b < 2(\pi - \vartheta_0), \quad \vartheta_0 = l/R_0.$$

Otherwise the head-on particles can interact with wakes left on the plate by tail-on particles. For the calculations below it is more relevant to perform the integration over k and n directly in an Eq. (5.6.3). For ultrarelativistic particles ($\gamma \gg 1$) the main contribution in an integral over k gives the residue at the simple pole $kc = -\omega_{mn}$, which corresponds to backward radiation. Then we can rewrite Eq. (5.6.3) in the form

$$\begin{aligned} \bar{L}_{m,\omega} = & \frac{iNe^2 U_m(I_z)}{2\Pi} \int d\Gamma'_1 d\Delta p' U^*(I'_z) \\ & \times \int_{\varphi}^{\varphi + 2\vartheta_0} d\varphi' f_{m,\omega} e^{im_z \nu_z (\varphi - \varphi')/2} \\ & \times \left[m_z \nu_z \left(\vartheta_0 - \frac{\varphi' - \varphi}{2} \right) - i \right]. \end{aligned} \quad (5.6.4)$$

Introducing now the momenta

$$\chi_{m,\omega}(\varphi) = \int d\Gamma_1 d\Delta p U_m^* \exp(im_z \nu_z \varphi/2) f_{m,\omega} \quad (5.6.5)$$

we can replace Eq. (5.6.2) by

$$\begin{aligned} \chi_{m,\omega}(\varphi) = & C_{m,\omega} + i\Lambda\rho(\varphi) \int_{\varphi}^{\varphi+2\vartheta_0} d\varphi' \chi_{m,\omega}(\varphi') e^{im_z\nu_z(\varphi-\varphi')/2} \\ & \times \left\{ m_z\nu_z \left(\vartheta_0 - \frac{\varphi' - \varphi}{2} \right) - i \right\}. \end{aligned} \quad (5.6.6)$$

Here,

$$C_{m,\omega} = i \exp(im_z\nu_z\varphi/2) \int d\Gamma_{\perp} d\Delta p \frac{U_m^* f_m^{(0)}}{\omega - \omega_m}, \quad (5.6.7)$$

$$\Lambda = \frac{Ne^2}{2\Pi} \int \frac{d\Gamma_{\perp} d\Delta p |U_m|^2}{\omega - \omega_m} \left(m_z \frac{\partial F_0}{\partial I_z} \right), \quad (5.6.8)$$

while the equilibrium distribution f_0 was assumed of the form

$$f_0 = F_0(I_{\perp}, \Delta p) \rho(\varphi).$$

5.6.2. Equation (5.6.6) describes the driving of the betatron oscillations of the particle at the point φ by the particles from the interval

$$\varphi \leq \varphi' \leq \varphi + 2\vartheta_0. \quad (5.6.9)$$

Since the limits of integration in this equation depend on variable φ , it presents the Volterra-type integral equation.⁴² It is well known⁴² that homogeneous integral equations of this type have neither eigensolutions nor eigenvalues. For Eq. (5.6.6) direct calculations* yield the dispersion equation in the form

*Let us consider first the Volterra-type equation of the general form

$$f(\varphi) = \Lambda \int_0^{\varphi} d\varphi' K(\varphi, \varphi') f(\varphi').$$

The eigenvalues of an integral equation are determined by zeros of its Fredholm determinant

$$D(\Lambda) = 0.$$

To find $D(\Lambda)$ for the case of interest, let us divide the interval of the integration $(0, \varphi)$ into samples of length h_j and replace the integral equation by the system of algebraic equations

$$f(\varphi_i) = \Lambda \sum_{j=1}^i h_j K(\varphi_i, \varphi_j) f(\varphi_j).$$

Its determinant obviously is

$$D^{(h)}(i) = \prod_{j=1}^i [1 - \Lambda h_j K(\varphi_i, \varphi_j)].$$

The Fredholm determinant is then obtained taking $\{h_j\} \rightarrow 0$:

$$D(\Lambda) = \lim_{\{h\} \rightarrow 0, i \rightarrow \infty} D^{(h)}(i).$$

This yields

$$\ln D(\Lambda) = -\Lambda \int_0^{\infty} d\varphi K(\varphi, \varphi),$$

or finally

$$\exp \left[-\Lambda \int_0^{\infty} d\varphi K(\varphi, \varphi) \right] = 0.$$

Substituting here the kernel of Eq. (5.6.6) yields the dispersion equation (5.6.10).

$$\exp \left[-\frac{Ne^2}{2\Pi} (1 + im_z v_z \vartheta_0) \int \frac{d\Gamma_\perp d\Delta p |U_m|^2}{\omega - \omega_m} \left(m_z \frac{\partial F_0}{\partial I_z} \right) \right] = 0. \quad (5.6.10)$$

This equation obviously has no roots at any finite ω .

Simple analytic solutions of Eq. (5.6.6) can be obtained only for special distributions $\rho(\varphi)$. Let us consider in more detail the case when

$$\rho(\varphi) = \begin{cases} \varphi_b^{-1}, & 0 \leq \varphi \leq \varphi_b, \\ 0, & \varphi < 0, \quad \varphi > \varphi_b, \end{cases} \quad (5.6.11)$$

while the bunch length $\varphi \ll 2\vartheta_0$. With these assumptions by the substitution $\chi_{m,\omega} = w(\varphi)\rho(\varphi)$ we can transform Eq. (5.6.6) into

$$w(\varphi) = C'_{m,\omega}(\varphi) + \frac{i\Lambda}{\varphi_b} \int_{\varphi}^{\varphi_b} d\varphi' w(\varphi') \left\{ m_z v_z \left(\vartheta_0 - \frac{\varphi' - \varphi}{2} \right) - i \right\} \quad (5.6.12)$$

or, with accuracy up to $(\varphi_b/2\vartheta_0) \ll 1$,

$$w(\varphi) = C'_{m,\omega}(\varphi) + i\bar{\Lambda} \int_{\varphi}^{\varphi_b} d\varphi' w(\varphi'), \quad (5.6.13)$$

where we introduced

$$\bar{\Lambda} = \frac{\Lambda}{\varphi_b} (m_z v_z \vartheta_0 - i), \quad C'_{m,\omega}(\varphi)\rho(\varphi) = C_{m,\omega}(\varphi).$$

The last equation can be transformed into an ordinary differential equation of the first order:

$$\frac{d}{d\varphi} (w - C'_{m,\omega}) = -i\bar{\Lambda} w(\varphi),$$

which obviously has the solution

$$w(\varphi) = C'_{m,\omega} + i\bar{\Lambda} \int_{\varphi}^{\varphi_b} d\varphi' C'_{m,\omega}(\varphi') \exp[i\bar{\Lambda}(\varphi' - \varphi)]. \quad (5.6.14)$$

As it was expected, the kernel of this solution (the resolvent of the equation)

$$R(\varphi, \varphi', \Lambda) = \exp[i\bar{\Lambda}(\varphi' - \varphi)]$$

is an entire function of Λ . Let us also note that the deviation of $w(\varphi)$ from initial perturbation $C'_{m,\omega}$ generally grows to the tail of the bunch.

5.6.3. Since Eq. (5.6.6) has no eigensolutions, the calculation of the dependence of w on time becomes of primary importance. It is determined by the integral

$$w(\varphi, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} w(\varphi) e^{-i\omega t}, \quad \text{Im}\omega > 0. \quad (5.6.15)$$

Let us calculate first $w(\varphi, t)$ neglecting the frequency spread in the beam. In this case we have

$$i\Lambda = \frac{\Omega_m}{\omega - \omega_m},$$

with

$$\Omega_m = -\frac{Ne^2}{2\Pi\varphi_b} \left\langle \frac{\partial |U_m|^2}{\partial I_z} \right\rangle m_z (1 + im_z v_z \vartheta_0) \quad (5.6.16)$$

being the complex coherent frequency shift of the short bunch. Shifting ω by ω_m and after substituting Eq. (5.6.14) in Eq. (5.6.15), we get

$$w(\varphi, t) e^{i\omega_m t} = C'_m(\varphi) + \int_{\varphi}^{\varphi_b} d\varphi' C'_m(\varphi') R_m(\varphi, \varphi', t),$$

$$R_m(\varphi, \varphi', t) = \Omega_m \int_{-\infty}^{\infty} \frac{d\omega}{2\pi\omega^2} \exp \left[-i\omega t + \frac{\Omega_m}{\omega} (\varphi' - \varphi) \right], \quad \text{Im}\omega > 0. \quad (5.6.17)$$

The integral R_m is determined by the specific singularity at $\omega=0$. It can be calculated, for instance, by the deformation of the integration contour in such a way that it starts from $\omega=-\infty$, goes around the point $\omega=0$ in a negative direction, and then returns to $\omega=-\infty$. (See Fig. 5.2.) This results in*

$$R_m(\varphi, \varphi', t) = \left(\frac{-i\Omega_m t}{\varphi' - \varphi} \right)^{1/2} J_1 \{ [i\Omega_m t (\varphi' - \varphi)]^{1/2} \}. \quad (5.6.18)$$

This equation clearly indicates that the dependence of w on time in general is nonexponential. With small time intervals $|\Omega_m|t \ll 1$ it linearly increases in time:

$$w(\varphi, t) = e^{-i\omega_m t} \left\{ C'_m(\varphi) - \frac{\Omega_m t}{2} \int_{\varphi}^{\varphi_b} d\varphi' C'_m(\varphi') \right\}. \quad (5.6.19)$$

The asymptote of the solution in the region $|\Omega_m|t \gg 1$ is determined by the asymptotic behavior of Bessel function $J_1(x)$. To simplify calculations we consider the special initial distribution when

*The same result can be obtained expanding the exponent

$$\exp \left(\frac{\Omega_m}{\omega} [\varphi' - \varphi] \right)$$

in the Taylor series and then calculating the residues of multiple poles of the integrand. The fact that multiple poles appear in calculations again describes the absence of eigenmodes in the problem as well as the resonant excitation of oscillations.

$$C'_m(\varphi) = C_m \delta(\varphi - \varphi_1), \quad 0 \leq \varphi_1 \leq \varphi_b. \quad (5.6.20)$$

In this case the solution coincides with the resolvent:

$$w(\varphi, t) = \begin{cases} 0, & \varphi > \varphi_1, \\ -C_m R_m(\varphi, \varphi_1, t) e^{-i\omega_m t}, & \varphi \leq \varphi_1. \end{cases} \quad (5.6.21)$$

The features of the time dependence of these solutions is determined by the ratio between the real and imaginary parts of $\Omega_m = \Omega'_m - i\delta_m$. Initially let us assume that $|\Omega'_m| \ll |\delta_m|$ (or $|m_z v_z \vartheta_0| \gg 1$). For the matched plates δ_m is positive, when Eq. (5.6.18) yields

$$w(\varphi, t) \simeq -iC_m e^{-i\omega_m t} (\varphi_1 - \varphi)^{-3/4} (\delta_m t)^{1/4}. \quad (5.6.22)$$

Note that this expression describes slow increases of oscillations in time. For other wakes δ_m can be negative. The relevant solution will grow rapidly

$$w \propto (|\delta_m| t)^{1/4} \exp\{-i\omega_m t + \sqrt{|\delta_m| t} (\varphi_1 - \varphi)\} \quad (5.6.23)$$

with "increments" depending on the distance from the excitation point:

$$\tau_m^{-1} \simeq |\delta_m| (\varphi_1 - \varphi). \quad (5.6.24)$$

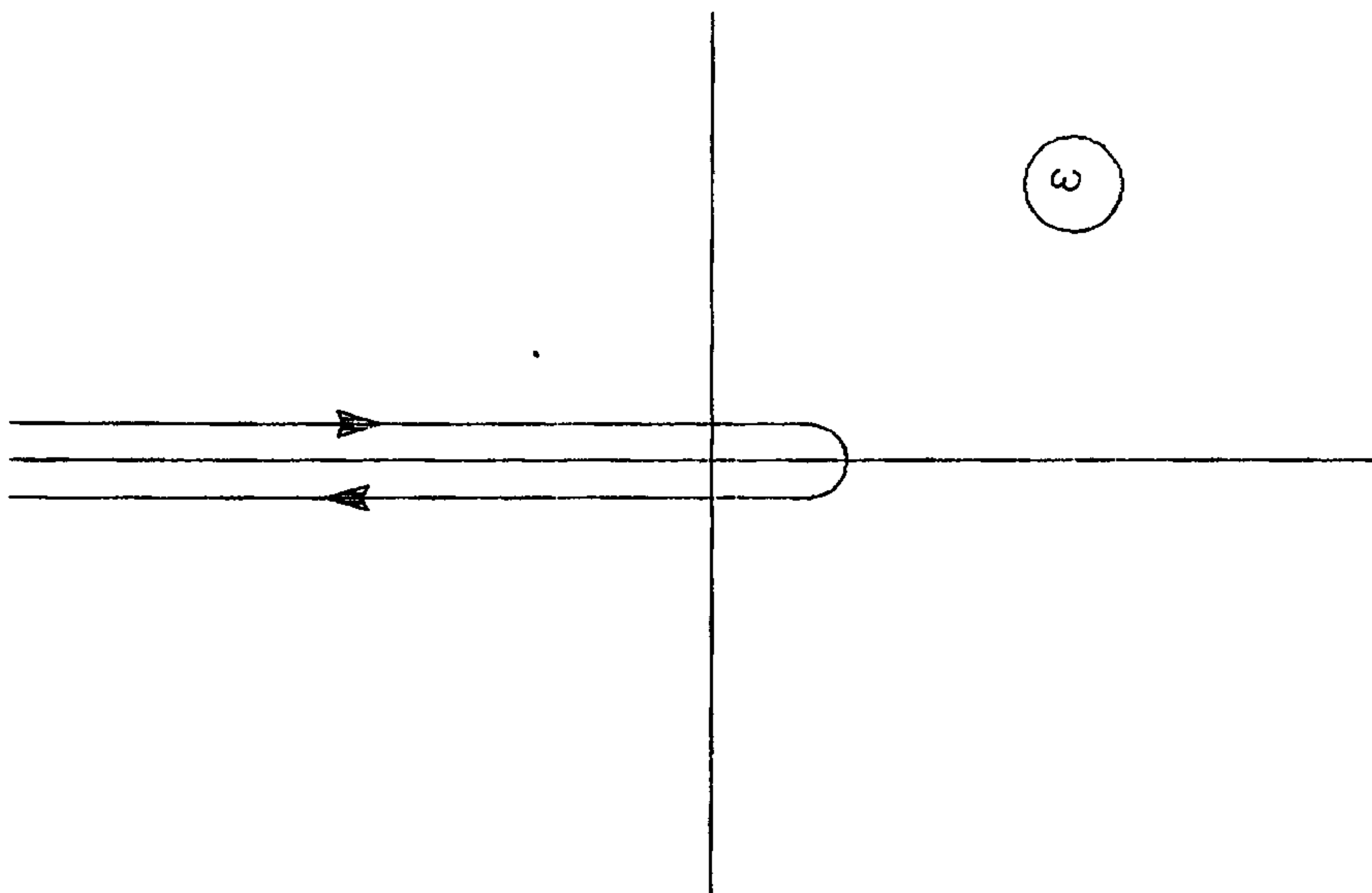


FIG. 5.2. The integration contour in Eq. (5.6.17).

Note also that in contrast with slow instabilities the index of the exponent in Eq. (5.6.23) grows only like $\sqrt{t}$.

The difference in asymptotes in Eq. (5.6.19) and Eqs (5.6.21) through (5.6.23) is caused by the fact that the last equations take into account the collective reaction of particles placed between φ_1 and φ . In the time interval $\sim |\delta_m|^{-1}$ the field perturbing the motion of a particle at point φ becomes the combination of wakes from all particles of that interval. As is indicated by Eq. (5.6.17) the phases of driven oscillations of intermediate particles are shifted relative to that at point φ by

$$\Delta\Phi(\omega) = \frac{\delta_m}{\omega}(\varphi_1 - \varphi). \quad (5.6.25)$$

Without those shifts one could expect exponential growth of oscillations at point φ . The interference of oscillations with phase shifts in Eq. (5.6.25) causes the decay of decrements with time. The formal reason for this decay is the following. In the asymptotic region $|\delta_m|t \gg 1$ the main contribution into the integral in Eq. (5.6.17) gives the vicinity of the stationary phase:

$$\Phi(\omega) = -i\omega t + \frac{\Omega_m}{\omega}(\varphi_1 - \varphi) \Big|_{\omega=\omega_0}, \quad (5.6.26)$$

$$\frac{d\Phi}{d\omega} \Big|_{\omega=\omega_0} = 0. \quad (5.6.27)$$

Here the second equation, giving the stationary point ω_0 , has the sense of a dispersion equation. The calculation of ω_0 for Eqs. (5.6.26) and (5.6.27) is easy:

$$\omega_0 = \pm \sqrt{i\Omega_m \frac{\varphi_1 - \varphi}{t}}.$$

This formula clearly indicates the decay of decrements $\text{Im}\omega_0$ with time like $t^{-1/2}$.

Let us note one more feature of an obtained solution. As can be seen from Eq. (5.6.17) it is presented by a superposition of waves, traveling along the beam with the dispersion law

$$\Phi(\omega) = -i\omega t + ik(\omega)(\varphi_1 - \varphi), \quad k(\omega) = -i\Omega_m/\omega.$$

Hence, the stationarity condition (5.6.27) can also be presented in the form

$$v_{gr} = \left(\frac{dk}{d\omega} \Big|_{\omega=\omega_0} \right)^{-1} = \frac{\varphi_1 - \varphi}{t}. \quad (5.6.28)$$

According to this equation the leading contribution to the oscillation at point φ gives only those of excited waves, which can pass the distance $\varphi_1 - \varphi$ during time t .

To conclude the discussion of the stability of monochromatic fast oscillations let us consider the case when $|\Omega'_m| \gg |\delta_m|$. Here, independently of signs of Ω'_m and δ_m , the argument of the Bessel function in Eq. (5.6.21) is the complex number. The asymptote of ω is then given by Eq. (5.6.23), where $|\delta_m|$ must be replaced by $|\Omega'_m|$. To this end, decrements of oscillations are determined by the real part of the coherent frequency shift

$$\tau_m^{-1} = |\Omega'_m| (\varphi_1 - \varphi). \quad (5.6.29)$$

All the mentioned properties are very specific for fast single-turn instabilities and are caused by the resonant excitation of oscillations along the beam. To cure that, one has to worry about the compensation of the real part of the coherent frequency shift in addition to the compensation of its imaginary part. Note also that the discussed asymptotes depend relatively weakly on the shape of initial deflection $C'_m(\varphi)$.

The choice of the linear density of the beam in the form of a step-function simplifies the calculations but is not significant for the discussed results. The substitution in Eq. (5.6.11) of any smooth function $\rho(\varphi)$ will replace $R_m(\varphi, \varphi')$ in Eq. (5.6.17) by

$$R(\varphi, \varphi', \omega) = \Lambda(\omega) \rho(\varphi) \exp \left\{ \Lambda(\omega) \int_{\varphi}^{\varphi'} d\varphi'' \rho(\varphi'') \right\}. \quad (5.6.30)$$

The time dependence of solutions will be determined respectively by the resolvent,

$$R_m(\varphi, \varphi', t) = \rho(\varphi) \sqrt{\frac{-i\Omega_m t}{\mathcal{N}(\varphi', \varphi)}} J_1 \left[\sqrt{i\Omega_m t \mathcal{N}(\varphi', \varphi)} \right],$$

$$\mathcal{N}(\varphi', \varphi) = \int_{\varphi}^{\varphi'} \rho(\varphi''),$$

which obviously has the same asymptotes.

5.6.4. The nonexponential dependence of the amplitudes of excited oscillations on time generally simplifies the damping of fast instabilities in storage rings. Two factors can stabilize linear oscillations: beam cooling and the frequency spread.

Let us consider first the damping due to beam cooling. For the sake of simplicity we assume that the cooling decrement λ is constant as in the case of cooling of electrons (positrons) due to synchrotron radiation. Then, the function $i\bar{\Lambda}$ in Eq. (5.6.14) reads

$$i\bar{\Lambda} = \frac{\Omega_m}{\omega - \omega_m + i|m_z|\lambda/2}. \quad (5.6.31)$$

Next we take $\lambda \gg |m_z|\Delta\omega$. ($\Delta\omega$ is the frequency spread in the beam). Otherwise the cooling is not important for the damping. Substituting Eq. (5.6.31) in Eq. (5.6.17) one can easily see that the only variation consists in the shift of the singularity in the integral over ω from the point $\omega=0$ into the point $\omega = -i|m_z|\lambda/2$. This results in the following modification of the time dependencies of the solutions:

$$w(\varphi, t)e^{i\bar{\omega}t} = C'_m(\varphi) + \int_{\varphi}^{\varphi_b} d\varphi' C'_m(\varphi') R_m(\varphi, \varphi', t),$$

$$\bar{\omega} = \omega_m - i|m_z|\lambda/2, \quad (5.6.32)$$

with the same resolvent R_m as in Eq. (5.6.18). Since the integral in Eq. (5.6.32) grows no faster than $\exp(\sqrt{t/\tau_m})$, starting with

$$t_0 = \frac{4}{m_z^2 \lambda^2 \tau_m} \quad (5.6.33)$$

oscillations will decay. This certainly implies $t_0\omega_c \ll 1$ or

$$|m_z|\lambda \gg 2\sqrt{\omega_c/\tau_m} = \frac{2}{\tau_m}\sqrt{\omega_c\tau_m}.$$

One can see that provided condition (5.6.1) is valid, the stabilization of oscillations generally does not demand a predominance of cooling decrements over $|\Omega_m|$. Note also that the cooling damps modes with high m_z especially effective. This is certainly true only for modes satisfying $\lambda \gg |m_z|\Delta\omega_z$.

Let us now discuss the stabilization of the fast oscillations due to the frequency spread of the beam or Landau damping. In the case of beam breakup instability, the necessary spread can be associated with nonlinearity of the ring lattice [$\nu_z = \nu_z(a_z)$] or its chromaticity [$\omega_z = \omega_z(\Delta p)$]. Generally, the stabilizing action of these factors depends on the properties of the stationary distribution function f_0 . To simplify the calculations we shall start with the simplest case, when $\Delta\omega = (d\omega_z/d\Delta p)\sigma_{||}$, while $f_0(\Delta p)$ is the Lorentz distribution. Now we can write

$$i\bar{\Lambda} = \frac{\Omega_m}{\omega - \omega_m + i|m_z|\Delta\omega},$$

which with obvious replacements coincides with Eq. (5.6.31). Hereby, we may state that the decay of amplitudes will be provided by the factor $\exp(-|m_z|\Delta\omega_z t)$ in front of the right hand side of Eq. (5.6.32). The "stability criterion" in Eq. (5.6.33) is then transformed into

$$t > t_0 = \frac{1}{(m_z \Delta \omega_z)^2 \tau_m}. \quad (5.6.34)$$

Similar results appear when f_0 is a one-peak, smooth distribution function. As we mentioned before, the asymptote of ω with $t \rightarrow \infty$ is determined by the vicinity of ω , where the phase

$$\Phi(\omega) = -i\omega t + \Omega_m(\varphi_1 - \varphi) \int \frac{d\varepsilon f(\varepsilon)}{\omega - \varepsilon}, \quad \text{Im}\omega > 0$$

is stationary. Here $f(\varepsilon)$ is the frequency distribution in the beam. The stationarity condition (5.6.27) yields the "dispersion equation" of the problem

$$1 = -\frac{i\Omega_m}{t}(\varphi_1 - \varphi) \frac{d}{d\omega} \int \frac{d\varepsilon f(\varepsilon)}{\omega - \varepsilon}. \quad (5.6.35)$$

For smooth distribution $f(\varepsilon)$ with frequency spread $\Delta\omega$ in the region

$$\frac{|\Omega_m|}{t}(\varphi_1 - \varphi) \gg (m_z \Delta\omega)^2$$

the roots of Eq. (5.6.35) read

$$\omega = \pm \sqrt{\Omega_m t (\varphi_1 - \varphi)}.$$

This indicates that initially, when criterion (5.6.34) is violated, oscillations will blow-up. However, after $t > t_0$ the oscillations enter in the region

$$\frac{|\Omega_m|}{t}(\varphi_1 - \varphi) \ll (m_z \Delta\omega)^2,$$

where the roots tend to $\omega = -i|m_z|\Delta\omega$ (see, for instance, §3.6) and oscillations will decay.

One more method of suppression of the beam breakup instability in linear colliders was suggested in Ref. 86. It is based on the detuning of the particles along the beam. Since in a linear accelerator the deviation of the particle energy can be coupled with its phase position, Ref. 86 proposed to use chromatic dependence of betatron frequencies $\omega_z(\Delta p)$ to detune particles from the resonance (so-called BNS-damping). Another possibility connected with the parasitic tune shift generated by wake along the bunch $[\Delta\nu_z(\varphi)]$ was recently mentioned in Ref. 88.

Let us calculate $\omega(\varphi, t)$ for the simplest case, when $\omega_z(\varphi) = \omega_z + \kappa\varphi$. The beam momentum spread is assumed to be zero. Since $\bar{\Lambda}$ is now the function of φ , Eq. (5.6.14) must be rewritten in the form

$$w = C'_m(\varphi) + i \int_{\varphi}^{\varphi_b} d\varphi' C'_m(\varphi') \bar{\Lambda}(\varphi') \exp\left(i \int_{\varphi}^{\varphi'} d\varphi'' \bar{\Lambda}(\varphi'')\right). \quad (5.6.36)$$

For $w(\varphi, t)$ this yields

$$we^{i\omega_m(\varphi)t} = C'_m(\varphi) + \int_{C_1} \frac{d\omega}{2\pi\omega} \Omega_m \int_{\varphi}^{\varphi_b} \frac{d\varphi' C'_m(\varphi')}{\omega - m_z \kappa(\varphi' - \varphi)} \times \exp\left\{-i\omega t + \Omega_m \int_{\varphi}^{\varphi'} \frac{d\varphi''}{\omega - m_z \kappa(\varphi'' - \varphi)}\right\}, \quad (5.6.37)$$

where the integration contour C_1 goes from $-\infty$ to ∞ above the singularities of the integrand. Let us study first the behavior of the resolvent of this solution. It reads

$$w = C_m e^{-i\omega_m(\varphi)t} \frac{d}{du} I(m_z \kappa ut), \quad \varphi < \varphi_1, \quad (5.6.38)$$

with

$$I(\psi_\kappa) = \int_{C_1} \frac{dz}{2\pi z} \exp\left\{\alpha_m \int_0^1 \frac{d\xi}{z - \xi} - i\psi_\kappa z\right\}, \quad \psi_\kappa = m_z \kappa ut, \quad (5.6.39)$$

$$u = \varphi_1 - \varphi, \quad \alpha_m = \Omega_m / m_z \Delta\omega.$$

$\Delta\omega = \kappa\varphi_b$ is the deviation of betatron frequencies between the head and the tail of the bunch. In contrast with Eq. (5.6.17) the integrand in Eq. (5.6.39) is an analytic function in the whole plane z except the cut coming from $z=0$ to $z=1$. Therefore, it can be calculated deforming the contour of integration to the cut.* This results in

$$I = -ie^{-i\psi_\kappa} F(1 - \alpha_m, 1, i\psi_\kappa), \quad (5.6.40)$$

where $F(\alpha, \gamma, x)$ is the confluent hypergeometric function.⁷⁹ The substitution of Eq (5.6.40) in Eq. (5.6.38) yields

$$w = -C_m \Omega_m t e^{-i\omega_m t - i\psi_\kappa} F(1 - \alpha_m, 2, i\psi_\kappa). \quad (5.6.41)$$

The direct substitution here of the Taylor expansion

* Another calculation of I uses the formula

$$x^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\xi \xi^{\alpha-1} \exp(-\xi/x),$$

which transforms the integral over z in Eq. (5.6.39) into one of representations of Bessel function $J_0(2\sqrt{i\xi\psi_\kappa})$. Subsequent integration over ξ yields Eq. (5.6.40).

$$F(\alpha, \gamma, x) = 1 + \frac{\alpha x}{\gamma} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots$$

indicates that Eq. (5.6.41) matches Eq. (5.6.21) when $\kappa \rightarrow 0$.

To derive a stability condition we need the asymptote of F when ψ_κ is large:

$$F \propto \frac{(-i\psi_\kappa)^{\alpha_m-1}}{\Gamma(1+\alpha_m)} + e^{i\psi_\kappa} \frac{(i\psi_\kappa)^{-\alpha_m-1}}{\Gamma(1-\alpha_m)}.$$

Substituting this expression in Eq. (5.6.41) yields

$$w(\varphi, t) = At^{\alpha_m} + Bt^{-\alpha_m}, \quad (5.6.42)$$

where A and B are the functions, which do not increase with time. One can see from Eq. (5.6.42) that oscillations will not grow only in the case when $\text{Re}\Omega_m = 0$. Otherwise with arbitrary sign and the value of the ratio $\Omega_m/m_z\Delta\omega$ one term in Eq. (5.6.42) will increase. It is clear, however, that the decrease in $|\alpha_m|$ substantially increases the rise-time of this instability.

We also have to note that the stabilizing action of detuning of particles along the beam generally depends on the distribution of initial perturbation. With arbitrary $C'_m(\varphi)$ solution w reads

$$we^{i\omega_m(\varphi)t} = C'_m(\varphi) - \Delta w, \\ \Delta w = \Omega_m t \int_0^{\varphi_b - \varphi} du C'_m(\varphi + u) F(1 - \alpha_m, 2, i\psi_\kappa). \quad (5.6.43)$$

If $C'_m(\varphi)$ is concentrated within a relatively short interval $\Delta\varphi$

$$\frac{\Delta\varphi}{\varphi_b} \ll \frac{1}{|m_z| \Delta\omega t}, \quad (5.6.44)$$

Eq. (5.6.43) predicts the same behavior as Eq. (5.6.42). If, on the contrary, C'_m is constant along the beam (or rather smooth function), the integration in Eq. (5.6.43) yields

$$\Delta w = C_m \{ F[\alpha_m, 1, im_z \kappa t(\varphi_b - \varphi)] - 1 \}. \quad (5.6.45)$$

In the region $|m_z| \kappa(\varphi_b - \varphi)t \gg 1$ has the asymptote

$$\Delta w = A't^{-\alpha_m} + B't^{\alpha_m-1} - 1.$$

It shows that oscillations do not grow provided the BNS-criterion is valid:

$$0 < \frac{\text{Re}\Omega_m}{m_z \Delta\omega} < 1. \quad (5.6.46)$$

The low-scale perturbation generally can be unstable even if this criterion is valid.

5.6.5. For the beam with linear density (5.6.11), one can construct the universal solution describing the oscillations of both short and long bunches. Since the general solution of, for instance, Eq. (5.6.6) can be expressed through its Green function, it is sufficient to find a solution with δ -like initial distribution (5.6.20).

Once the oscillation at point $\varphi < \varphi_1$ is driven only by particles from the interval $\varphi < \varphi' < \varphi_1$, we can formally spread the value $\rho = 1/\varphi_b$ over the points outside this interval. This transforms Eq. (5.6.6) into

$$\chi_{m,\omega} = C_{m,\omega} \delta(\varphi - \varphi_1) + i \int_{\varphi}^{\infty} d\varphi' \chi_{m,\omega}(\varphi') \bar{\Lambda}_m(\varphi' - \varphi). \quad (5.6.47)$$

Here

$$\bar{\Lambda}_m = \Lambda_m \begin{cases} m_z v_z \left(\vartheta_0 - \frac{\varphi' - \varphi}{2} \right) - i, & \varphi' < \varphi + 2\vartheta_0, \\ 0, & \varphi' > \varphi + 2\vartheta_0. \end{cases} \quad (5.6.48)$$

Equation (5.6.47) can be solved using the Fourier transformation

$$\chi_{m,\omega}(\varphi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \chi_{m,\omega}(k) e^{ik\varphi}, \quad (5.6.49)$$

which yields

$$\chi_{m,\omega}(k) = \frac{C_{m,\omega} e^{-ik\varphi_1}}{\epsilon(k,\omega)}. \quad (5.6.50)$$

Here the introduced function

$$\begin{aligned} \epsilon(k,\omega) &= 1 - i\bar{\Lambda}(k) \\ &= 1 + \Omega_m \int \frac{d\Gamma_{\perp} d\Delta p |U_m|^2}{\omega - \omega_m} \left(m_z \frac{\partial F_0}{\partial I_z} \right) \left\langle m_z \frac{\partial |U_m|^2}{\partial I_z} \right\rangle^{-1} \end{aligned} \quad (5.6.51)$$

formally coincides with the dielectric constant of the coasting beam, which has intensity $N^* = 2\pi N/\varphi_b$ in the bunching factor higher than N .

The substitution of Eqs (5.6.49) and (5.6.50) in Eq. (5.6.15) gives the solution in the form of Eq. (5.6.17):

$$\chi_{m,\omega} = \int_{C_1} \frac{d\omega}{2\pi} \frac{C_{m,\omega} \exp[-i\omega t + ik(\omega)(\varphi - \varphi_1)]}{\partial \epsilon(k,\omega)/\partial k|_{k=k(\omega)}}, \quad (5.6.52)$$

where the dispersion law $k = k(\omega)$ is defined by the equation $\epsilon(k,\omega) = 0$.

5.6.6. Equations describing the fast single-turn (or the beam breakup) instability can be obtained directly from the synchrotron mode coupling equations of the type of Eq. (5.2.13) as asymptotic expansions in powers of ω_c/Ω_m . Let us calculate now the first corrections, describing the self-consistent parts of relevant solutions due to synchrotron oscillations of the bunch particles. Since we expect Volterra-type integral equations, we have to start with equations taking into account initial conditions. Then Eq. (5.2.13) reads

$$(\Delta\omega_m - m_c\omega_c)\chi_m = i\chi_m^{(0)}(\varphi_c) + \rho(\varphi_c) \int_{-\infty}^{\infty} dn \Omega_{m,n} J_{m_c}(n_1\varphi_c) \\ \times \sum_{m_c=-\infty}^{\infty} \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1\varphi'_c) \chi_m(\varphi'_c). \quad (5.6.53)$$

In the linear approximation Eqs (5.6.53) enables the most comprehensive study of the stability of synchrotron modes of a single bunch with an arbitrary ratio between coherent frequency shift Ω_m and the frequency of synchrotron oscillations. In the region $|\Omega_m| \ll \omega_c$ these equations split into the system of independent equations describing the uncoupled synchrotron modes. In the region $|\Omega_m| \sim \omega_c$ the mode coupling becomes significant and calculation of the beam stability criteria requires the solution of the infinite set of integral equations (5.6.53). Except for some simple cases (see, for instance, in § 5.2) this must be done numerically. Finally, in the asymptotic region $|\Omega_m| \gg \omega_c$ the mode coupling becomes so strong that, as we saw, Eqs. (5.6.53) can lose the spectrum of eigenvalues.

The asymptotes of Eqs. (5.6.53) in the region $\omega_c \rightarrow 0$, can be calculated directly. Using the substitution

$$\chi(n) = \sum_{m_c=-\infty}^{\infty} \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(n_1\varphi'_c) \chi_m(\varphi'_c), \quad (5.6.54)$$

we transform Eqs. (5.6.53) into the integral equation for azimuthal harmonics of the bunch linear density $\chi(n)$. If, as before in this section, we assume that the azimuthal length of the device interacting with the bunch is rather short, we may neglect in Eqs. (5.6.53) the dependence of $\Omega_{m,n}$ on the subscript n . The resulting equation for $\chi(n)$ can be written in the form

$$\chi(n) = \sum_{m_c=-\infty}^{\infty} \frac{i}{\Delta\omega_m - m_c\omega_c} \int_0^{\infty} d\varphi_c \varphi_c \chi_m^{(0)}(\varphi_c) J_{m_c}(n\varphi_c) \\ + \sum_{m_c=-\infty}^{\infty} \frac{1}{\Delta\omega_m - m_c\omega_c} \int_{-\infty}^{\infty} dn' \rho_m(n, n') w_m(n' - \xi) \chi(n'). \quad (5.6.55)$$

Here, $w_m(n) = \Omega_m(n)/\omega_c$ and

$$\rho_m(n, n') = \int_0^\infty d\varphi_c \varphi_c \rho(\varphi_c) J_{m_c}(n\varphi_c) J_{m_c}(n'\varphi_c). \quad (5.6.56)$$

First, let us calculate for the homogeneous part of Eq. (5.6.53). Using Eq. (5.1 2), we replace in Eq. (5.6.55) the summation over m_c by integration and separate in the resulting sum over l the term with $l=0$, describing interaction of particles during one period of their synchrotron oscillations. Then the resulting homogeneous integral equation reads

$$\chi(n) = \int_{-\infty}^\infty dn' [K_{sp}(n, n') + K_{mp}(n, n')] w_m(n' - \xi) \chi(n'), \quad (5.6.57)$$

where the kernel

$$K_{sp}(n, n') = \int_{-\infty}^\infty dm_c \frac{\rho_{m_c}(n, n')}{x_m - m_c}, \quad x_m = \frac{\Delta\omega_m}{\omega_c}, \quad \text{Im}\omega > 0 \quad (5.6.58)$$

describes single-period effects, while

$$K_{mp}(n, n') = \sum_{l \neq 0} \int_{-\infty}^\infty dm_c \frac{\rho_{m_c}(n, n') \exp(2\pi i l m_c)}{x_m - m_c}, \quad \text{Im}\omega > 0 \quad (5.6.59)$$

describes the multiperiod effects and, therefore, the feedback between the head-on and tail-on parts of the bunch. The kernel K_{mp} is determined by the residue of the integrand in the simple pole $m_c = \Delta\omega_m/\omega_c$. Since $\text{Im}\omega > 0$ and for all particles we have

$$|\psi - \psi'| \leq 2\pi$$

the integration contour for the terms with $l < 0$ can be looped in the lower half-plane of the complex variable m_c and thus gives no contribution to K_{mp} . The remaining part of the sum in Eq. (5.6.59) reads ($T_c = 2\pi/\omega_c$):

$$K_{mp}(n, n') = -2\pi i \rho_{x_m}(n, n') \sum_{l=1}^\infty \exp(2\pi i l x_m) = \frac{2\pi i}{1 - \Lambda} \rho_{x_m}(n, n'),$$

$$\Lambda = \exp(-2\pi i x_m) = \exp(-iT_c \Delta\omega_m), \quad \text{Im}\omega > 0. \quad (5.6.60)$$

We can rewrite Eq. (5.6.57) in the form

$$\begin{aligned}\chi(n) = & \frac{2\pi i}{1-\Lambda} \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \chi(n') \\ & + \int_{-\infty}^{\infty} dn' K_{sp}(n, n') w_m(n' - \xi) \chi(n').\end{aligned}\quad (5.6.61)$$

By its definition, the value Λ contributing to these equations has the physical sense of the amplification coefficient of the amplitude $\chi(n)$ after its transformation over the period of synchrotron oscillations:

$$\chi(n, t + T_c) = \Lambda \chi(n, t).$$

In this sense the first part of Eq. (5.6.61) is in general agreement with circuit theory: the output signal of the system with the feedback coupling is inversely proportional to the amplification coefficient.

Equation (5.6.61) indicates that depending on the value of $|\text{Im}x_m|$ the first and second terms in its right-hand side have different weights. In the region $\text{Im}x_m \ll 1$ and, therefore, $\delta_m = \text{Im}\omega \ll \omega_c$, the first addend in the right-hand side of this equation dominates

$$\chi(n) \simeq \frac{2\pi i}{1-\Lambda} \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \chi(n'), \quad (5.6.62)$$

if $|x_m - m_c| \ll 1$. The eigenfrequencies of coherent modes in this region are close to harmonics of synchrotron oscillations:

$$\omega = m_c \omega_c + \Delta\omega_m, \quad |\Delta\omega_m| \ll \omega_c.$$

In some simple cases the values $\Delta\omega_m$ can be calculated analytically. (See, for instance, in §5.1). If, for example, $\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2)$, the value $\rho_{x_m}(n, n')$ reads

$$\rho_{x_m}(n, n') = J_{m_c}(n\varphi_0) J_{m_c}(n'\varphi_0), \quad x_m \simeq m_c$$

and therefore $\chi(n) \propto J_{m_c}(n\varphi_0)$. This yields

$$\Delta\omega_m \simeq \int_{-\infty}^{\infty} dn J_{m_c}^2(n\varphi_0) \Omega_m(n - \xi).$$

More generally, if α marks the solutions of Eq. (5.6.62), the upper limit of eigenvalues in Eq. (5.6.62) can be estimated by the sum

$$\begin{aligned}
\sum_{\alpha} (1 - \Lambda_{\alpha}) &\simeq 2\pi i \sum_{m_c = -\infty}^{\infty} \int_{-\infty}^{\infty} dn \rho_{m_c}(n, n) w_m(n - \xi) \\
&= 2\pi i \int_0^{\infty} d\varphi_c \varphi_c \rho(\varphi_c) \int_{-\infty}^{\infty} dn w_m(n - \xi) \\
&\quad \times \sum_{m_c = -\infty}^{\infty} J_{m_c}^2(n\varphi_c).
\end{aligned}$$

Using here

$$\int_0^{\infty} d\varphi_c \varphi_c \rho(\varphi_c) = 1$$

and

$$\sum_{m_c = -\infty}^{\infty} J_{m_c}^2(n\varphi_c) = 1,$$

we can write

$$\sum_{\alpha} (1 - \Lambda)_{\alpha} \simeq \frac{2\pi i \Omega_m}{\omega_c} = 2\pi i \int_{-\infty}^{\infty} dn w_m(n - \xi). \quad (5.6.63)$$

According to these equations, the condition $\text{Im}\omega \ll \omega_c$ holds in the region $|\Omega_m| \ll \omega_c$, where $|\Lambda_{\alpha}| \simeq 1$.

The inverse asymptote $|\Lambda| \gg 1$ can be associated with the region $\text{Im}x_m \gg 1$, where, due to the relationship

$$|\Lambda| = \exp(2\pi \text{Im}x_m) \gg 1$$

the contribution of the first term in the right-hand side of Eq. (5.6.61) into eigensolutions $\chi(n)$ is exponentially suppressed. Respectively, in the limit $\omega_c \rightarrow 0$, neglecting these small terms, we can write

$$\chi(n) \simeq \int_{-\infty}^{\infty} dn' K_{sp}(n, n') w_m(n' - \xi) \chi(n'), \quad \text{Im}x_m \gg 1. \quad (5.6.64)$$

Now, after direct calculation of the integral over m_c in K_{sp} we can write

$$\begin{aligned}
K_{sp}(n, n') &= \int_0^\infty d\varphi_c \varphi_c \rho(\varphi_c) \\
&\quad \times \int_{-\infty}^\infty dm_c \frac{J_{m_c}(n\varphi_c) J_{m_c}(n'\varphi_c)}{x_m - m_c}, \quad \text{Im} x_m > 0, \\
&= \int_0^\infty d\varphi_c \varphi_c \rho(\varphi_c) \int_0^{2\pi} \frac{d\psi_1 d\psi_2}{(2\pi)^2} e^{i(n\varphi_1 - n'\varphi_2)} \\
&\quad \times \int_{-\infty}^\infty dm_c \frac{e^{-im_c(\psi_1 - \psi_2)}}{x_m - m_c} \\
&= \int_0^\infty d\varphi_c \varphi_c \int_0^{2\pi} \frac{d\psi}{2\pi i} \rho(\varphi_c) e^{-in'\varphi(\psi)} \\
&\quad \times \int_0^\psi du \exp\{in\varphi(\psi - u) + ix_m u\}. \tag{5.6.65}
\end{aligned}$$

In the region $\text{Im} x_m \gg 1$, due to fast convergency of the integral over u , we can extend the upper limit in this integral until infinity. If we replace in this equation $\rho(\varphi_c)$ by a distribution function $f_{\parallel}(\varphi, \Delta p) = \lambda_0(\varphi) \Phi(\Delta p)$, with $\Phi(\Delta p) = \Phi(-\Delta p)$, this yields

$$\begin{aligned}
K_{sp}(n, n') &= \omega_c \int d\Gamma_{\parallel} \frac{f_{\parallel}(\varphi, \Delta p) e^{i(n-n')\varphi}}{\Delta\omega_m - n\omega'_0 \Delta p}, \quad \text{Im} \omega \gg \omega_c \\
&\simeq \frac{\omega_c}{\Delta\omega_m} \left[\int_{-\infty}^\infty \frac{d\varphi}{2\pi} \lambda_0(\varphi) e^{i(n-n')\varphi} + O\left(\frac{\omega_c^2}{\Delta\omega_m^2}\right) \right]. \tag{5.6.66}
\end{aligned}$$

The substitution of this kernel into Eq. (5.6.64) results in

$$\begin{aligned}
\chi(n) &= \frac{1}{\Delta\omega_m} \int_{-\infty}^\infty dn' \Omega_m(n - \xi) \chi(n') \int_{-\infty}^\infty d\varphi \lambda_0(\varphi) e^{i(n-n')\varphi} \\
&= \int_{-\infty}^\infty d\varphi \lambda_0(\varphi) e^{-in\varphi} \int_{-\infty}^\infty d\varphi' \chi(\varphi') W_m(\varphi - \varphi'), \tag{5.6.67}
\end{aligned}$$

where

$$W_m(\varphi) = \int_{-\infty}^\infty dn \Omega_m(n) e^{in\varphi} \tag{5.6.68}$$

gives the transverse wake function of the bunch (multiplied by N). The inverse transformation:

$$\chi(\varphi) = \int_{-\infty}^\infty dn e^{in\varphi} \chi(n) \tag{5.6.69}$$

converts Eq. (5.6.67) into the integral equation

$$\chi(\varphi) = \frac{\lambda_0(\varphi)}{\Delta\omega_m} \int_{-\infty}^{\infty} d\varphi' W_m(\varphi - \varphi') \chi(\varphi'). \quad (5.6.70)$$

Due to the causality condition

$$W_m(\varphi) = 0, \quad \varphi > 0,$$

Equation (5.6.70) is a homogeneous integral equation of Volterra type and therefore has no solutions except the trivial. As we saw in the beginning of this section, nonhomogeneous integral equations of this type appear in calculations if one takes into account initial conditions for coherent oscillations. In the approximation ($\omega_c \rightarrow 0$) considered here it describes the beam breakup-type instability of a single bunch in a storage ring. As the homogeneous part of the equation has no solutions, the corresponding coherent oscillations of the beam have neither eigenfrequency spectrum nor eigenmodes, while their amplitudes can depend on time by essentially nonexponential laws.

Using Eqs (5.6.66) through (5.6.70), we can rewrite Eq. (5.6.61) in the asymptotic region $|\Lambda| \gg 1$ as

$$\chi(\varphi) = \chi_\Lambda(\varphi) + \frac{\lambda_0(\varphi)}{\Delta\omega_m} \int_{\varphi}^{\infty} d\varphi' W_m(\varphi - \varphi') \chi(\varphi'), \quad (5.6.71)$$

where

$$\chi_\Lambda(\varphi) = \int_{-\infty}^{\infty} dn \chi_\Lambda(n) e^{in\varphi}, \quad (5.6.72)$$

$$\chi_\Lambda(n) = \frac{2\pi i}{1 - \Lambda} \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \chi(n'). \quad (5.6.73)$$

In contrast with Eq. (5.6.70), the total Eq. (5.6.71) is a nonhomogeneous integral equation of the Volterra type. Its nontrivial solution can be written using the resolvent kernel of this equation:

$$\chi(\varphi) = \chi_\Lambda(\varphi) + \int_{\varphi}^{\infty} R_m(\varphi, \varphi', \Delta\omega_m) \chi_\Lambda(\varphi'). \quad (5.6.74)$$

Combining Eqs. (5.6.73) and (5.6.74) we can write the resulting integral equation in the form

$$\Lambda \chi_{\Lambda}(n) = -2\pi i \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \\ \times \left\{ \chi_{\Lambda}(n') + \int_{-\infty}^{\infty} dn'' R_m(n', n'', \Delta\omega_m) \chi_{\Lambda}(n'') \right\}, \quad (5.6.75)$$

$$|\Lambda| \gg 1,$$

where

$$R_m(n, n', \Delta\omega_m) = \int_{-\infty}^{\infty} \frac{d\varphi}{2\pi} e^{-in\varphi} \int_{\varphi}^{\infty} d\varphi' R_m(\varphi, \varphi', \Delta\omega_m) e^{in'\varphi'}. \quad (5.6.76)$$

In some simple cases Eq. (5.6.75) can be solved in analytic form. We have discussed the model $\rho(\varphi_c) = \delta(\varphi_0^2 - \varphi_c^2)$ and, therefore,

$$\lambda_0(\varphi) \propto \frac{1}{\sqrt{\varphi_0^2 - \varphi^2}}.$$

For this model we can write

$$\rho_{x_m}(n, n') = J_{x_m}(n\varphi_0) J_{x_m}(n'\varphi_0).$$

Substituting in Eq. (5.6.75) the solution

$$\chi_{\Lambda} \propto J_{x_m}(n\varphi_0),$$

we obtain the dispersion equation of the problem in the form

$$\Lambda(x_m) = -2\pi i \int_{-\infty}^{\infty} dn J_{x_m}^2(n\varphi_0) w_m(n - \xi) - 2\pi i \int_{-\infty}^{\infty} dn w_m(n - \xi) \\ \times J_{x_m}(n\varphi_0) \int_{-\infty}^{\infty} dn' J_{x_m}(n'\varphi_0) R_m(n, n', \Delta\omega_m). \quad (5.6.77)$$

This equation can be simplified even more if the bunch wake $W_m(\varphi)$ is the step-function*

$$W_m(\varphi) = 2\Omega_m \begin{cases} 0, & \varphi > 0 \\ 1, & \varphi < 0, \end{cases}$$

*However, as it was shown in § 5.2, in this particular case the homogeneous part of Eqs. (5.6.53) can be easily solved directly.

where Ω_m is the coherent frequency shift of the short bunch ($\varphi \rightarrow 0$). In this case the resolvent R_m can be easily calculated:

$$R_m(\varphi, \varphi', \Delta\omega_m) = \frac{2\Omega_m}{\Delta\omega_m} \lambda_0(\varphi) \exp \left\{ \frac{2\Omega_m}{\Delta\omega_m} \int_{\varphi}^{\varphi'} d\varphi'' \lambda_0(\varphi'') \right\}. \quad (5.6.78)$$

Since for such a wake the first term in the right-hand side of Eq. (5.6.77) vanishes, we can replace Eq. (5.6.77) in the asymptotic region by

$$\Lambda(x_m) = 2\Omega_m \int_{-\infty}^{\infty} \frac{dn}{n} J_{x_m}(n\varphi_0) \int_{-\infty}^{\infty} dn' J_{x_m}(n'\varphi_0) R_m(n, n', \Delta\omega_m).$$

Equation (5.6.75) is an integral equation of the Fredholm type and thus definitely has the spectrum of eigenvalues. In this sense the spectrum of eigenmodes of coherent oscillations of a bunch always exists. If α marks the particular modes, and once eigenfrequencies of the modes only logarithmically depend on Λ_α ,

$$\Delta\omega_{m,\alpha} = \frac{i\omega_c}{2\pi} \ln \Lambda_\alpha, \quad (5.6.79)$$

the distance in the spectrum generally decreases when $|\Lambda|$ increases. From Eq. (5.6.79) we can estimate the rate of this distance in the asymptotic region $|\Lambda| \gg 1$ as the exponential one:

$$\Delta(\omega_\alpha) = \omega_{\alpha+1} - \omega_\alpha \sim \omega_c \exp(-\delta_{m,\alpha} T_c), \quad (5.6.80)$$

where $\delta_{m,\alpha} = \text{Im} \omega_{m,\alpha}$ is the increment of the mode α .

The dependence of the beam dipole momentum on the time is determined by the Fourier integrals

$$\chi(\varphi, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \chi(\varphi), \quad \text{Im} \omega > 0. \quad (5.6.81)$$

Here the Fourier amplitudes $\chi(\varphi)$ must be calculated as the solutions of the nonhomogeneous Eq. (5.6.53). If we define here

$$\chi^{(0)}(n, \Delta\omega_m) = \sum_{m_c=-\infty}^{\infty} \frac{i}{\Delta\omega_m - m_c \omega_c} \int_0^\infty d\varphi_c \varphi_c \chi_m^{(0)}(\varphi_c) J_{m_c}(n\varphi_c), \quad (5.6.82)$$

then using Eqs. (5.6.72) and (5.6.74) we can write the amplitudes $\chi(\varphi)$ in the form

$$\begin{aligned} \chi(\varphi) = & \chi_\Lambda(\varphi) + \chi^{(0)}(\varphi, \Delta\omega_m) + \int_{\varphi}^{\infty} d\varphi' R_m(\varphi, \varphi', \Delta\omega_m) [\chi_\Lambda(\varphi') \\ & + \chi^{(0)}(\varphi', \Delta\omega_m)]. \end{aligned} \quad (5.6.83)$$

As can be seen from Eqs. (5.6.73) and (5.6.83), in the asymptotic region $|\Lambda| \gg 1$ the contributions of eigenmodes in the beam signal will be exponentially small compared to the contribution from the fast development of initial perturbation $\chi^{(0)}$.

Now, for the sake of convenience let us select in Eq. (5.6.83) the part responsible for the single-period effects:

$$\chi(\varphi) = \chi_{sp}(\varphi) + \chi_{\Lambda}(\varphi). \quad (5.6.84)$$

Repeating here the calculations that led from Eq. (5.6.57) to Eq. (5.6.61) and then to Eq. (5.6.66), we can rewrite $\chi^{(0)}(\varphi, \Delta\omega_m)$ in the form

$$\begin{aligned} \chi^{(0)}(n, \Delta\omega_m) &= \frac{2\pi}{\omega_c(1-\Lambda)} \int_0^\infty d\varphi_c \varphi_c \chi_{x_m}^{(0)}(\varphi_c) J_{x_m}(n\varphi_c) \\ &\quad + i \int d\Gamma_{\parallel} \frac{\chi_m^{(0)}(\varphi, \Delta p)}{\Delta\omega_m - n\omega'_0 \Delta p} \\ &\equiv \chi_{mp}^{(0)} + \chi_{sp}^{(0)}. \end{aligned} \quad (5.6.85)$$

Now we may define

$$\chi_{sp}(\varphi) = \chi_{sp}^{(0)}(\varphi, \Delta\omega_m) + \int_{\varphi}^{\infty} d\varphi' R_m(\varphi, \varphi', \Delta\omega_m) \chi_{sp}^{(0)}(\varphi', \Delta\omega_m). \quad (5.6.86)$$

This part of the solution coincides with that obtained before and yields the leading term of its asymptotic expansion in powers of ω_c/Ω_m . As we know, the main features of the dependence on the time of the function $\chi_{sp}(\varphi, t)$ are determined by the specific singularity of the resolvent kernel R_m as a function of the complex variable $\Delta\omega_m$. The time-dependence of the function $\chi_{sp}(\varphi, t)$ is scaled by the value of the coherent frequency shift of the zeroth-length bunch:

$$\Omega_m = \int_{-\infty}^{\infty} dn \Omega_m(n) \quad (5.6.87)$$

and generally has the asymptotes

$$\chi_{sp}(\varphi, t) \sim \begin{cases} \Omega_m t \mathcal{N}(\varphi), & \Omega_m t \ll 1, \\ \exp[\sqrt{\Omega_m t \mathcal{N}(\varphi)}], & \Omega_m t \gg 1, \end{cases} \quad (5.6.88)$$

where

$$\mathcal{N}(\varphi) = \int_{\varphi}^{\infty} d\varphi' \lambda_0(\varphi'). \quad (5.6.89)$$

The multiperiod part of $\chi(\varphi)$ in the asymptotic region $|\Lambda| \gg 1$ satisfies the equation

$$\Lambda\chi_\Lambda(n) = \tilde{\chi}_{mp}^{(0)} - 2\pi i \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \\ \times \left\{ \chi_\Lambda(n') + \int_{-\infty}^{\infty} dn'' R_m(n', n'', \Delta\omega_m) \chi_\Lambda(n'') \right\}, \quad (5.6.90)$$

where

$$\tilde{\chi}_{mp}^{(0)} = -2\pi i \int_{-\infty}^{\infty} dn' \rho_{x_m}(n, n') w_m(n' - \xi) \\ \times \left\{ \chi_{mt}^{(0)}(n') + \int_{-\infty}^{\infty} dn'' R_m(n', n'', \Delta\omega_m) \chi_{mt}^{(0)}(n'') \right\}, \quad (5.6.91)$$

and

$$\chi_{mp}^{(0)}(n, \Delta\omega_m) \simeq -\frac{2\pi}{\omega_c \Lambda} \int_0^\infty d\varphi_c \varphi_c \chi_{x_m}^{(0)}(\varphi_c) J_{x_m}(n\varphi_c). \quad (5.6.92)$$

The dependence of the multi-period part $\chi_\Lambda(t)$ on the time generally can be represented by its expansion in the series in eigenmodes

$$\chi_{mp}(\varphi, t) = \sum_{\alpha} A_{\alpha} \chi_{\Lambda, \alpha}(\varphi) \exp(-i\Delta\omega_{\alpha} t). \quad (5.6.93)$$

By Eq. (5.6.79) the increments in this series are proportional to ω_c . For these reasons we may conclude that in the asymptotic region $\Omega_m \gg \omega_c$ initial behavior ($\omega_c t \ll 1$) of transverse coherent oscillations of a single bunch will be mainly determined by the beam breakup phenomenon. Self-consistent behavior of oscillations might be expected on rather longer time intervals ($\omega_c t \gg 1$). To this end, if the oscillations are unstable due to single-period effects, the calculation of the self-consistent spectrum of the beam in the asymptotic region can contradict the assumed application of the perturbation theory.

5.6.7 It is clear that fast instabilities of a bunch due to interaction with other low-Q systems differ from the discussed only by the Green function of the relevant electrodynamic problem. Due to resonant excitation of particle oscillations the main specific feature of those instabilities is the variation of the rise time along the beam

$$\tau_m^{-1} \sim \Omega_m \int_{\varphi}^{\infty} d\varphi' \rho(\varphi'),$$

as well as the equal importance for beam stability of both real and imaginary parts of Ω_m .

The fast longitudinal instabilities can develop slightly differently. In the region (5.6.1) the longitudinal variables of particles are mainly changed by wakes.

Therefore, the solutions $w(\varphi, t)$ can contain the factors $\exp[-i\Omega_m(\varphi)t]$, where $\Omega_m(\varphi) = \Omega_m \rho(\varphi)$ is the local coherent frequency shift. If $\text{Im}\Omega_m(\varphi) > 0$, among the amplification along the beam there can be the common growth of its longitudinal oscillations with the increment $\text{Im}\Omega_{\text{max}}$:

$$\tau_{m,\parallel}^{-1} = |\delta_m|/\varphi_b.$$

We do not calculate here the intermediate case $|\Omega_m| \sim \omega_c$, when the interaction couples a relatively small number of synchrotron modes. Since the number of coupled modes is estimated by the ratio $|\Omega_m|/\omega_c$ different behavior of oscillations could be expected when it is close to 1. Relevant calculations can be found, for instance, in Ref. 89.

In this chapter we have intentionally focused the discussion on the collective interaction of a bunch with the matched plate. Historically, this problem is associated with the first rigorous description of collective oscillations of the bunched beam.^{26,27,28} A rare possibility of analytic calculation of the wake-fields was very important for this purpose. Then, a matched plate presents a manifold application device, which is frequently installed inside the vacuum pipe. Thus, the study of its effect on the beam stability is of practical interest. Finally, the possibility to damp coherent oscillations using these plates, which was initially demonstrated by measurements in VEPP-2, gives sufficient reasons for detailed study of this problem.

Low-Q Collective Instabilities

The instabilities discussed in the two previous chapters present very important, but still limited classes of collective instabilities limiting the currents of beams in storage rings. More generally, even interacting with the low-Q value element the beam induces the fields, which do not decay completely after one turn. In this case one may expect that decrements or increments of coherent modes will consist of two parts: one describing the single-turn, and the other describing multiturn phenomena. For bunched beams the instabilities of this type were considered first in Refs. 90 and 91 in connection with the so-called resistive wall instability. Later on it was found out that many collective instabilities have similar properties.

For the sake of simplicity in this chapter we assume that the Q-factors are so low that the contributions of single-turn and multiturn effects to increments are well separated.

6.1. RESISTIVE WALL INSTABILITY OF A SHORT BUNCH

Heuristically it seems useful to start the study of resistive wall instability from the simple model assuming the zero bunch length.⁹⁰ When finding electromagnetic fields inside the chamber we shall neglect both the curvature of the beam trajectory and wave effects. For the sake of simplicity we shall assume no additional elements inside the chamber. The electric field inside the chamber will be described by the sum

$$E = E_b + E_i, \quad (6.1.1)$$

where E_b is an electric field of the beam in free space, and E_i is an electric field of charges induced on the conducting walls of the chamber. Similarly, for the magnetic field we have

$$H = H_b + H_i, \quad (6.1.2)$$

where H_i is the magnetic field of currents induced in the conducting walls of the chamber.

For a relativistic beam moving inside the vacuum chamber with ideally conducting walls, the induced fields are proportional to the local instantaneous density of a beam and disappear just when the beam passed. If the wall conductivity σ is not infinite, the fields E_b and H_b continue to follow the instantaneous density. As to the fields associated with the finite value of σ , after the beam the

component $E_{i\perp}$ vanishes very rapidly (for the times of charge relaxation); but the magnetic field related to the currents, induced inside the layer of finite thickness, can exist inside the vacuum chamber for a long time. This slow decay of this residual magnetic field may cause the instability of coherent oscillations of the beam.

Let us calculate the magnetic field remaining inside the tube of rectangular cross section after a single pass (parallel to the tube axis) of the short bunch (Fig. 6.1). We shall start with the supplementary problem. Assuming that the transverse bunch sizes are small ($a_x, a_z \ll h \ll w$), the wake field of the continuous beam with the charge linear density $e\rho$, moving along the tube with ideally conductive walls and displaced along z , can be calculated, for example, by summing up the reflection fields. Its x -component is

$$H_i^{\sigma=\infty} \simeq \frac{4e\rho}{h^2}(z+2z_b), \quad (6.1.3)$$

where z is the coordinate of the observation point, z_b is the coordinate of the beam center, and $z, z_b \ll h$.

Just after the appearance of the beam, H_i will be of the same density as in the case of finite conductivity of walls. But in this case, H_i will reduce with time because of graduate "diffusion" of the magnetic field into the walls. For the time t the field penetrates into each wall to the depth

$$\delta \simeq c \sqrt{\frac{t}{4\pi\sigma}},$$

which increases the distance between walls by the order of $2(\delta/2) = \delta$. Correspondingly at that moment H_i equals

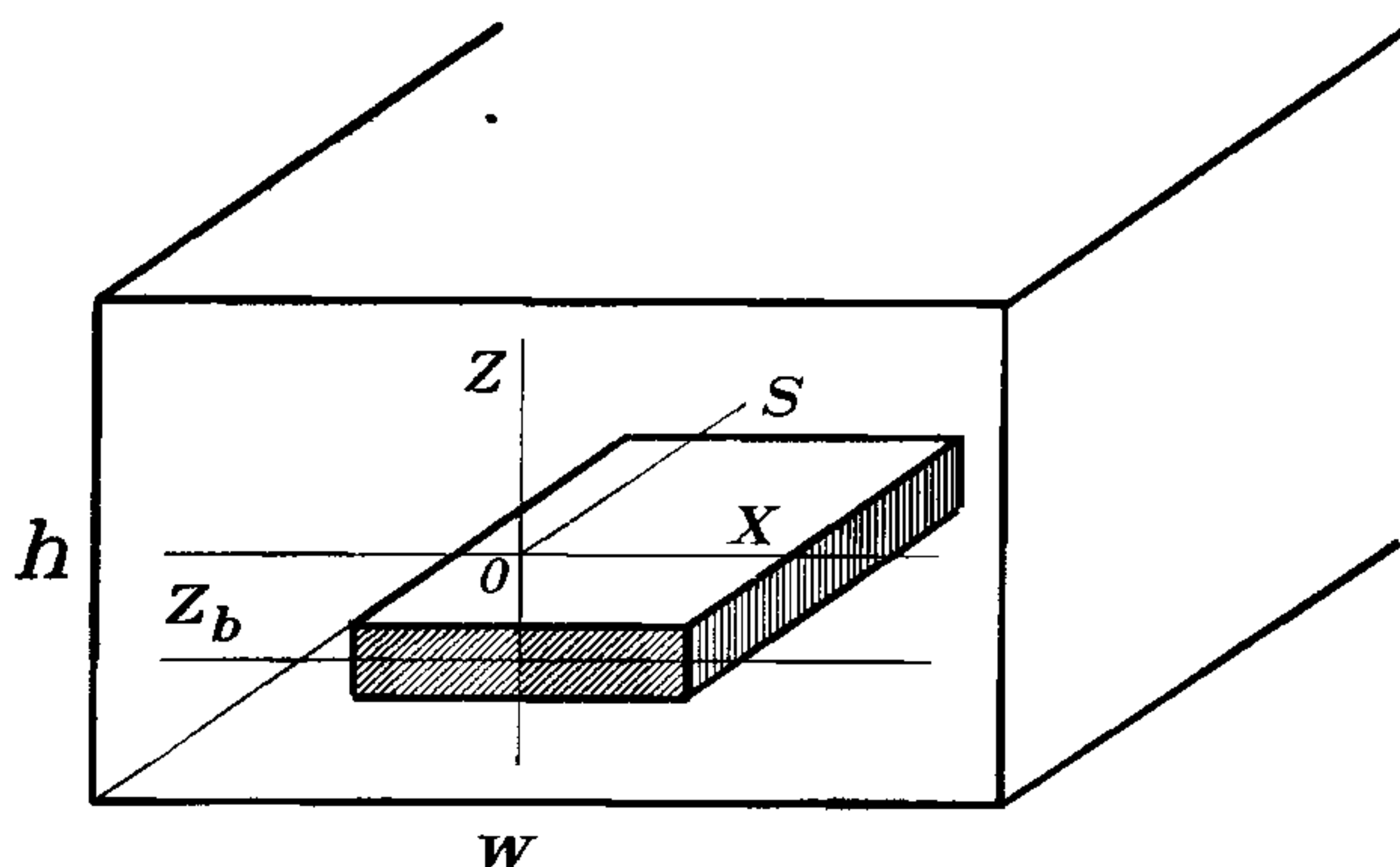


FIG. 6.1. The bunch in the piece of the vacuum chamber with the rectangular cross section.

$$H_i(t) \simeq H_i^{\sigma=\infty} + \frac{dH_i^{\sigma=\infty}}{dh} \delta \simeq H_i^{\sigma=\infty} - \frac{4ec\rho}{\sqrt{\pi\sigma}h^3}(z+2z_b)\sqrt{t}. \quad (6.1.4)$$

Hence, one can immediately see that after the passing of the bunch with the length τ (which can be presented as two steps of density of inverse polarity shifted in time by τ), the wake field is

$$\Delta H^\sigma(t) \simeq -\frac{4ec\rho}{(\pi\sigma)^{1/2}h^3}(z+2z_b)(\sqrt{t}-\sqrt{t-\tau}). \quad (6.1.5)$$

(The reference point of time is the moment when bunch appeared.) At $t \gg \tau$ the expansion of this expression over τ gives

$$\Delta H^\sigma(t) \simeq -\frac{2c}{(\pi\sigma)^{1/2}h^3}(e\rho\tau)(z+2z_b)t^{-1/2}. \quad (6.1.6)$$

As seen, the field is determined now by the total charge, $e\rho\tau c$, and does not depend on the bunch length. Note that the sign of the residual magnetic field has a sign opposite to the sign of the reflection field.

In the same approximation, the field remaining after the passage of the beam with an arbitrary $\rho(t)$ and $z_b(t)$ can be written in the usual way, as a superposition of fields remaining after short bunches when any beam could be divided, that is, in the kind of the integral

$$\Delta H^\sigma(t) \simeq \frac{\pi ec}{3h^3\sqrt{\sigma}} \int_{t_0}^t d\tau \frac{\rho(\tau)[z+2z_b(\tau)]}{\sqrt{t-\tau}}, \quad (6.1.7)$$

where t_0 is initial moment of the time

Note that the integrals in Eq. (6.1.7) in the general case are divergent at $t_0 \rightarrow -\infty$ which is connected with the nonvalidity of the Leontovich conditions for too low frequencies or which is equivalent with the comparative thickness of the skin layer with respect to the wall thickness and other characteristic dimensions.

Taking into account the magnetic field remaining on each azimuth by particles passed up to the given moment of time t , we can write the equation describing the betatron oscillations of a particle in the following form:

$$\ddot{z} + \omega_z^2 z(t) = \frac{A}{2} z(t) \int_{t_0}^t d\tau \frac{\rho(\tau)}{\sqrt{t-\tau}} + A \int_{t_0}^t d\tau \frac{z_b \rho(\tau)}{\sqrt{t-\tau}},$$

$$A = \frac{2\pi e^2 c}{3\gamma m h^3 \sqrt{\sigma}}. \quad (6.1.8)$$

The first term in the right-hand side of this equation yields the variation of the betatron frequency by the wake. For the rough estimation of the effect it can be omitted in the following discussion. Assuming no constant orbit distortion, we

find that the integral over τ converges when $t_0 \rightarrow -\infty$. Results for the betatron oscillations equation then take the form

$$\ddot{z} + \omega_z^2 z = A \int_{-\infty}^t \frac{d\tau z_b(\tau) \rho(\tau)}{\sqrt{t-\tau}} = A \int_0^\infty d\tau \frac{z_b(t-\tau) \rho(t-\tau)}{\sqrt{\tau}}. \quad (6.1.9)$$

Let us consider now the case of the dipole oscillation of a short bunch. The equation of motion of the dipole moment of the bunch in this approximation has the form

$$\ddot{z}_b + \omega_z^2 z_b = \frac{N}{c} A \sum_{n=1}^{\infty} \frac{z_b(t-nT_s)}{(nT_s)^{1/2}}. \quad (6.1.10)$$

Assuming that the unperturbed dipole oscillations are described by

$$z_b = a \cos(\omega_z t + \psi_0),$$

we find in the first approximation of the averaging method the following expression for the increment of unstable oscillations:

$$\delta = \left(\frac{d \ln a}{dt} \right) = - \frac{N r_0 c \pi R_0}{3 \gamma h^3 \nu_z (T_s \sigma)^{1/2}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n \nu_z)}{\sqrt{n}}, \quad r_0 = e^2 / mc^2. \quad (6.1.11)$$

The dependence of the increment (δ) on ν_z is defined by the function (Fig. 6.2)

$$\varphi(\nu) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n \nu)}{\sqrt{n}}.$$

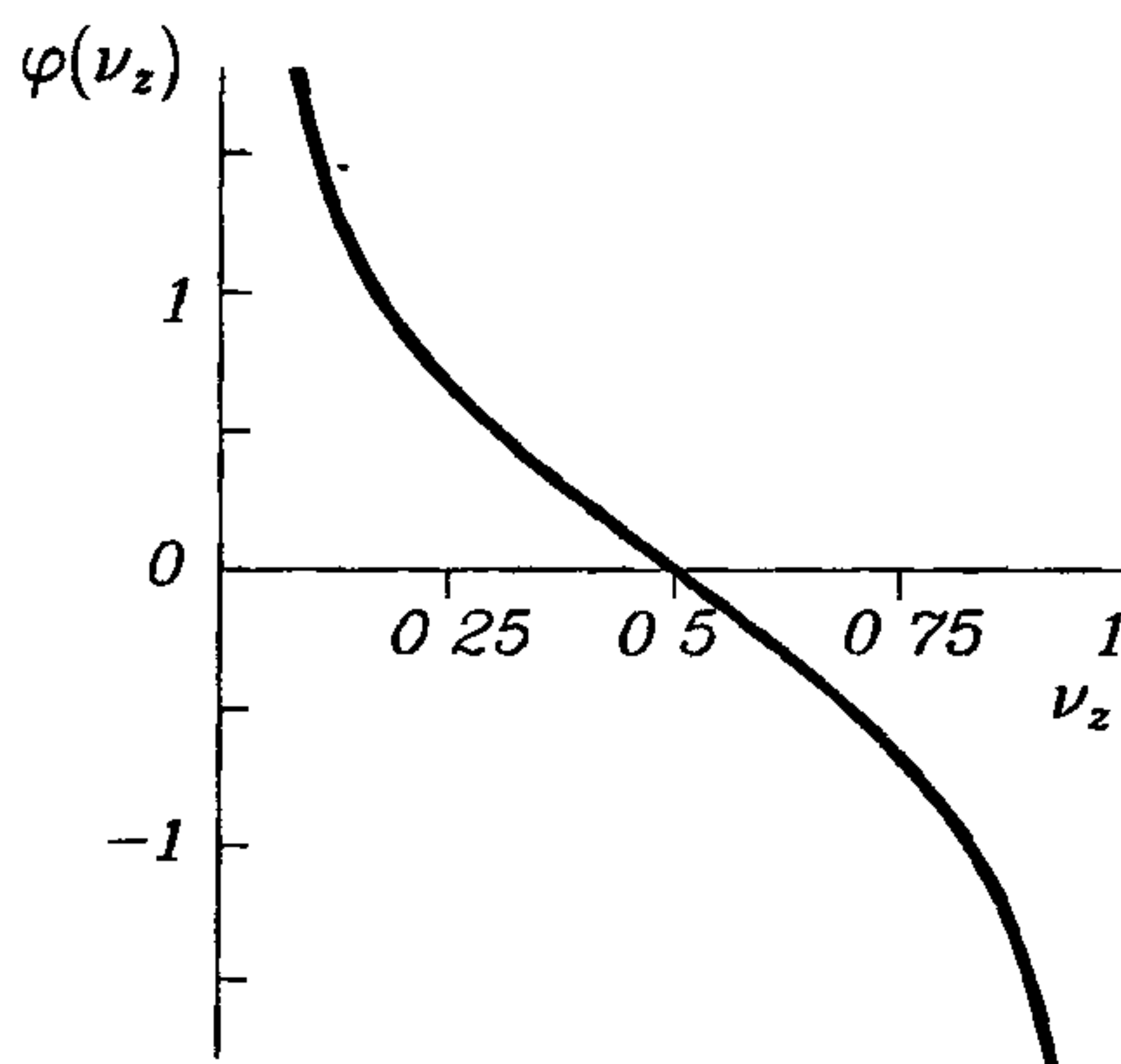


FIG. 6.2. Dependence of the increment of the resistive wall instability on the betatron tune.

Because of the periodicity and antisymmetry of each term of the given row, an increment will be positive (the case of instability) when

$$k - \frac{1}{2} < v_z < k,$$

and negative when

$$k < v_z < k + \frac{1}{2}, \quad k=0,1,2,\dots$$

6.2. THE FORMAL THEORY OF RESISTIVE WALL INSTABILITY

Let us now use the technique given in Chapter 3 to study collective stability of bunches with more or less arbitrary length interacting with vacuum chamber walls of finite conductivity σ . Here we shall follow Ref. 33 where this problem has been solved for oscillations of arbitrary multipolarity m and an arbitrary distribution of σ over the vacuum chamber walls.

With the use of the Lorenz gauge condition, the Green functions of a chamber with walls of finite conductivity satisfy the system of equations

$$\begin{aligned} \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G_{00} &= -4\pi \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \\ \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G_{\alpha\beta} &= -4\pi \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \end{aligned} \quad (6.2.1)$$

and should be constructed in such a way as to let the Fourier amplitudes E_ω and H_ω satisfy the Leontovich boundary conditions⁹ on the chamber walls:

$$\mathbf{n} \times \mathbf{E}_\omega = \left(\frac{-i\omega}{4\pi\sigma} \right)^{1/2} \mathbf{n} \times [\mathbf{H} \times \mathbf{n}], \quad (6.2.2)$$

where $\mathbf{n}$ is a unit vector of the normal to the chamber walls directed inside the metal; $\delta_{\alpha\beta}$ is the Kronecker's symbol, Greek indices are running over the values x , z , and s , and the Latin symbols are running over the values x and z . On the boundary surface we can define three orths: vector $\mathbf{n}$, the tangential to the cross-section contour unit vector $\mathbf{e}$ and the binormal vector, which is parallel to the tangential to the orbit $\boldsymbol{\tau} = \mathbf{n} \times \mathbf{e}$.

Neglecting the chamber curvature, let us write the solution of Eqs. (6.2.1) in the following form

$$G_{\alpha\beta} = \int \frac{dk d\omega}{(2\pi)^2} e^{ik(s-s') - i\omega(t-t')} G_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}'_1, k, \omega), \quad (6.2.3)$$

$$\mathbf{r}_\perp = \mathbf{r} - \tau \mathbf{s}.$$

The Fourier amplitude of the Green's function $G_{\alpha\beta}(k, \omega)$ is presented by the sum

$$G_{\alpha\beta} = G_{\alpha\beta}^\infty + G_{\alpha\beta}^\sigma, \quad (6.2.4)$$

where $G_{\alpha\beta}^\infty$ is the Green's function of the ideally conducting chamber

$$\begin{aligned} \left(\Delta_\perp - k^2 + \frac{\omega^2}{c^2} \right) G_{\alpha\beta}^\infty &= -4\pi \delta_{\alpha\beta} \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp), \\ \left(\Delta_\perp - k^2 + \frac{\omega^2}{c^2} \right) G_{00}^\infty &= -4\pi \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp), \\ G_{s,\beta} &= \delta_{s,\beta} G_{s,s}^\infty, \quad G_{s,\beta}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) = 0, \quad e_i G_{i,\beta}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) = 0, \end{aligned} \quad (6.2.5)$$

where $\mathbf{r}_\Sigma$ is $\mathbf{r}_\perp$ on the chamber surface.

The term $G_{\alpha\beta}^\sigma$ satisfies the homogeneous equation

$$\left(\Delta_\perp - k^2 + \frac{\omega^2}{c^2} \right) G_{\alpha\beta}^\sigma = 0 \quad (6.2.6)$$

and the inhomogeneous boundary conditions

$$G_{s\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}_\perp) e^{iks} = \delta(\omega) (\mathbf{e} \times \nabla)_\lambda G_{\lambda\alpha}(\mathbf{r}_\Sigma, \mathbf{r}_\perp) e^{iks}, \quad (6.2.7)$$

$$e_i G_{i\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}_\perp) e^{iks} = -\delta(\omega) (\boldsymbol{\tau} \times \nabla)_\lambda G_{\lambda\alpha}(\mathbf{r}_\Sigma, \mathbf{r}_\perp) e^{iks}. \quad (6.2.8)$$

The complex function introduced here

$$\delta(\omega) = \frac{c}{\sqrt{-4\pi i \sigma \omega}} = \frac{c}{\sqrt{2\pi \sigma |\omega|}} \frac{1 + i \operatorname{sgn}(\omega)}{2} \quad (6.2.9)$$

is defined in such a way that its module is equal to the thickness of the skin layer at frequency ω . Since $G_{\alpha\beta}^\sigma$ describes the fields without sources, with the special choice of calibration

$$\frac{\partial}{\partial x_\alpha} G_{\alpha\beta}^\sigma(\mathbf{r}'_\perp, \mathbf{r}'_\perp) e^{iks} = 0,$$

the scalar components of G^σ can be taken equal to zero.

Using Green's theorem

$$\begin{aligned} G_{\alpha\beta}^\sigma(\mathbf{r}_\perp, \mathbf{r}'_\perp) &= \frac{1}{4\pi} \oint dl_\Sigma \left\{ G_{\lambda\alpha}^\infty[\mathbf{r}(l), \mathbf{r}_\perp] \frac{\partial}{\partial n} G_{\lambda\beta}^\sigma[\mathbf{r}(l), \mathbf{r}_\perp] \right. \\ &\quad \left. - G_{\lambda\beta}^\sigma[\mathbf{r}(l), \mathbf{r}_\perp] \frac{\partial}{\partial n} G_{\lambda\alpha}^\infty[\mathbf{r}(l), \mathbf{r}_\perp] \right\} \end{aligned} \quad (6.2.10)$$

(an integration is carried out along the waveguide cross section contour, $\partial/\partial n = \mathbf{n} \cdot \nabla$), one can transform Eqs. (6.2.5) and (6.2.6) into the system of integral equations for $G_{\alpha\beta}^\sigma$. Taking into account that on the boundary surface

$$G_{s,\beta}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) = 0,$$

for the longitudinal component $G_{s\beta}^\sigma$ Eq. (6.2.10) is transformed into

$$G_{s,\alpha}^\sigma = -\frac{1}{4\pi} \oint dl_\Sigma G_{s,\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp). \quad (6.2.11)$$

Substituting here $G_{s,\alpha}^\sigma$ on the contour from Eq. (6.2.7) we get

$$\begin{aligned} G_{s,\alpha}^\sigma = & - \oint dl_\Sigma \frac{\delta(\omega)}{4\pi} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) (\mathbf{e} \times \nabla)_\lambda G_{\lambda\alpha}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) \\ & - \oint dl_\Sigma \frac{\delta(\omega)}{4\pi} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) (\mathbf{e} \times \nabla)_\lambda G_{\lambda\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}'_\perp). \end{aligned} \quad (6.2.12)$$

For the transverse components $G_{i\alpha}^\sigma$ Eq. (6.2.10) after some transformation can be rewritten in the form

$$\begin{aligned} G_{i\alpha}^\sigma = & - \oint dl_\Sigma \frac{\delta(\omega)}{4\pi} \{ ik G_{s,\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) n_j G_{ji}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) \\ & + \mathbf{e}_j G_{j\alpha}^\sigma(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) (\boldsymbol{\tau} \times \nabla)_k G_{k,i}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) \}. \end{aligned} \quad (6.2.13)$$

In these transformations we used the boundary conditions for $G_{\lambda\alpha}^\infty$, relations of the completeness $e_i e_k + n_i n_k = \delta_{ik}$ and the calibration condition for $G_{\alpha\beta}^\sigma$. After substituting in Eq. (6.2.13) the values of $G_{\alpha\beta}^\sigma$ from Eqs. (6.2.7) and (6.2.8), we get the system of integral equations for transverse components $G_{i\alpha}^\sigma$:

$$\begin{aligned} G_{i\alpha}^\sigma = & - \oint \frac{dl_\Sigma \delta(\omega)}{4\pi} [ik n_j G_{ji}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) (\mathbf{e} \times \nabla)_\lambda G_{\lambda\alpha}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_\perp) \\ & - (\boldsymbol{\tau} \times \nabla)_k G_{k,i}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) (\boldsymbol{\tau} \times \nabla)_\lambda G_{\lambda\alpha}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_\perp)]. \end{aligned} \quad (6.2.14)$$

Equations (6.2.12) and (6.2.14) form the complete system of inhomogeneous integral equations for calculation of $G_{\alpha\beta}^\sigma$. Taking into account the fact that for typical ranges of frequencies the value $|\delta(\omega)|$ is small compared to the characteristic transverse sizes of a vacuum chamber, we may find the solution of Eqs. (6.2.12), (6.2.14) as a power series of $\delta(\omega)$. In the first approximation this yields

$$G_{s\alpha}^\sigma = - \oint \frac{dl_\Sigma \delta(\omega)}{4\pi} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_\perp) (\mathbf{e} \times \nabla)_\lambda G_{\lambda\alpha}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_\perp), \quad (6.2.15)$$

$$G_{ia}^\sigma = - \oint \frac{dl_\Sigma \delta(\omega)}{4\pi} [ik\mathbf{n}_j G_{ji}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_1) (\mathbf{e} \times \nabla)_\lambda G_{\lambda a}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_1) - (\boldsymbol{\tau} \times \nabla)_k G_{ki}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_1) (\boldsymbol{\tau} \times \nabla)_j G_{ja}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_1)]. \quad (6.2.16)$$

This solution has simple spectral properties. In the low-frequency region $G_{\alpha\beta}^\sigma(\omega)$ behaves like $\omega^{-1/2}$. This results in the asymptotical decrease in induced electric and magnetic fields proportional to $t^{-3/2}$ and $t^{-1/2}$ respectively. Except for the root singularity in the low-frequency region $G_{\alpha\beta}^\sigma(\omega)$ has simple and double poles on the eigenfrequencies of the chamber. The appearance of double poles is connected with the use of perturbation theory for solving Eqs. (6.2.12) and (6.2.14). But in the frequency range of our interest the pole features cannot contribute substantially to oscillation decrements.

As an example of the use of the formula obtained let us calculate $G_{\alpha\beta}^\sigma$ for the chamber of rectangular cross section shown in Fig. 6.1.⁹¹ In this case, the Green's function for the chamber with ideally conducting walls is written in the form

$$G_{\alpha\beta}^\infty = -\delta_{\alpha\beta} \frac{2}{hw} \sum_{k_1} \frac{g_{\alpha\alpha}(\mathbf{r}_1, \mathbf{r}'_1, k_1)}{\omega^2 - k^2},$$

$$g_{xx} = C_x S_z C_{x'} S_{z'}, \quad g_{zs} = S_x S_z S_{x'} S_{z'},$$

$$g_{zz} = S_x C_z S_{x'} C_{z'}. \quad (6.2.17)$$

Here we introduced

$$C_x = \cos k_x x, \quad S_x = \sin k_x x, \quad k_x = \frac{m\pi}{w}, \quad m = 0, 1, \dots,$$

$$C_z = \cos k_z z, \quad S_z = \sin k_z z, \quad k_z = \frac{n\pi}{h}, \quad n = 0, 1, \dots,$$

$$k^2 = k_\parallel^2 + k_x^2 + k_z^2.$$

To avoid cumbersome calculations let us first get $G_{s,\lambda}^\sigma$. Using

$$(\mathbf{e} \times \nabla)_\alpha = ik_\parallel \mathbf{n}_\alpha - \tau_\alpha \frac{\partial}{\partial n},$$

and rewriting Eq. (6.2.15) in the form

$$G_{s,\lambda}^\sigma = \oint \frac{dl_\Sigma \delta(\omega)}{4\pi} \left\{ \delta_{s,\lambda} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_1) \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_1) - ik_\parallel \mathbf{n}_\lambda G_{\lambda\lambda}^\infty(\mathbf{r}_\Sigma, \mathbf{r}'_1) \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, \mathbf{r}_1) \right\}. \quad (6.2.18)$$

Taking into account that on the horizontal walls of the chamber $n_\lambda = \pm \delta_{z,\lambda}$ (the sign + is related to the upper wall) and on the vertical walls $n_\lambda = \pm \delta_{x,\lambda}$ (the sign + is related to the wall where $x=w$), we can transform Eq. (6.2.18) into

$$G_{s,\lambda}^\sigma = -\frac{2}{wh} \sum_{k_1} \frac{g_{s,\lambda}^\sigma(\mathbf{r}_1, \mathbf{r}'_1)}{\omega^2 - k^2}. \quad (6.2.19)$$

The kernel $g_{s,\lambda}^\sigma(\mathbf{r}_1, \mathbf{r}'_1)$ is obtained by substituting Eq. (6.2.17) in Eq. (6.2.18). The result is

$$\begin{aligned} g_{s,\lambda}^\sigma(\mathbf{r}_1, \mathbf{r}'_1) = & \frac{1}{2\pi wh} \sum_{k'_1} \frac{1}{k'^2 - \omega^2} \\ & \times \frac{\delta_x(\omega)}{2} \delta_{s,\lambda} \delta_{k_x k'_x} h [1 - (-1)^{n+n'}] k_z k'_z S_x S_{x'} S_z S_{z'} \\ & + \frac{\delta_z(\omega)}{2} \delta_{s,\lambda} \delta_{k_z k'_z} w [1 - (-1)^{m+m'}] k_x k'_x S_x S_{x'} S_z S_{z'} \\ & - ik_{\parallel} \frac{\delta_x(\omega)}{2} \delta_{x,\lambda} \delta_{k_z k'_z} h k_x [1 - (-1)^{n+n'}] S_x S_z C_{x'} S_{z'} \\ & - ik_{\parallel} \frac{\delta_z(\omega)}{2} \delta_{z,\lambda} \delta_{k_x k'_x} w k_z [1 - (-1)^{m+m'}] S_x C_z S_{x'} S_{z'}, \\ & \delta_{x,z}(\omega) = \delta(\omega, \sigma_{x,z}). \end{aligned} \quad (6.2.20)$$

After the summation over k'_1 we get

$$g_{s,\lambda}^\sigma = \frac{1}{8\pi} (S_x B_z^s Q_z + S_z B_x^s Q_x),$$

where

$$\begin{aligned} Q_z = & -\delta(\omega) (\delta_{s,\lambda} k_z S_{x'} S_{z'} - ik_{\parallel} \delta_{z,\lambda} S_{x'} C_{z'}), \\ B_z^s = & \frac{\sin q_z(z-h) + (-1)^n \sin q_z z}{\sin q_z h}, \\ q_z^2 = & \omega^2 - k_{\parallel}^2 - k_x^2. \end{aligned} \quad (6.2.21)$$

Performing similar calculations one can reconstruct from Eq. (6.2.16) the other components $G_{\alpha\beta}^\sigma$. Omitting the intermediate calculation, the results is

$$G_{\alpha\beta}^\sigma = -\frac{1}{2wh} \sum_{k_1} \frac{g_{\alpha\beta}^\sigma(\mathbf{r}_1, \mathbf{r}'_1)}{\omega^2 - k^2},$$

$$\begin{aligned}
g_{s,\beta}^{\sigma} &= S_x B_z^s Q_z + S_z B_x^s Q_x, \\
g_{x\beta}^{\sigma} &= C_x B_z^s P_z + S_z B_x^c R_x, \\
g_{z\beta}^{\sigma} &= C_z B_x^s P_x + S_x B_z^c R_z,
\end{aligned} \tag{6.2.22}$$

where we introduced

$$\begin{aligned}
P_z &= -\delta(\omega) (\delta_{x\beta} k_z C_{x'} S_{z'} - \delta_{z\beta} S_{x'} C_{z'}), \\
R_z &= -\frac{\delta(\omega)}{q_z} [(k_x^2 + k_{\parallel}^2) \delta_{z\beta} S_{x'} C_{z'} - k_x k_z \delta_{x\beta} C_{x'} S_{z'} + i k_{\parallel} k_z S_{x'} S_{z'}],
\end{aligned}$$

$$B_z^c = \frac{1}{q_z} \frac{d}{dz} B_z^s.$$

The remaining expressions determining g^{σ} can be obtained with the corresponding replacement of subscripts $x \rightarrow z$ and $z \rightarrow x$; the values σ_x and σ_z are the conductivities of the horizontal and vertical walls respectively.

As a conclusion, let us note that for the closed chamber, because of the field periodicity over the azimuth, the integration in Eq. (6.2.3) over k should be replaced by the summation

$$\int \frac{dk}{2\pi} f(k) \rightarrow \sum_n \frac{f(k_n)}{\Pi}, \quad k_n = n/R_0.$$

Using Eqs. (6.2.15) and (6.2.16) for the Green function of the chamber with finite conductivity of walls, let us calculate the decrements of synchrotron coherent oscillations. Additionally we assume that

$$\gamma l_b \gg l_1, \quad \gamma \gg l_1/R_0,$$

where l_1 again is the transverse size of the chamber and l_b is the bunch length. When these inequalities are satisfied in the frequency denominators of $G_{\alpha\beta}^{\infty}$ [see, for example, Eq. (6.2.17)]

$$k_1^2 + n^2/R_0^2 - (\omega + n\omega_s)^2/c^2 = k_1^2 - \frac{n^2}{\gamma^2 R_0^2} - \frac{\omega^2}{c^2} - 2 \frac{n\omega_0 \omega}{c^2}$$

one can neglect all the terms except for k_1 . As a result, the azimuthal dependence of $G_{\alpha\beta}^{\infty}$ becomes the δ -like dependence

$$G_{\alpha\beta}^{\infty}(\mathbf{r}_1, \mathbf{r}'_1, \vartheta - \vartheta') \rightarrow G_{\alpha\beta}^{\infty}(\mathbf{r}_1, \mathbf{r}'_1) \delta(\vartheta - \vartheta'),$$

that corresponds to the strong shielding of the relativistic beam by the walls of the vacuum chamber.

For the sake of simplicity below we shall do the calculations for the vertical synchrotron oscillations $\omega \simeq m_z \omega_z + m_c \omega_c$. Assuming that the distributions

over amplitudes of the betatron and synchrotron oscillations are factored, let us rewrite Eq. (3.1) in the form

$$\Delta\omega_m f_m = Ne^2 \rho(\varphi_c) m_z \frac{\partial F_0}{\partial I_z} \sum_{u=-\infty}^{\infty} J_{m_c}(n_1 \varphi_c) \times \int d\Gamma' J_{m_c}(n_1 \varphi') K_m(\Gamma_1, \Gamma'_1) f_m(\Gamma'_1, \varphi'_c).$$

Taking into account Eq. (6.2.4) let us present the kernel K_m of this equation by the sum:

$$K_m = K_m^\infty + K_m^\sigma,$$

where the term K_m^∞ is defined by the Fourier harmonics G^∞ , and K_m^σ is defined by the Fourier harmonics G^σ . In the absence of coherent radiation and far from the lines of machine resonances K_m^∞ leads only to the shift of frequencies of coherent oscillations and cannot cause directly either the instability itself or the damping of coherent oscillations. Having in mind the evolution of oscillation decrements due to the conductivity of chamber walls, the term K_m^∞ will be omitted below.

According to Eqs. (6.2.15) and (6.2.16) the term K_m^σ reads

$$K_m^\sigma = \frac{v_s^2}{8\pi^2 R_0} \oint dl_\Sigma \delta(\omega) \left\{ \theta_z \theta'_z (\tilde{\tau}_\lambda G_{\lambda z}^\infty) (\tilde{\tau}_\lambda G_{\lambda z}^\infty)' + \left[\frac{\partial}{\partial n} G_{s,s}^\infty + ik_n \theta_z n_\lambda G_{\lambda z}^\infty \right] \left[\frac{\partial}{\partial n} G_{s,s}^\infty - ik_n \theta_z G_{\lambda z}^\infty \right]' \right\}_{m,n},$$

where $\theta_z = v_z/v_s$, $\tilde{\tau}_\lambda = (\tau \times \nabla)_\lambda$, and $k_n = n/R_0$.

After calculating the Fourier harmonics over the betatron phases we get

$$K_m^\sigma = \frac{v_s^2}{8\pi} \left(\frac{a_z a'_z}{4} \right)^{|m_z|} \frac{1}{(m_z!)^2} \frac{\Phi_m^{(0)} + 2m_z v_z n (l_1/R_0)^2 \Phi_m^{(1)}}{\sqrt{-i(m_z v_z + n)}}, \quad (6.2.23)$$

where the leading terms of the expansion over l_1/R_0 were kept. The geometric factors $\Phi_m^{(0)}$ and $\Phi_m^{(1)}$ in Eq. (6.2.23) are

$$\Phi_m^{(0)} = \oint dl_\Sigma \left(\frac{\omega_s}{4\pi\sigma} \right)^{1/2} \left\{ \frac{\partial^{|m_z|}}{\partial z^{|m_z|}} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, 0) \right\}^2, \\ \Phi_m^{(1)} = \oint \frac{dl_\Sigma}{l_1^2} \left(\frac{\omega_s}{4\pi\sigma} \right)^{1/2} \left(\frac{\partial^{|m_z|}}{\partial z^{|m_z|}} \frac{\partial}{\partial n} G_{s,s}^\infty(\mathbf{r}_\Sigma, 0) \right) \left(\frac{\partial^{|m_z|-1}}{\partial z^{|m_z|-1}} n_\lambda G_{\lambda z}^\infty \right). \quad (6.2.24)$$

It is easy to see that $\Phi_m^{(0)}$ and $\Phi_m^{(1)}$ are of the same order of magnitude.

Taking $f_{m,\omega}(I, \varphi_c)$ in the form

$$f_{m,\omega}(I, \varphi_c) = \left(\frac{a_z}{2}\right)^{|m|} \frac{\partial F_0}{\partial I_z} \chi_m(\varphi_c),$$

we obtain for $\chi_m(\varphi_c)$ the integral equation

$$\Lambda \chi_m = -\rho(\varphi_c) \sum_{n=-\infty}^{\infty} V_{mn} J_{m_c}(n_1 \varphi_c) \int_0^{\infty} d\varphi'_c J_{m_c}(n_1 \varphi'_c) \chi_m(\varphi'_c), \quad (6.2.25)$$

where we introduced

$$\Lambda = \frac{\Delta \omega_m 8\pi \gamma v_z (m_z!)^2}{N r_0 c R_0^2 \text{sgn}(m_z)} \left\{ \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \right\}^{-1}, \quad (6.2.26)$$

$$V_{mn} = \frac{\Phi_m^{(0)} + 2m_z v_z n \Phi_m^{(1)} (l_1 / R_0)^2}{\sqrt{-i(m_z v_z + n)}}. \quad (6.2.27)$$

Since Eq. (6.2.25) still is very difficult for the general analytic solution, the following steps need some approximation. Note that in many cases the estimation of the leading eigenvalue, which determines the threshold current of instability, is of practical importance. We can calculate this eigenvalue with the following approach. Multiplying Eq. (6.2.25) by $\varphi_c^{m_c+k+1}$, $k=0,1,2,\dots$, and integrating it over φ_c from 0 to ∞ we get the system of equations for the momenta

$$\begin{aligned} X_k &= \int_0^{\infty} d\varphi_c \varphi_c^{m_c+k+1} \chi_m(\varphi_c), \\ \Lambda X_k &= - \sum_n V_{mn} \int_0^{\infty} d\varphi_c \varphi_c^{m_c+k+1} J_{m_c}(n_1 \varphi_c) \\ &\quad \times \rho(\varphi_c) \sum_{k'=0}^{\infty} \left(\frac{n_1}{2}\right)^{m_c+2k'} \frac{(-1)^{k'} X_{2k'}}{k'!(m_c+k')!}. \end{aligned} \quad (6.2.28)$$

For a Gaussian distribution over the amplitudes of synchrotron oscillations

$$\rho(\varphi_c) = (2/\varphi_0^2) \exp(-\varphi_c^2/\varphi_0^2)$$

the momenta X_k grow with k not faster than $\varphi_0^k \Gamma(m_c+k)$. In this case, in the sum over k in Eq. (6.2.28) one can neglect the contribution of the momenta with numbers $2k' > k$ (i.e., by coupling the momentum X_k with the higher ones). This yields

$$\Lambda X_k = - \sum_n V_{mn} \int_0^\infty d\varphi_c \varphi_c^{m_c+k+1} J_{m_c}(n_1 \varphi_c) \times \rho(\varphi_c) \sum_{k'=0}^{[k/2]} \left(\frac{n_1}{2}\right)^{m_c+2k'} \frac{(-1)^{k'} X_{2k'}}{k'!(m_c+k')!}. \quad (6.2.29)$$

Formally such an approximation assumes that in the first excited state in the vicinity of $\omega = m_z \omega_z + m_c \omega_c$ the perturbation of the longitudinal phase space density $\chi_m(\varphi_c)$ has no roots except for the origin, where it has the $|m_c|$ -multiple root, and the point $\varphi_c = \infty$.

Equations (6.2.29) have the "triangle" matrix so it can be solved easily. To calculate the eigenvalue Λ we take in Eq. (6.2.29) $k=0$:

$$\Lambda = -\frac{2}{m_c!} \sum_n V_{mn} \left(\frac{n_1 \varphi_0}{2}\right)^{2m_c} \exp\left(-\frac{n_1^2 \varphi_0^2}{4}\right). \quad (6.2.30)$$

The remaining equations define the momenta X_k . The value X_0 is determined by the normalization of the solution $\chi_m(\varphi_c)$.

Using the summation formula in Eq. (3.1.26) we present the sum over n in Eq. (6.2.30) in the form

$$\Lambda = \Lambda_{st} + \Lambda_{mt} = - \int_{-\infty}^{\infty} dn(\dots) - \sum_{l \neq 0} \int_{-\infty}^{\infty} dne^{i2\pi nl}(\dots). \quad (6.2.31)$$

The first summand Λ_{st} in this equation corresponds to the single-turn interaction of particles, while the second one describes the interaction of a bunch with the fields induced on the previous turns.

Let us first calculate Λ_{st} . Replacing in Eq. (6.2.30) the summation over n by the integration and taking as the new variable of integration the value

$$2u = \varphi_0 n - \nu - \xi = \varphi_0 n - \xi$$

we write

$$\Lambda_{st} = -\frac{2}{m_c!} \int_{-\infty}^{\infty} du \frac{a + 2bu}{\sqrt{-i(2u - \xi)}} u^{2m_c} e^{-u^2},$$

$$a = \frac{2}{\sqrt{\varphi_0}} \left[\Phi_m^{(0)} - \frac{\bar{\xi}}{\varphi_0} 2\nu_z \Phi_m^{(1)} \left(\frac{l_\perp}{R_0}\right)^2 \right],$$

$$b = \frac{4\nu}{\varphi_0^{3/2}} \left(\frac{l_\perp}{R_0}\right)^2 \Phi_m^{(1)}, \quad (6.2.32)$$

all other designations are the same as those in § 5.4:

$$v = m_z v_z \varphi_0; \quad \xi = m_z \varphi_0 \frac{dv_z}{d \ln \omega_0}.$$

Using the formula

$$u^{2m} e^{-u^2} = (-1)^m \frac{d^m}{d\alpha^m} e^{-\alpha u^2} \Big|_{\alpha=1},$$

we rewrite Eq. (6.2.32) in the form

$$\Lambda_{st} = -\frac{2}{m_c!} (-1)^{m_c} \frac{d^{m_c}}{d\alpha^{m_c}} I(\alpha) \Big|_{\alpha=1},$$

$$I(\alpha) = \int_{-\infty}^{\infty} du \frac{a + 2bu}{\sqrt{-i(2u - \xi)}} e^{-\alpha u^2}, \quad (6.2.33)$$

or after the integration by parts in $I(\alpha)$ we have

$$I(\alpha) = \left(a - \frac{2b}{\alpha} \frac{d}{d\xi} \right) \int_{-\infty}^{\infty} du \frac{e^{-\alpha u^2}}{\sqrt{-i(2u - \xi)}}. \quad (6.2.34)$$

The integral after the simple transformations can be expressed in terms of the parabolic cylinder function⁵²

$$\begin{aligned} \int_{-\infty}^{\infty} du \frac{e^{-\alpha u^2}}{\sqrt{-i(2u - \xi)}} &= \frac{1}{(2\alpha)^{1/4}} \int_0^{\infty} \frac{du}{\sqrt{u}} \exp \left\{ -\frac{u^2}{2} - i\xi \sqrt{\frac{\alpha}{2}} u \right\} \\ &= \frac{1}{(2\alpha)^{1/4}} \exp \left(-\frac{\alpha \xi^2}{8} \right) D_{-1/2} \left(i\xi \sqrt{\frac{\alpha}{2}} \right) = \mathcal{D}_{-1/2} \left(i\xi \sqrt{\frac{\alpha}{2}} \right). \end{aligned} \quad (6.2.35)$$

Substituting this expression into Eq. (6.2.34) and taking into account that⁵²

$$\frac{d}{d\xi} \mathcal{D}_{-1/2} \left(i\xi \sqrt{\frac{\alpha}{2}} \right) = -\frac{\alpha \xi}{2} \mathcal{D}_{-1/2} - i \sqrt{\frac{\alpha}{2}} \mathcal{D}_{1/2},$$

for Λ_{st} instead of Eq. (6.2.33) we obtain

$$\begin{aligned} \Lambda_{st} &= -\frac{2}{m_c!} (-1)^{m_c} \frac{d^{m_c}}{d\alpha^{m_c}} \left\{ (a + b\xi) \mathcal{D}_{-1/2} \left(i\xi \sqrt{\frac{\alpha}{2}} \right) \right. \\ &\quad \left. + ib \sqrt{\frac{2}{\alpha}} \mathcal{D}_{1/2} \left(i\xi \sqrt{\frac{\alpha}{2}} \right) \right\}. \end{aligned} \quad (6.2.36)$$

Simple formulas for decrements $\delta_{st} \propto -\text{Im} \Lambda_{st}$ can be found in two limiting cases.

Case 1: $|\xi| \ll 1$. (Small Phase Advance of Betatron Oscillations over the Bunch Length). Expanding Eq. (6.2.36) into a power series over ξ and separating of the imaginary part of Λ , we have

$$\begin{aligned}\delta_{st} &= \frac{Nr_0 c R_0^2 |m_z|}{8\pi\gamma \sqrt{\varphi_0} \Gamma^2(|m_z|)} \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \right\rangle (\Delta_1 + \Delta_2), \\ \Delta_1 &= \varphi_0 \frac{d \ln v_z}{d \ln \omega_s} \frac{\pi 2^{1/4} \Gamma(|m_c| - 1/4)}{\Gamma(1/4) \Gamma(-1/4) |m_c|!} \Phi_m^{(0)}, \\ \Delta_2 &= \left(\frac{l_\perp}{R_0} \right)^2 \frac{4\pi \Gamma(|m_c| + 1/4)}{2^{1/4} \varphi_0 \Gamma^2(1/4) |m_c|!} \Phi_m^{(1)}.\end{aligned}\quad (6.2.37)$$

The first summand Δ_1 describes the effect of modulation of a betatron phase by the longitudinal oscillations (the head-tail effect). It gives the contribution of different signs to the decrements of the betatron $m_c=0$ and synchrobetatron $m_c \neq 0$ oscillations. As was already discussed, this behavior characteristic for the head-tail effect is caused by the redistribution of decrements between the betatron and synchrobetatron modes because of the modulation of betatron oscillation phases by the longitudinal ones. The appearance of this summand is due to the excitation of the fields by the longitudinal motion of particles.

The second summand Δ_2 occurred due to the excitation of fields by the transverse motion of particles. Its sign obviously does not depend on the multipolarity of synchrotron oscillations m_c . The value Δ_2 comprises the small factor $(l_\perp / R_0)^2$, so the ratio Δ_1 / Δ_2 on the order of magnitude is

$$\frac{\Delta_1}{\Delta_2} \approx \left(\frac{l_b}{l_\perp} \right)^2 \frac{d \ln v_z}{d \ln \omega_s}.$$

One can see that the second summand becomes decisive when

$$\left| \frac{d \ln v_z}{d \ln \omega_s} \right| < \left(\frac{l_\perp}{l_b} \right)^2.$$

Case 2: $\xi \gg 1$. (Large Betatron Phase Advance on the Bunch Length)

Using an asymptotic value of $\mathcal{D}_{-1/2}$,⁵² we get

$$\begin{aligned}\Delta_1 &\approx \frac{2^{5/4} \text{sgn}(dv_z / \ln \omega_s)}{v_z |\xi|^{1/2}} \frac{\Gamma(|m_c| + 1/2)}{|m_c|!} \Phi_m^{(0)}, \\ \Delta_2 &\approx \left(\frac{l_\perp}{R_0} \right)^2 \frac{28 \sqrt{\pi} 2^{3/4}}{|\xi|^{3/2}} \Phi_m^{(1)}.\end{aligned}\quad (6.2.38)$$

The ratio Δ_1 / Δ_2 by the order of magnitude can be estimated by

$$\left| \frac{\Delta_1}{\Delta_2} \right| \sim \frac{|\xi| l_b R_0}{l_\perp^2} > 1,$$

if the machine chromaticity is not too high:

$$\left| \frac{d \ln v_z}{d \ln \omega_s} \right| < \frac{1}{|m_z|} \frac{l_\perp^2}{l_b R_0}.$$

The second term in Eq. (6.2.31) describes the interaction of the bunch with wakes from the previous turns. Keeping the leading terms of the expansion over powers of l_1/R_0 , one can write

$$\Lambda_{mt} = -\frac{2\Phi_m^{(0)}}{m_c!} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dn \frac{\exp(-n_1^2 \varphi_0^2 - 2\pi i l n)}{\sqrt{-i(m_z \nu_z + n)}} \left(\frac{n_1 \varphi_0}{2}\right)^{2|m_c|},$$

or

$$\Lambda_{mt} = -\frac{2\Phi_m^{(0)}}{|m_c|!} \sqrt{\frac{2\pi}{\varphi_0}} \sum_{l=1}^{\infty} e^{2\pi i l m_z \nu_z} \frac{d^{|m_c|}}{d(-\alpha)^{|m_c|}} \Lambda_l(\alpha \xi), \quad (6.2.39)$$

$$\Lambda_l = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{-ix}} \exp\left[-\alpha \left(x - \frac{\xi}{2}\right)^2 - \frac{4\pi i l x}{\varphi_0}\right]. \quad (6.2.40)$$

If the machine chromaticity is small ($|\xi| \ll |m_c|$), Eq. (6.2.40) mainly contributes into increments of betatron modes ($|m_c| = 0$). Otherwise ($|\xi| \gg |m_c|$) its contribution into increments of both betatron and synchrobetatron modes reads

$$\delta_{mt} = \frac{N r_0 c R_0 |m|}{8\pi \gamma \nu_z \Gamma^2(|m_z|)} \left\langle \left(\frac{a_z}{2}\right)^{2|m_z|-2} \right\rangle \Phi_m^{(0)} \Delta_{mt}, \quad (6.2.41)$$

where

$$\Delta_{mt} = \frac{e^{-\xi^2/4}}{|m_c|!} \left(\frac{\xi}{2}\right)^{2|m_c|} \sum_{l=1}^{\infty} \frac{\sin(2\pi i l m_z \nu_z)}{l^{1/2}}. \quad (6.2.42)$$

Note that Δ_{mt} takes its maximum value at $|\xi| = 4|m_c|$, when one has

$$\Delta_{mt}^{\max} \simeq \frac{1}{|m_c|^{1/2}} \sum_{l=1}^{\infty} \frac{\sin(2\pi i l m_z \nu_z)}{l^{1/2}}.$$

If the machine chromaticity is small ($|\xi| \rightarrow 0$), the multiturn part of the decrement of synchrotron oscillations δ_{mt} is proportional to the small numbers $(\varphi_0/2\pi)^{2|m_z|} \ll 1$. Therefore, from the point of view of stability of oscillations the only issue of practical interest is the case of betatron oscillations $|m_c| = 0$ when the decrement δ_{mt} is weakly dependent on the bunch length:

$$\Delta_{mt} \simeq \varphi(m_z \nu_z) = \sum_{l=1}^{\infty} \frac{\sin(2\pi i l m_z \nu_z)}{l^{1/2}}. \quad (6.2.43)$$

The behavior of the function $\varphi(x)$ was already discussed in § 6.1:

$$\varphi(x) = \varphi(x+1),$$

$$\varphi(x) > 0, \quad k < x < k + 1/2,$$

$$\varphi(x) < 0, \quad k + 1/2 < x < k + 1, \quad k = 0, 1, 2, \dots$$

Taking into account these properties, the "stability condition" ($\delta_{mt} > 0$) has the form

$$k < |m_z v_z| < k + 1/2. \quad (6.2.44)$$

For the dipole betatron oscillations $|m_z| = 1$ this inequality yields what we obtained in § 6.1:

$$k < v_z < k + 1/2.$$

Let us consider also an instability of two-dimensional betatron oscillations with respect to the multiturn effects. For oscillations of this type the multiturn part of the decrement is defined by³³

$$\begin{aligned} \delta_{mt} &= \frac{N r_0 c R_0^2 B_{m_1} \Phi_{m_1}^{(0)}}{8 \pi \gamma \Gamma^2(|m_z|) \Gamma^2(|m_x|)} \varphi(\mathbf{m}_1 \mathbf{v}_1), \\ B_{\mathbf{m}_1} &= \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \left(\frac{a_x}{2} \right)^{2|m_x|-2} \left[\frac{\text{sgn}(m_x) \left(\frac{a_z}{m_z} \right)^2}{v_x} \right. \right. \\ &\quad \left. \left. + \frac{\text{sgn}(m_z) \left(\frac{a_x}{m_x} \right)^2}{v_z} \right] \right\rangle \text{sgn}(\mathbf{m}_1 \mathbf{v}_1). \end{aligned} \quad (6.2.45)$$

The value δ_{mt} increases the decrements of oscillations if the following inequalities are satisfied:

$$k - \frac{1 - \text{sgn}(B_{\mathbf{m}_1})}{4} < |\mathbf{m}_1 \mathbf{v}_1| < k + \frac{1 + \text{sgn}(B_{\mathbf{m}_1})}{4}. \quad (6.2.46)$$

For the sum-type excitation $m_x m_z > 0$ the condition in Eq. (6.2.46) determines the bands similar to those defined by the condition in Eq. (6.2.44). For the differential-type excitation $m_x m_z < 0$ the position of the instability bands depends on the sign of B_m . In this case, in the plane v_x, v_z on the line

$$\frac{v_x}{v_z} = \frac{m_x^2 \left\langle \left(\frac{a_x}{2} \right)^{2|m_x|-2} \left(\frac{a_z}{2} \right)^{2|m_z|} \right\rangle}{m_z^2 \left\langle \left(\frac{a_z}{2} \right)^{2|m_z|-2} \left(\frac{a_x}{2} \right)^{2|m_x|} \right\rangle} \quad (6.2.47)$$

the stability band shifts from the region

$$k < |m_x v_x + m_z v_z| < k + 1/2$$

to the region

$$k + 1/2 < |m_x v_x + m_z v_z| < k + 1.$$

As was already said, oscillation stability requires a positive sign for the sum of oscillation decrements. The invariance properties of the total sum of decrements of coherent oscillations with respect to the mode coupling were discussed in §3.4. In the linear approximation [i.e., within the accuracy of the value of order $|\Delta\omega_m|/\min(\omega_\alpha)$] apart from the total sum of decrements, the system invariants are also the partial sums of decrements (summed over not all the multipole numbers but over group m_α). Therefore, their positive sign can also be the necessary condition for the stability of coherent oscillations.

To this end, let us calculate the sums of decrements of synchrobetatron oscillations with the given multipole of betatron motion (m_x and m_z). In the low-frequency range [within the accuracy of values of order $(l_1/l_b) \ll 1$] the sum of decrements of synchrobetatron oscillations has the form

$$\delta_{m_1} = \sum_{m_c=-\infty}^{\infty} \delta_{m_1, m_c} = \frac{Nr_0 c R_0^2 B_{m_1} \Phi_{m_1}^{(0)}}{8\pi\gamma \Gamma^2(|m_z|) \Gamma^2(|m_x|)} \varphi(m_1, v_1). \quad (6.2.48)$$

Note that in contrast with the partial decrements, the sum of decrements in Eq. (6.2.48) does not depend on the machine chromatism $d v_1 / d \ln \omega_s$. Equation (6.2.48) coincides with the decrement of a beam of zero length; therefore it is totally defined by the multiturn effects. The sum of decrements δm_1 is positive when the condition in Eq. (6.2.46) is satisfied. Therefore, the satisfaction of the condition of the beam stability with respect to the multiturn interaction is necessary in any case.

6.3. THE INSTABILITY OF A BUNCHED BEAM DUE TO UNCAPTURED IONS

Another example of instability, that occurs due to interaction of the beam with the quasistationary induced fields is an instability of a proton beam caused by the ionization by protons of the residual gas nuclei in the storage ring. An instantaneous position of the ion trace coincides with the instantaneous position of protons. At low velocities of the ion escaping from the proton beam this leads to the generation of a long memory in the system.

Following Ref. 34 we assume that the production of ions with velocities close to thermal velocities is the most probable. Then the ion generation rate in the unit of the phase space volume reads

$$\frac{dF_i}{dt} = \eta \delta(\mathbf{p}_i) \int d^3 p' [f_0 + \tilde{f}(\mathbf{r}_i, \mathbf{p}', t)], \quad (6.3.1)$$

where the index i corresponds to the variables related to ions; $\eta = v_s \sigma_i n_0$ is the ionization rate per one proton; σ_i is the ionization cross section; and n_0 is the density of residual gas. The first term in the right-hand side of Eq. (6.3.1) describes the stationary flux of ions from the proton beam to the vacuum chamber walls. The action of such a flux on protons leads to some shift in tunes ν_α but does not cause a systematic variation in energy of collective oscillations. Therefore, in further calculations it can be neglected. Then the initial linearized system of equations can be written in the form

$$\frac{\partial \tilde{f}}{\partial t} + \omega \frac{\partial \tilde{f}}{\partial \psi} = e \frac{\partial \tilde{U}}{\partial \psi} \frac{\partial f_0}{\partial \mathbf{I}}, \quad (6.3.2)$$

$$\frac{dF_i}{dt} + [\overline{\mathcal{H}_i}, F_i] = \eta \delta(\mathbf{p}_i) \int d^3 p' \tilde{f}(\mathbf{r}_i, \mathbf{p}', t). \quad (6.3.3)$$

Here $U(\mathbf{r}, t)$ is the field potential of the ion trace acting on protons

$$U(\mathbf{r}, t) = e \int (d\Gamma_\perp dp_\parallel) F_i(\mathbf{r}_i, \mathbf{p}_i, t) \ln |\mathbf{r}_\perp - \mathbf{r}_{\perp i}|. \quad (6.3.4)$$

The horizontal line over U and $\mathcal{H}$ in Eqs. (6.3.2) and (6.3.3) means the averaging over the equilibrium azimuth ϑ_s ; $\mathcal{H}_i$ is a Hamiltonian describing the ion motion in the field of a proton beam. The leading contribution is the discussed instability yield ions produced inside the proton beam where their density is maximal. In this case, the Hamiltonian $\mathcal{H}_i$ is

$$\mathcal{H}_i = \sum_{\alpha=x,z} \left(\frac{p_\alpha^2}{2M_i} - \frac{M_i \Omega_\alpha^2 r_\alpha^2}{2} \right), \quad \Omega_\alpha^2 = \omega_s^2 \nu_{i\alpha}^2 = \frac{2Ne^2}{M_i R_0 a_{\alpha 0}^2},$$

where $a_{\alpha 0}$ is the proton beam size over the degrees of freedom α ; $M_i = A_i m_p$ is the ion mass; A_i is an atomic weight of residual gas; e and m_p are the charge and mass of a proton. The common solution of Eqs. (6.3.2) and (6.3.3) with respect to $\tilde{F}_i$ leads to the integral equation for the Fourier amplitudes $\tilde{f}_{m,\omega}$:

$$\begin{aligned} \Delta \omega_m \tilde{f}_{m,\omega} = & -i\eta \frac{2\pi Ne^2}{R_0} \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \sum_{n=-\infty}^{\infty} \frac{J_{m_c}(n_1 \varphi_c)}{(\mathbf{m} \mathbf{v} + n + i\nu_t)} \\ & \times \int \frac{d^2 k_\perp}{k_\perp^2} b_{\mathbf{m}}(\mathbf{k}_\perp, \mathbf{I}) \int d\Gamma' J_{m_c}(n_1 \varphi'_c) b_{\mathbf{m}}^*(\mathbf{k}_\perp, \mathbf{I}') \tilde{f}_{m,\omega} \\ b_{\mathbf{m}}(\mathbf{k}_\perp, \mathbf{I}) = & \{ \exp(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp) \}_{\mathbf{m},n}, \quad \nu_t = \nu_{ix} + \nu_{iz}, \end{aligned} \quad (6.3.5)$$

whose spectrum determines the spectrum of collective oscillations of protons interacting with the ion trace. For further calculations let us specify the distribution form f_0 :

$$f_0 = \frac{\delta(\varphi_c^2 - \varphi_0^2)}{I_{x0} I_{z0}} \exp\left(-\frac{I_x}{I_{x0}} - \frac{I_z}{I_{z0}}\right), \quad (6.3.6)$$

where the values I_{x0} , I_{z0} , and φ_0 are related to the beam sizes

$$a_\perp^2 = \frac{2R_0 I_{\perp 0}}{p_s v_\perp}, \quad l_b = 2R_0 \varphi_0.$$

Equation (6.3.5) with the distribution from Eq. (6.3.6) can be parameterized:

$$\Lambda_m C_m(\mathbf{k}) = \int \frac{d^2 k_\perp}{k_\perp^2} g_m(k, k') C_m(\mathbf{k}'), \quad (6.3.7)$$

where

$$C_m(k) = \int d\Gamma b_m(\mathbf{k}_\perp, \mathbf{I}) \tilde{f}_{m,\omega},$$

$$g_m(k, k') = \exp\left(-\frac{k_\perp^2 + k_\perp'^2}{4}\right) I_{|m_z|}\left(\frac{k_z k_z'}{2}\right) I_{|m_x|}\left(\frac{k_x k_x'}{2}\right),$$

[$I_m(x)$ is the Bessel function of imaginary argument], while the complex shift of a frequency $\Delta\omega_m$ is expressed in terms of the eigennumbers of Eq. (6.3.7):

$$\Delta\omega_m = i\Lambda_m \frac{2\pi A_i \eta}{\gamma} \left(\frac{m_x v_{ix}^2}{v_x} + \frac{m_z v_{iz}^2}{v_z} \right) \sum_{n=-\infty}^{\infty} \frac{J_{m_c}^2(n_1 \varphi_0)}{m v + n + i v_t}. \quad (6.3.8)$$

The eigennumbers of Eq. (6.3.7) depend only on the multipole numbers m_α . Since $g_m(\mathbf{k}, \mathbf{k}')$ is the positive kernel, all the eigenvalues Λ_m are positive. In this case, the maximum eigenvalue is limited from above by the value

$$\Lambda_m^{\max} \leq \frac{2}{\pi} \frac{\psi\left(m_x + \frac{1}{2}\right) - \psi\left(m_z + \frac{1}{2}\right)}{m_x^2 - m_z^2}, \quad \psi(x) = \frac{d \ln \Gamma(x)}{dx}.$$

We present the oscillation decrements in terms of the sums of contributions from the single-turn and multiturn effects,

$$\delta = \delta_{st} + \delta_{mt}.$$

Let us first estimate the influence of the single-turn effects:

$$\delta_{st} = -\Lambda_m \frac{2\pi A_i \eta}{\gamma} \left(\frac{m_x v_{ix}^2}{v_x} + \frac{m_z v_{iz}^2}{v_z} \right) \int_{-\infty}^{\infty} dn \operatorname{Re} \left[\frac{J_{m_c}^2(n_1 \varphi_0)}{m v + n + i v_t} \right].$$

The simple expressions for the decrement δ_{st} can be obtained in two ultimate cases.

Case 1: $|\zeta| \ll 1$ (Ring with a Small Chromaticity). In this case, the integrand in Eq. (6.3.8) can be expanded into a power series of ζ . After the integration over n we get

$$\delta_{st} = -\eta \frac{4\pi\Lambda_m A_i}{\gamma} \zeta B'_{m_1} \Delta_{m_c}(q_i), \quad (6.3.9)$$

$$q_i = \varphi_0 v_t, \quad B'_{m_1} = \left(\frac{m_x v_{ix}^2}{v_x} + \frac{m_z v_{iz}^2}{v_z} \right)$$

$$\Delta_{m_c}(q_i) = \begin{cases} \frac{2}{\pi} \frac{1}{m_c^2 - 1/4} & q_i \ll 1, \\ \frac{\ln q_i}{q_i} & q_i \gg 1. \end{cases} \quad (6.3.10)$$

Note, that the single-turn summand of decrement δ_{st} is related to the head-tail mixing. In this case, the decrement has the already mentioned characteristic dependence on the multipole number of synchrotron motion: the decrement of betatron oscillations $m_c = 0$ has the sign reverse to decrements of synchrotron oscillations $m_c \neq 0$. It is easy to see that the sum of the decrements δ_{st} over all the longitudinal modes m_c equals zero:

$$\sum_{m_c = -\infty}^{\infty} \delta_{st}(m_c) = 0.$$

Let us mention one more feature of the instability under consideration. At low densities of protons, $q_i \ll 1$, when the ions are in a beam quite a long time, the decrements δ_{st} depend linearly on the number of protons N . With further increase in the proton beam density, the generated ions can escape the beam for the time of order l_b/v_s . Under these conditions the decrement δ_{st} becomes independent of the proton current and it is only determined by the density of nuclei in the residual gas n_0 . Let us give an expression for the decrement δ_{st} for the case of the dipole betatron oscillations:

$$\delta_{st} = -\eta \frac{4\pi\Lambda_1 A_i v_{iz}^2}{\gamma} \varphi_0 \frac{d \ln v_z}{d \ln \omega_s} \Delta_{m_c}(q_i),$$

$$\Lambda_1 < \frac{2}{\pi} [\psi(3/2) - \psi(1/2)] = \frac{4}{\pi}, \quad m_z = 1, \quad m_x = 0. \quad (6.3.11)$$

If the rate of ion escape from the proton beam is not large ($q_i \ll 1$), Eq. (6.3.11) gives

$$\delta_{st} = -\eta \frac{8Nr_p \Lambda_1}{\gamma \beta^2 a_0^2} l_b \frac{d \ln v_z}{d \ln \omega_s} \frac{1}{m_c^2 - 1/4}. \quad (6.3.12)$$

For the dipole betatron oscillations $m_c = 0$ the decrement reads

$$\delta_{st} \simeq 41 \eta \frac{Nr_p}{\gamma \beta^2 a_0^2} l_b \frac{d \ln v_z}{d \ln \omega_s}. \quad (6.3.13)$$

At high densities of the proton beam $q_i \ll 1$ the decrement from Eq. (6.3.11) is

$$\delta_{st} = -\eta \frac{4\pi \Lambda_1 A_i}{\gamma q_i} v_{iz}^2 \ln q_i \xi \simeq -\frac{\pi n_0 \sigma_i v_s \Lambda_1 A_i}{\gamma \varphi_0} \ln q_i \frac{d \ln v_z}{d \ln \omega_s}. \quad (6.3.14)$$

Case 2: $|\xi| \gg 1$. (Ring with a Large Chromaticity). If $|\xi| \gg 1$ the value δ_{st} can be estimated with a logarithmic accuracy by the formula

$$\delta_{st} = -\eta \frac{8\Lambda_m A_i B'_{m_1}}{\gamma |\xi|} L_m, \quad (6.3.15)$$

$$L_m \simeq \begin{cases} \ln(|\xi/m_c|), & |\xi| \gg q_b \\ \ln(q_i/|m_c|), & |\xi| \ll q_i \end{cases}$$

The contribution of multiturn effects into the decrement

$$\delta_{mt} = -\frac{2\pi \Lambda_m A_i B'_{m_1}}{\gamma} \operatorname{Re} \sum_{l \neq 0} \int_{-\infty}^{\infty} dn \frac{e^{2\pi i l n} J_{m_c}^2(n_1 \varphi_0)}{m\nu + n + i\nu_t} \quad (6.3.16)$$

is only substantial for the betatron modes $m_c = 0$ as for the synchrotron modes the value of decrement δ_{mt} , as seen from Eq. (6.3.8), is proportional to the small number $(\varphi_0/2\pi)^{2|m_z|} \ll 1$. Performing in Eq. (6.3.16) the integration over n and summation over l we get the expression for the decrement δ_{mt} :

$$\delta_{mt} = -\eta \frac{2\pi \Lambda_m A_i \Re_m}{\gamma} \begin{cases} 2\pi \frac{\sin(2\pi |\mathbf{m}_1 \mathbf{v}_1|)}{\exp(2\pi |\mathbf{m}_1 \mathbf{v}_{i1}|)}, & 2\pi \nu_t \gg 1, \\ \frac{\pi}{2} \cot(\pi |\mathbf{m}_1 \mathbf{v}_1|), & 2\pi \nu_t \ll 1, \end{cases} \quad (6.3.17)$$

$$\Re_m = \operatorname{sgn}(\mathbf{m}_1 \mathbf{v}_1) B'_{m_1}.$$

The contribution of multiturn effects δ_{mt} into the oscillation decrement is positive if

$$k - \frac{1 + \operatorname{sgn}(\Re_m)}{4} \leq |\mathbf{m}_1 \mathbf{v}_1| \leq k + \frac{1 - \operatorname{sgn}(\Re_m)}{4}. \quad (6.3.18)$$

For the sum-type $m_x m_z > 0$ or one-dimensional $m_x m_z = 0$ modes the value $\Re_m$ is positive. In this case, the stability condition (6.3.18) can be written in a simpler form:

$$\begin{aligned} m_x m_z > 0, \quad k - 1/2 \leq |m_x v_x + m_z v_z| \leq k, \\ m_x = 0, \quad m_z > 0, \quad k - 1/2 \leq m_z v_z \leq k. \end{aligned} \quad (6.3.19)$$

Note that the stability conditions (6.3.18) and (6.3.19) are opposite to the bunch stability conditions with respect to the resistive-wall instability. This is connected with the fact that in the case considered, due to repulsion of the proton and ion, the induced fields act against the deflection of protons from the equilibrium orbit.

The sum of decrements δ_m summed over all longitudinal modes m_c coincides with the decrement of betatron oscillations, calculated assuming the zero length of the bunch (δ_{mt}). This means that for the stability of excitations with the given m_x and m_z conditions (6.3.18) and (6.3.19) should be satisfied in any case.

6.4. INTERACTION OF A BUNCHED BEAM WITH LOW-Q CAVITY

The coherent interaction of a beam with the resonant elements of the vacuum chamber were already discussed in § 4.1 and § 4.2. But the major attention there was paid to the case where such an interaction is the resonant one. Now we consider a case of the same importance: when the resonator wakes decay faster than the particle revolution period ($\lambda_k \gg \omega_s$) and the working point of the ring is far from the machine resonances.

The equation defining the spectrum of the beam coherent oscillations interacting with the resonator was obtained in § 4.2 and has the form

$$\begin{aligned} \Delta \omega_m f_m = N e^2 \left(\mathbf{m} \frac{\partial f_0}{\partial \mathbf{I}} \right) \sum_{n=-\infty}^{\infty} \frac{(\nabla \mathbf{A}_k)_{mn}}{\omega_k^2 - (\omega_{mn} + i\lambda_k)^2} \\ \times J_{m_c}(n_1 \varphi_c) \int d\Gamma' (\nabla \mathbf{A}_k)_m^{n*} J_{m_c}(n_1 \varphi'_c) f_{m,\omega}. \end{aligned} \quad (6.4.1)$$

To estimate the decrements let us use the model with

$$f_0(\mathbf{I}_1, \varphi_c) \propto \delta(\varphi_c^2 - \varphi_0^2),$$

where we assume the bunch length from the region

$$\omega_k \varphi_0 \gg \omega_s$$

or, which is the same,

$$l_b \gg \bar{\lambda}_k.$$

$\bar{\lambda}_k$ is the wave length of the cavity mode excited by the beam. Taking into account that harmonics $(\mathbf{vA}_k)_{mn}$ are indeed the power functions of $I_\perp$,

$$(\mathbf{vA}_k)_{mn} \propto (I_\alpha)^{m_\alpha/2},$$

from Eq. (6.4.1) we obtain the formula defining the eigenfrequencies of oscillations

$$\Delta\omega_m = - \sum_{n=-\infty}^{\infty} \frac{Ne^2 \left\langle \mathbf{m}_\perp \frac{\partial}{\partial I_\perp} |(\mathbf{vA}_k)_{mn}|^2 J_{m_c}^2(n_1\varphi_0) \right\rangle}{\omega_k^2 - (\omega_{mn} + i\lambda_k)^2}. \quad (6.4.2)$$

Hence, for the decrements, for instance, of synchrobetatron oscillations $m_\perp^2 \neq 0$ we have

$$\delta_m = \sum_{n=-\infty}^{\infty} V_{mn} J_{m_c}^2(n_1\varphi_0) \left[\frac{\lambda_k}{(\omega_k - \omega_{mn})^2 + i\lambda_k^2} - \frac{\lambda_k}{(\omega_k + \omega_{mn})^2 + i\lambda_k^2} \right],$$

$$V_{mn} = \frac{Ne^2}{2\omega_k} \left\langle \mathbf{m}_\perp \frac{\partial}{\partial I_\perp} |(\mathbf{vA}_k)_{mn}|^2 \right\rangle. \quad (6.4.3)$$

Using Eq. (3.1.26) let us represent δ_m as a sum

$$\delta_m = \delta_{st} + \delta_{mt}$$

of the single-turn term

$$\delta_{st} = \int_{-\infty}^{\infty} dn V_{mn} J_{m_c}^2(n_1\varphi_0) \left[\frac{\lambda_k}{(\omega_k - \omega_{mn})^2 + i\lambda_k^2} - \frac{\lambda_k}{(\omega_k + \omega_{mn})^2 + i\lambda_k^2} \right] \quad (6.4.4)$$

and the multiturn term

$$\delta_{mt} = \sum_{l \neq 0} \int_{-\infty}^{\infty} dne^{2\pi i l n} V_{mn} J_{m_c}^2(n_1\varphi_0) \times \left[\frac{\lambda_k}{(\omega_k - \omega_{mn})^2 + i\lambda_k^2} - \frac{\lambda_k}{(\omega_k + \omega_{mn})^2 + i\lambda_k^2} \right]. \quad (6.4.5)$$

Let us first calculate δ_{st} . Taking as the new variable in the integrand of Eq. (6.4.4) $u = n_1\varphi_0$, we rewrite Eq. (6.4.4) in the form

$$\delta_{st} = \int_{-\infty}^{\infty} du v_m(u - v - \xi) J_{m_c}^2(u) \left[\frac{\lambda}{(v_k - u + \xi)^2 + \lambda^2} - \frac{\lambda}{(v_k + u - \xi)^2 + \lambda^2} \right],$$

$$v_m(u - v - \xi) = \frac{1}{\omega_s} V_{m, (u - v - \xi)/\varphi_0}, \quad \lambda = \frac{\lambda_k}{\omega_s} \varphi_0,$$

$$v = m_{\perp} v_{\perp} \varphi_0, \quad v_k = \frac{\omega_k \varphi_0}{\omega_s}, \quad \xi = m_{\perp} \frac{dv_{\perp}}{d \ln \omega_s} \varphi_0. \quad (6.4.6)$$

We shall perform further calculations with the natural assumption of $\lambda \ll 1$ which means that the induced fields cannot vanish for the time required for the bunch to pass the resonator, $l_b/v \gg 1/\bar{\lambda}_k$. In addition, let us assume that $v_k \gg |\xi|$ and the beam length is substantially larger than the resonator azimuthal length $\varphi_0 \gg \vartheta_0$.

The function in brackets in Eq. (6.4.6) is concentrated in the vicinity of maxima at

$$u = v_k + \xi, \quad u = -v_k + \xi$$

with the width of $\Delta u \simeq \lambda$. In the given assumptions the functions $v_m(u)$ and $J_{m_c}^2(u)$ vary slowly in these intervals of Δu . Replacing in the integrand

$$\frac{\lambda}{\lambda^2 + x^2} \rightarrow \pi \delta(x),$$

instead of Eq. (6.4.6) we get

$$\delta_{st} = \pi [J_{m_c}^2(v_k + \xi) v_m(v_k - v) - J_{m_c}^2(v_k - \xi) v_m(v_k + v)]$$

$$\simeq -\pi J_{m_c}^2(v_k) v \frac{dv_m(v_k)}{dv_k} + \pi \xi v_m(v_k) \frac{d}{dv_k} J_{m_c}^2(v_k). \quad (6.4.7)$$

The first summand in this expression describes the effect of fast damping of the coherent oscillations. It does not depend on the sign of the machine chromatism and is positive, when the inequality

$$v \frac{d}{dv_k} v_m(v_k) < 0 \quad (6.4.8)$$

is satisfied. The physical meaning of this condition is obvious. From Eq. (6.4.6) it follows that δ_{st} is defined by the difference in contributions of the differential-type $\omega_{mn} = \omega_k$ and the sum-type $\omega_{mn} = -\omega_k$ resonances. In § 4.2 it was found that the coupling of the beam and resonator stabilized the coherent oscillations in the vicinity of the differential-type resonance and it introduces an instability in the vicinity of the sum-type resonance. In this way the condition in Eq. (6.4.8) requires that the contribution of the differential-type resonance in δ_{st} be the leading one.

The second term in Eq. (6.4.7) is proportional to ζ . It describes the head–tail effect. Since

$$\sum_{m_c=-\infty}^{\infty} J_{m_c}^2(x) = 1,$$

the sign of the contribution in δ_{st} depends on m_c . In the range of parameters $|m_c| + 1 \gg \nu_k \gg \lambda$, the function $J_{m_c}^{2(\nu_k)}$ in Eq. (6.4.7) can be replaced by

$$J_{m_c}^2(\nu_k) \simeq \begin{cases} 1 - \frac{\nu_k^2}{2}, & m_c = 0, \\ \frac{1}{(|m_c|!)^2} \left(\frac{\nu_k}{2}\right)^{2|m_c|}, & |m_c| \neq 0, \end{cases}$$

which after substitution into Eq. (6.4.7) gives

$$\begin{aligned} \delta_{st} &\simeq -\pi \nu \frac{dv_m}{d\nu_k} - \pi v_m(\nu_k) \zeta \frac{\nu_k}{2}, & m_c = 0, \\ \delta_{st} &\simeq \pi v_m(\nu_k) \frac{\zeta}{|m_c|} \frac{(\nu_k/2)^{2|m_c|-1}}{[(|m_c|-1)!]} 2, & |m_c| \neq 0. \end{aligned} \quad (6.4.9)$$

As was already mentioned, the contributions of the head–tail effect in the decrements of betatron and synchrobetatron oscillations of a short bunch have the opposite signs. At $\zeta v_m(\nu_k) > 0$ it decreases the decrements of betatron mode due to an increase in decrements of synchrobetatron modes. If the variations of $v_m(\nu_k)$ are fast

$$\frac{d \ln |v_m(\nu_k)|}{d\nu_k} > \left| \frac{\nu_k}{2\nu} \zeta \right|,$$

this can be used for the simultaneous damping of the betatron and synchrobetatron oscillations of a beam.

In the region $\nu_k \gg |m_c| + 1$ for the estimation of decrements one can use the asymptotic approximation

$$J_{m_c}^2(\nu_k) \simeq \frac{2}{\pi \nu_k} \cos^2 \left(\nu_k - \frac{\pi}{2} \left[|m_c| + \frac{1}{2} \right] \right).$$

Then the contribution of the main terms in ν_k^{-1} yields

$$\delta_{st} \simeq -\frac{2\nu}{\nu_k} \frac{dv_m}{d\nu_k} + (-1)^{m_c} \frac{2\zeta v_m(\nu_k)}{\nu_k} \cos(2\nu_k). \quad (6.4.10)$$

The fact that the head-tail effect in Eq. (6.4.10) does not increase with the growth of m_c is connected with the high frequency of a field oscillator interacting with a beam ($\nu_k \gg 1$) assumed here.

Let us give one more formula for δ_{st} for the most important oscillations, $m_1^2 = 1$. Taking in Eq. (6.4.9), for instance, $m_x = 0$, $m_z^2 = 1$, we obtain

$$\delta_{st} = -\frac{Nr_0c^3}{\gamma l_b \omega_k^2} \left\{ \frac{d\Phi(n)}{dn} + (-1)^{m_c} \frac{d \ln \nu_z}{d \ln R_0} \Phi(n) \cos(2\nu_k) \right\}_{n=\nu_k},$$

$$\Phi(n) = \left| \left(R_0 \frac{\partial A_{||}}{\partial z} + m_z \nu_z A_z \right)_{nm} \right|_{r_1=0}^2,$$

$$l_b = 2\varphi_0 R_0, \quad r_0 = e^2/mc^2, \quad \gamma \gg 1. \quad (6.4.11)$$

For the calculation of the multiturn summand of the decrement δ_{mt} it is convenient to start directly from Eq. (6.4.2):

$$\delta_{mt} = \text{Im} \left[\int_{-\infty}^{\infty} dn \sum_{l \neq 0} \frac{2\omega_k V_{m\alpha} J_{m_c}^2(n_1 \varphi_0) e^{2\pi i l n}}{\omega_k^2 - (\omega_{mn} + i\lambda_k)^2} \right]. \quad (6.4.12)$$

The integral over n in Eq. (6.4.12) is defined by the residues of the integrand in the simple poles

$$n\omega_s = -m_\alpha \omega_\alpha - i\lambda_k \pm \omega_k.$$

Neglecting the values of order $|\xi|/\nu_k$ and ν/ν_k we rewrite Eq. (6.4.12) in the form

$$\delta_{mt} = 2\pi J_{m_c}^2(\nu_k) V_{m,\nu_k} \text{Re} \sum_{l=-\infty}^{-1} e^{i\lambda_k T_s} \{ \exp[il(-2\pi m_\alpha \nu_\alpha + \omega_k T_s)] - \exp[il(-2\pi m_\alpha \nu_\alpha - \omega_k T_s)] \}. \quad (6.4.13)$$

The positive l do not contribute to Eq. (6.4.12). The series over l in Eq. (6.4.13) can be calculated exactly, though there is no need to do this since by definition $\lambda_k \gg \omega_s$ so $\exp(\lambda_k T_s) \ll 1$. Therefore, within an exponential accuracy from Eq. (6.4.13) we obtain

$$\delta_{mt} \simeq \frac{2\pi N e^2}{\omega_k} J_{m_c}^2(\nu_k) \left\langle \mathbf{m}_1 \frac{\partial}{\partial \mathbf{I}_1} |(\mathbf{v} \mathbf{A}_k)_{m,\nu_k}|^2 \right\rangle$$

$$\times e^{-\lambda_k T_s} \sin(\omega_k T_s) \sin(2\pi m_\alpha \nu_\alpha). \quad (6.4.14)$$

From this expression it is seen that in the general case the sign of δ_{mt} depends on the oscillation phase advances per one turn of both the particle and field. For the

one-dimensional betatron oscillations $m_x m_z = 0$ the values δ_{mt} are positive when the conditions^{28,91}

$$k - \frac{1 - \text{sgn}(\sin(\omega_k T_s))}{4} < |m_\alpha v_\alpha| < k + \frac{1 - \text{sgn}[\sin(\omega_k T_s)]}{4} \quad (6.4.15)$$

are satisfied, where k are integer numbers, and $\alpha = x, z$. In particular, for the dipole oscillations $m_\alpha = \pm 1$ and $\sin \omega_k T_s > 0$ the condition (6.4.15) coincides with the stability condition for the short bunch with respect to the resistive instability

$$k < v_\alpha < k + 1/2.$$

The particular expressions for the decrements δ_{mt} can easily be obtained from Eq. (6.4.15).

6.5. MULTIBUNCH INSTABILITIES

If the harmonic number q of the RF system of a ring is larger than 1, the beam can contain q bunches. Each bunch may generate wakes in its surroundings. Thus, provided those wakes are long enough, inside the beam bunches can interact through the external system (or talk to each other). Since this generally increases the number of freedoms of the system, such an interaction can substantially modify coherent spectra. This scope of questions was first considered in Refs. 92 and 93. The formalism developed here enables one to extend the results of Refs. 92 and 93 to the case of oscillations of the arbitrary configurations.

As was already mentioned, the interaction of bunches is provided by the noticeable duration of memory of the external system. This circumstance may result in resonance multi-bunch instabilities and in multi-bunch instabilities due to beam interaction with a low- Q external system. When calculating, one actually takes into account two factors. First, the beam current in corresponding equations of § 3.1 equals the sum of the currents of individual bunches

$$\mathbf{j} = \sum_{a=1}^q \mathbf{j}_a(\mathbf{r}, t). \quad (6.5.1)$$

Second, $\vartheta_a = 2\pi a/q + \omega_s t + \varphi_a$, where a marks the bunches in the beam. Since in this section we describe the interaction of the beam with a resonant cavity, the main integral equation (3.1.11) can be written

$$\begin{aligned} \Delta \omega_m f_{m,\omega}^a = & N e^2 \left(\mathbf{m} \frac{\partial f^a}{\partial \mathbf{I}} \right) \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\pi i n a}{q}\right) \frac{[V_n \exp(in\varphi)]_m}{\omega_k^2 - (\omega + n\omega_s + i\lambda_k)^2} \\ & \times \int d\Gamma' \sum_{b=1}^q \exp(-2\pi i n b/q) (V_n e^{in\varphi})_m^* f_{m,\omega}^b. \end{aligned} \quad (6.5.2)$$

6.5.1. The resonance interaction of a beam with the high-Q resonator corresponds to the mode with $\omega_{mn} \simeq \omega_k$. Therefore, neglecting off-resonance terms in Eq. (6.5.2), we obtain

$$\Delta\omega_m f_m^a = N e^2 \left(\mathbf{m} \frac{\partial f^a}{\partial \mathbf{I}} \right) e^{2\pi i n a / q} \frac{(V_n e^{i n \varphi})_m}{2\omega_k (\Delta - \Delta\omega_m - i\lambda_k)} \times \sum_{b=1}^q e^{-2\pi i n b / q} \int d\Gamma' f_m^b (V_n e^{i n \varphi'})_m^* \quad (6.5.3)$$

By the substitution

$$\chi_{mn} = \sum_{b=1}^q \exp(-2\pi i n b / q) f_m^b \quad (6.5.4)$$

and in subsequent summation of Eqs. (6.5.3) over a , Eq. (6.5.3) is transformed into the following integral equation

$$\Delta\omega_m \chi_{mn} = - \frac{e^2 (V_n e^{i n \varphi})_m}{2\omega_k (\Delta\omega_m - \Delta + i\lambda_k)} \sum_{a=1}^q N_a \left(\mathbf{m} \frac{\partial f^a}{\partial \mathbf{I}} \right) \int d\Gamma (V_n e^{i n \varphi})_m^* \chi_{mn}$$

Only the factor

$$\sum_{a=1}^q N_a \left(\mathbf{m} \frac{\partial f^a}{\partial \mathbf{I}} \right)$$

distinguishes this equation from Eq. (4.2.11) whose spectrum was detailed in §4.2. Hence, one can conclude that at the resonant interaction the coherent oscillation spectra of the single bunch and the system of bunches are described by the same equation. This result could be expected beforehand. The dispersion equations of the resonance instability for both the bunched and the coasting beams also coincide.

6.5.2. The oscillation spectrum describing the beam interaction with the low-Q resonator has more complicated properties. For the sake of simplicity let us carry out the calculation for the axially longitudinal oscillations. In this case, Eq. (6.5.2) has the form

$$\Delta\omega_m f_m^a = N_a e^2 m_z \frac{\partial f^a}{\partial I_z} \sum_n \exp(2\pi i n a / q) \frac{V_{mn}(\mathbf{I}_\perp) J_{m_c}(n_1 \varphi_c)}{\omega_k^2 - (\omega_{mn} + i\lambda_k)^2} \times \int d\Gamma \sum_{b=1}^q \exp(-2\pi i n b / q) V_{mn}^* J_{m_c}(n_1 \varphi_c') f_m^b \quad (6.5.5)$$

Let us divide in Eq. (6.5.5) the summation over n into two parts:

$$\sum_n \alpha_n = \sum_{l=-\infty}^{\infty} \sum_{p=0}^{q-1} \alpha_{lq+p}.$$

Then, taking into account the fact that V_{mn} actually is the power function of $I_{\perp}$,

$$V_{mn} = I_z^{|m_z|/2} g_{mn},$$

using these substitutions and

$$\chi_{mp} = I_z^{|m_z|/2} \frac{\partial f^a}{\partial I_z} \bar{\chi}_{mp}(\varphi_c),$$

we find the integral equation for $\bar{\chi}_{mp}(\varphi_c)$

$$\begin{aligned} \Delta\omega_m \bar{\chi}_{mp} = & \sum_{p'=0}^{q-1} \rho(p-p', \varphi_c) \sum_{l=-\infty}^{\infty} |g_m(lq+p)| \frac{J_{m_c}^2(\varphi_c[lq+p])}{\omega_k^2 - (m\omega + \omega_s[lq+p'] + i\lambda_k)^2} \\ & \times \int_0^{\infty} d\varphi'_c \varphi'_c J_{m_c}(\varphi'_c[lq+p]) \bar{\chi}_{m,p'}(\varphi'_c), \end{aligned} \quad (6.5.6)$$

$$\rho(\Delta p, \varphi_c) = -\rho(\varphi_c) m_z |m_z| \langle I_z^{|m_z|-1} \rangle \sum_{a=1}^q N_a e^2 e^{(-2\pi i a/q) \Delta p},$$

$$\Delta p = p - p'. \quad (6.5.7)$$

In Eq. (6.5.7) we assumed that the distribution functions f^a are the same for all the bunches. More than that, if the populating of bunches $N_a = N/q$ is the same, Eq. (6.5.6) can be simplified even more:

$$\rho(p-p') \propto \delta_{p,p'}$$

and consequently

$$\begin{aligned} \Lambda_m \bar{\chi}_{m,p}(\varphi_c) = & \rho(\varphi_c) \sum_{l=-\infty}^{\infty} |g_m(lq+p)|^2 J_{m_c}(\varphi_c[lq+p]) \\ & \times \frac{\int_0^{\infty} d\varphi' \varphi' J_{m_c}(\varphi'[lq+p]) \bar{\chi}_{m,p'}(\varphi'_c)}{\omega_k^2 - [m_z \omega_z + \omega_s(lq+p) + i\lambda_k]^2}. \end{aligned} \quad (6.5.8)$$

The eigenvalues Λ_m of this equation are related to $\Delta\omega_m$ through the equation

$$\Delta\omega_m = -Ne^2 m_z |m_z| \langle I_z^{|m_z|-1} \rangle \Lambda_m.$$

The fact that the values p classify the solutions of Eq. (6.5.8) reflects the symmetry of the stationary state of the beam with respect to the discrete rotations on $\Delta\vartheta \propto 2\pi/q$.

The main features of a spectrum of Eq. (6.5.8) can be studied in quite a general form. Using Eq. (3.1.26), we can change the summation over harmonics numbers l by the summation over periods (over bunches and turns). In this case, the kernel in Eq. (6.5.8) can be written in the form

$$K = K_{sb} + K_{mb}, \quad (6.5.9)$$

where

$$K_{sb} = \rho(\varphi_c) \int_{-\infty}^{\infty} dl \frac{|g_m(lq+p)|^2 J_{m_c}(\varphi_c[lq+p]) J_{m_c}(\varphi_c[lq+p])}{\omega_k^2 - [m_z \omega_z + \omega_s(lq+p) + i\lambda_k]^2} \quad (6.5.10)$$

describes the single-turn interaction of a bunch with its own wake, while

$$K_{mb} = \rho(\varphi) \sum_{\mu \neq 0} \int_{-\infty}^{\infty} dl \frac{e^{2\pi i \mu l} |g_m(lq+p)|^2 J_{m_c}(\varphi_c[lq+p]) J_{m_c}(\varphi_c[lq+p])}{\omega_k^2 - [m_z \omega_z - \omega_s(lq+p) + i\lambda_k]^2} \quad (6.5.11)$$

describes both the interaction of bunches and multiturn effects. The influence of the single-turn effects due to the interaction of a single bunch with low-Q cavity on the spectrum of its coherent oscillations was detailed in §6.4. Therefore, let us discuss the influence of the second term in Eq. (6.5.9). As the azimuthal dimensions of resonator ϑ_0 and the bunch φ_0 are small compared to $2\pi/q$ the integration over l in Eq. (6.5.11) is reduced to calculating the residues in poles

$$l = -\frac{i\lambda_k}{q\omega_s} - \frac{p}{q} - \frac{m_z \nu_z}{q} \pm \frac{\omega_k}{q\omega_s}.$$

In the absence of resonances $\omega_k = q\omega_s$ and provided $|m_z|$ is not too large ($|m_z|\omega_z \ll \omega_k$) the result can be written in the form

$$\begin{aligned} K_{mb} = & \rho(\varphi_c) \left| g_m\left(\frac{\omega_k}{\omega_s}\right) \right|^2 J_{m_c}\left(\varphi_c \frac{\omega_k}{\omega_s}\right) J_{m_c}\left(\varphi_c \frac{\omega_k}{\omega_s}\right) \\ & \times \frac{2\pi}{q\omega_s \omega_k} \sum_{\mu=1}^{\infty} \sin\left(\frac{2\pi\mu\omega_k}{q\omega_s}\right) \\ & \times \exp\left\{ -\frac{2\pi\mu\lambda_k}{q\omega_s} - i\frac{2\pi\mu}{q}(p + m_z \nu_z) \right\}. \end{aligned} \quad (6.5.12)$$

The kernel $\rho(\varphi_c) J_{m_c}(\bar{\nu}\varphi_c) J_{m_c}(\bar{\nu}\varphi'_c)$ is positively determined. Therefore, the signs of eigennumbers K_{mb} are determined by the sign of the factor

$$\sum_{\mu=1}^{\infty} \sin\left(\frac{2\pi\mu\omega_k}{q\omega_s}\right) \exp\left\{-\frac{2\pi\mu\lambda_k}{q\omega_s} - \frac{2\pi i\mu}{q}(p+m_z v_z)\right\} \frac{u \sin\left(\frac{2\pi\omega_k}{q\omega_s}\right)}{1-2u\cos\left(\frac{2\pi\omega_k}{q\omega_s}\right)+u^2},$$

$$u = \exp\left\{-\frac{2\pi\lambda_k}{q\omega_s} - \frac{2\pi i}{q}(p+m_z v_z)\right\}. \quad (6.5.13)$$

Correspondingly, their contributions into decrements of oscillations will be positive if the following conditions are satisfied:

$$m_z \sin\left(\frac{2\pi\omega_k}{q\omega_s}\right) \text{Im}(u) < 0,$$

or

$$m_z \sin\left(\frac{2\pi}{q}(p+m_z v_z)\right) \sin\frac{2\pi\omega_k}{q\omega} > 0. \quad (6.5.14)$$

The latter can be rewritten in the form of inequalities

$$k - \frac{1 - \text{sgn}(\phi_m)}{4} < \frac{m_z v_z + p}{q} < k + \frac{1 + \text{sgn}(\phi_m)}{4},$$

$$\phi_m = m_z \sin\frac{2\pi\omega_k}{q\omega_s}, \quad k=0,1,\dots \quad (6.5.15)$$

If $q=1$ and $p=0$, respectively, Eq. (6.5.15) is transformed into Eq. (6.5.14). It is seen that for a large number of bunches the stability condition Eq. (6.5.15) is more strong.

Note also that if in Eq. (6.5.14) q tends to ∞ , the stability condition of q bunches is transformed to the stability condition for the coasting beam (3.1.21),

$$m_z(p+m_z v_z) > 0. \quad (6.5.16)$$

This shift is quite natural. The wave number p describes the relative motion of bunches. But with the given circumference of the machine Π and $q \rightarrow \infty$ such a motion becomes the main collective motion of a beam.

As could be expected, the contribution of bunch interactions through the external system is determined by the factor $\exp(-2\pi\lambda_k/q\omega_s)$ in Eqs. (6.5.12) and (6.5.13). Therefore, if the Q -factor of a system is so small that

$$\lambda_k \gg q\omega_s/2\pi,$$

the spectra of coherent oscillations of q bunches will, within an exponential accuracy, be the same as spectra for coherent oscillations of individual bunches. The latter property is actually the corollary of the theorem on the sum of

decrements of coherent modes. The system memory relates the oscillations of individual bunches. As a consequence, the stability of the beam coherent oscillations can be provided by the special filling of separatrices.

Let, for example, the first q_1 separatrices be equipopulated ($N_a = N/q$, $a = 1, \dots, q_1$) and the rest $(q - q_1)$ be empty. If, moreover, the system Q-factor satisfies ($\lambda_k \gg \omega_s/2\pi$), then neglecting the terms of the order of $\exp(-2\pi\lambda_k/\omega_s)$, the system of integral equations (6.5.3) can be transformed into the system having the triangular matrix of kernels. The eigenvalues of such a system are the eigenvalues of diagonal kernels. Apart from singlebunch instabilities, the interaction of bunches in such a beam may result in the beam break-up instability (see in §5.6).

6.6. COHERENT PHENOMENA IN STORAGE RING VEPP-2

The development of the theory of coherent oscillations in this and the three preceding chapters was to a great extent stimulated by the search for ways to increase the stored current already in the first storage rings. Here, for the sake of illustration, we describe the results of experimental study of coherent oscillations of beams carried out on the storage ring VEPP-2 at the Institute of Nuclear Physics (Novosibirsk), where nearly all the instabilities of coherent oscillations of the bunched beam described here were first observed and thoroughly studied. The characteristic feature of the coherent effect observed on VEPP-2 was the high level of motion coherence. On other installations (such as ACO and ADONE) the coherence level of beam excitation was of a few percent order. The reason of such a difference in the behavior of beams in the weak-focusing ring of VEPP-2 and in the strong focusing storage rings was not revealed.

On the storage ring VEPP-2 at sufficiently large currents the following coherent effects have been observed: the phase, radial, and vertical instability and the fast damping of vertical coherent oscillations.

6.6.1. The design of the electron-positron storage ring VEPP-2 was detailed in Ref. 94 and 95. Therefore, here we shall only discuss some of its details.

The storage ring was the weak focusing racetrack with four similar straight sections. Two straight sections of the ring [the first and third; (See Fig. 6.3)] were used for the injection of electrons and positrons, the second for experiments. Some parameters of the ring VEPP-2 are

Bending radius	1.5 m
$B\rho$	2.25 Tm
Straight sections	4×0.632 m
The field index n	0.6
The aperture:	
Δz	± 40 mm
Δx	± 70

The single-turn injection into the storage ring was performed using the elec-

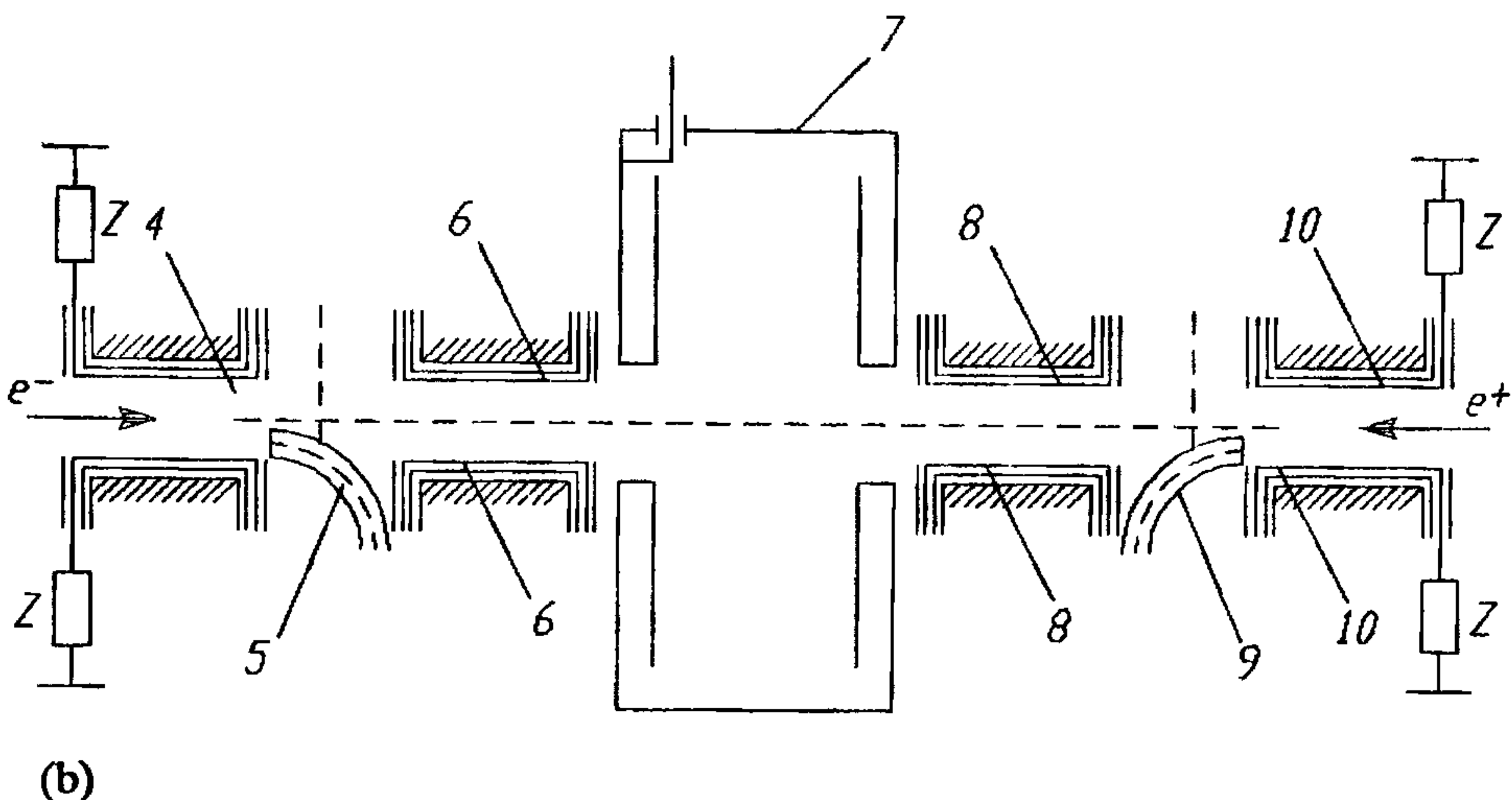
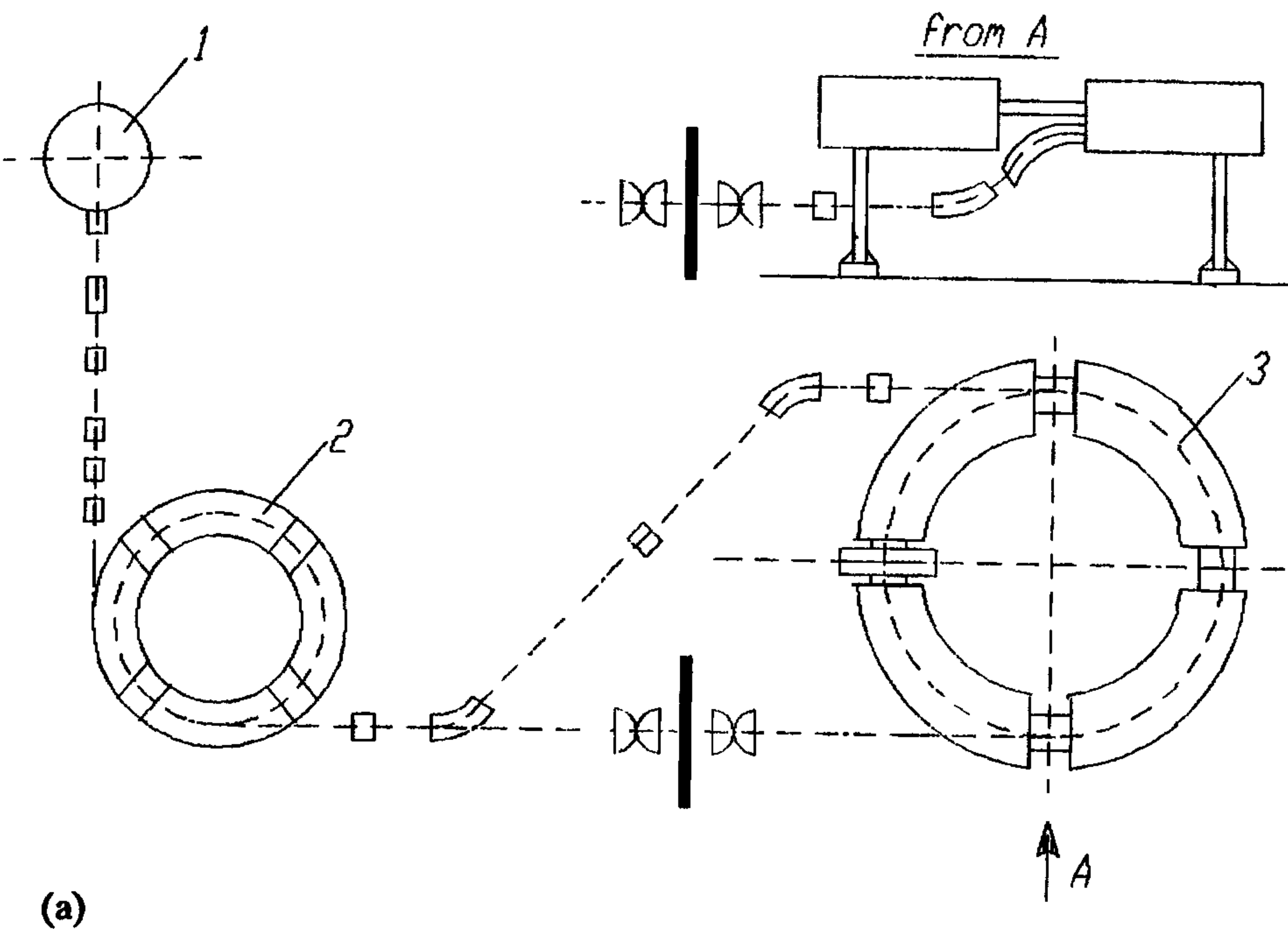


FIG. 6.3. General layout (a) and the longitudinal developed views (b) of the ring VEPP-2, 1—the pulsed linac, 2—booster B-3M, 3—storage ring, 4—the electron inflector, 5—the positron septum, 6, 8—separation plates ($Z=25$ ohm, 7—accelerating cavity, 9—the electron septum; 10—the positron inflector

tromagnetic inflector. The inflector plates were located in the third and fourth quadrants (See Fig. 6.3), and each plate had the characteristic impedance of 25 ohm. In addition (See Figs. 6.4 and 6.5), in the storage ring there was a system of plates (with wave resistance of 25 ohm) for the vertical beams separation and the ion clearing field electrodes ($Z=50$ ohm).

The fine tuning of the frequencies of betatron oscillations was performed using eight current buses introduced into the vacuum chamber of the fourth section. The system of commutation enabled the switching of these bases in both sextupole and quadrupole modes. In addition, the betatron oscillation frequencies could be varied by the plates for the ion section. The joint action of these plates and current corrections provided suppression of the ring chromaticity as well as the required tuning of the operation frequencies of betatron oscillations ($\Delta\nu = \pm 0.02$, $d\nu/dx=0.012$). The last option was especially important for obtaining the maximum accumulation rate of positrons.

This complex VEPP-2 was upgraded twice. After the second modernization the system of correction coils was changed and a new cavities was installed. During its design, the experience of operation with large currents was taken into account. The frequency characteristic was carefully studied up to the frequency 1700 MHz and all the unfavorably tuned parasitic cavities were studied retuned to avoid beam instabilities. The frequency characteristics of the plates were

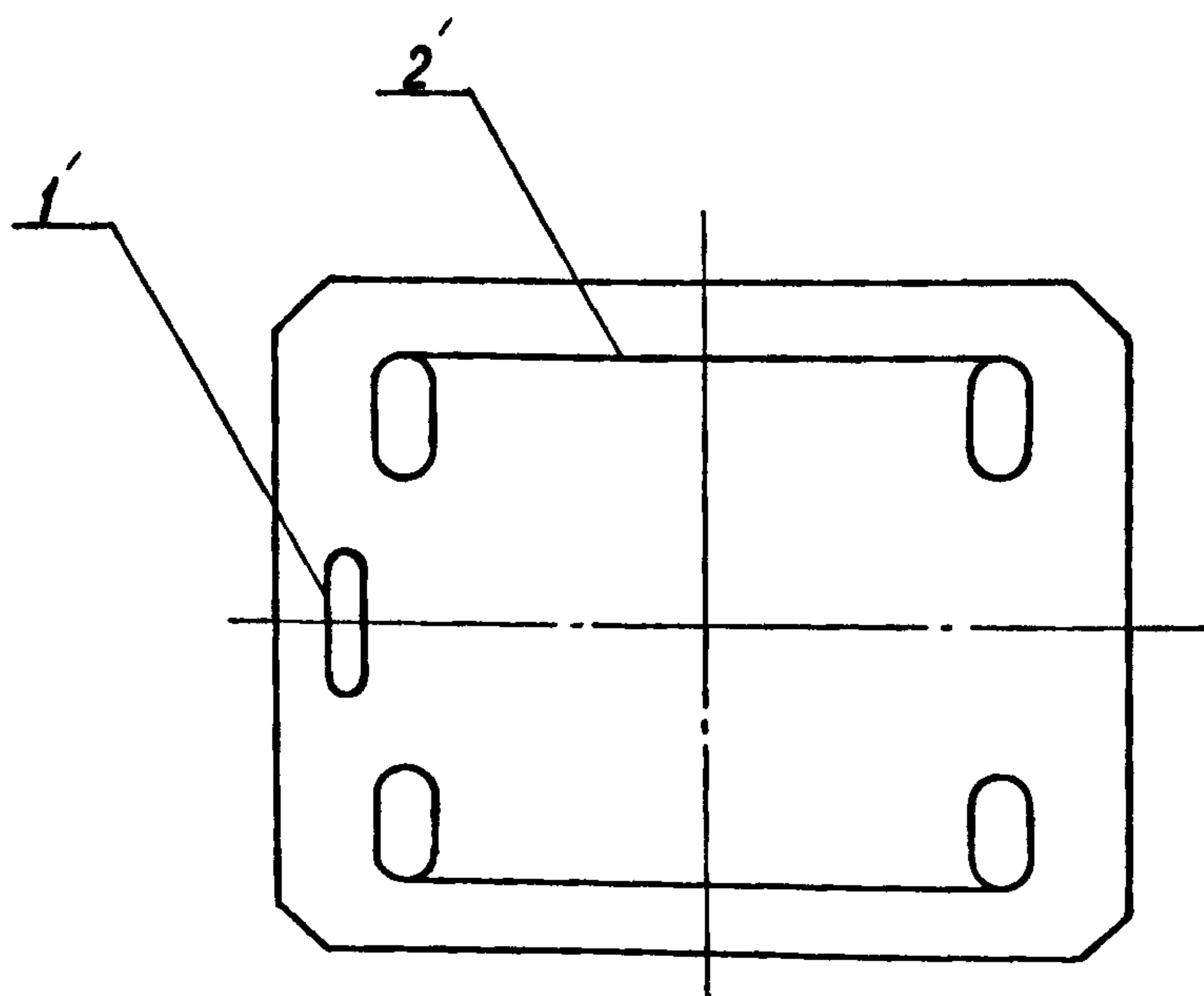


FIG. 6.4. The sketch of the cross section of the vacuum chamber of the ring. 1'—the clearing field electrode, 2'—separation plates, three plates in each dipole, except the fourth one, where there was no the bottom plate

studied to improve their termination and to suppress the excitation of unfavorable modes. These options suppress the instability of horizontal oscillations, which was detected in primary operations. With the operation of the first harmonics RF-generator the maximum value of the stored current of electron beam reached 4 A (10^{12} particles), and with the switched-on generator of the third harmonics of revolution frequency it was 700 mA. The storage of such large currents while operating on the first harmonics turned out to be possible due to the damping of dipole synchrotron oscillations by the resonator of the third harmonics.

6.6.2. The amplitudes of coherent oscillations as well as the level of the coherence of oscillations were measured using the system of pickup electrodes and two wideband amplifiers with the amplification coefficients $\kappa = 15000$. The sensitivity of pickup electrodes in the z-channel was 2.5 MkV/(mA · mm) and in the x-channel was 1 MkV/(mA · mm). The plates of pickup electrodes could be commutated to provide either the differential or the sum signal (i.e., for the observation of both the dipole and quadrupole oscillations). (See Fig. 6.6.) The

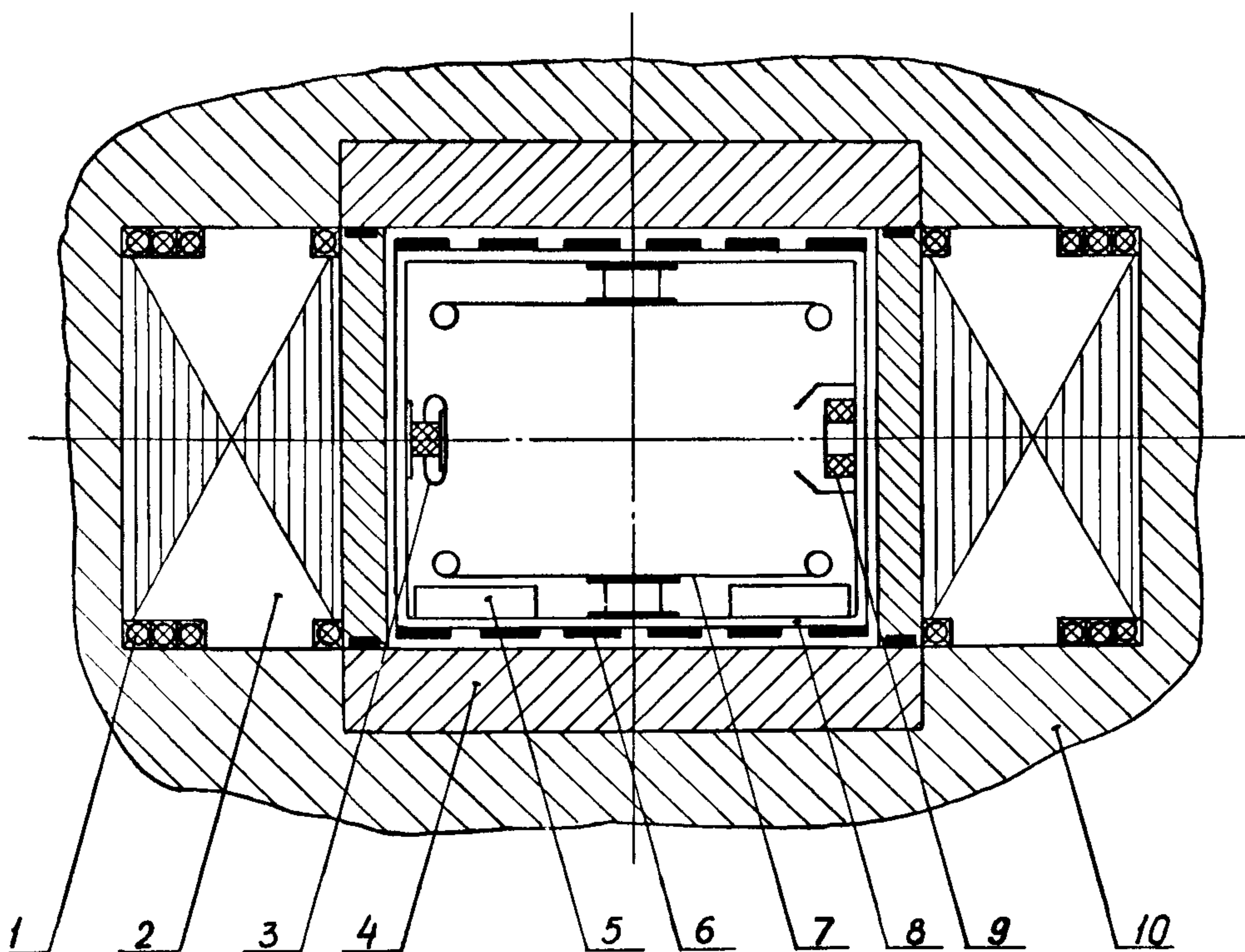


FIG. 6.5. The cross section of the vacuum chamber of VEPP-2 after the upgrade. 1, 6—back and forward current correcting coils; 2—the guiding magnetic field coil; 3—clearing field electrode; 4—dipole pole; 5—ion pump; 7—the separating or inflecting plate; 8—inner chamber; 9—synchrotron radiation absorber.

sum signal was used for the normalization on the beam current in the monitor, measuring the amplitudes of betatron oscillation, as well as for observing the phase motion of bunches. The amplifiers of x - and z - pickups had bandwidth $2\Delta f_z = 3$ MHz and $2\Delta f_x = 2.5$ MHz respectively. Later on, the bandwidth for both amplifiers was extended up to 7 MHz. Operation frequencies of amplifiers were $(1 - \nu_z, 1 - \nu_x)/T_s$ (i.e., 4 and 7 MHz). These monitors and their parameters are detailed in Ref. 94 and 96.

In addition, beam behavior was observed with several optical systems with the use of electron-optic converters, photomultipliers with slits, and TV cameras. (See Fig. 6.7.) Within the current range 2-5 mA the instabilities of vertical coherent oscillations observed in the storage ring did not lead to beam loss. An instability was ceased by the development in the beam of coherent oscillations of large amplitudes ($a_z \approx 1$ cm, See in Fig. 6.8.) Due to radiation damping the particles captured into the region of betatron separatrix, were concentrated in the vicinity of equilibrium amplitude a_s . Usually, the low-coherency effects lead to the diffusion of the beam picture. After development of this technique even the radial and synchrotron coherent oscillations could be easily recognized on the TV screen.

We would like to draw special attention to the advantage of using the light indicators as probes for the systems of feedback. When suppressing the radial instability we encountered the difficulties in using the large amplification factors in the feedback systems. The main cause of parasitic couplings was a large number of the capacitive couplings between the plates inside the vacuum

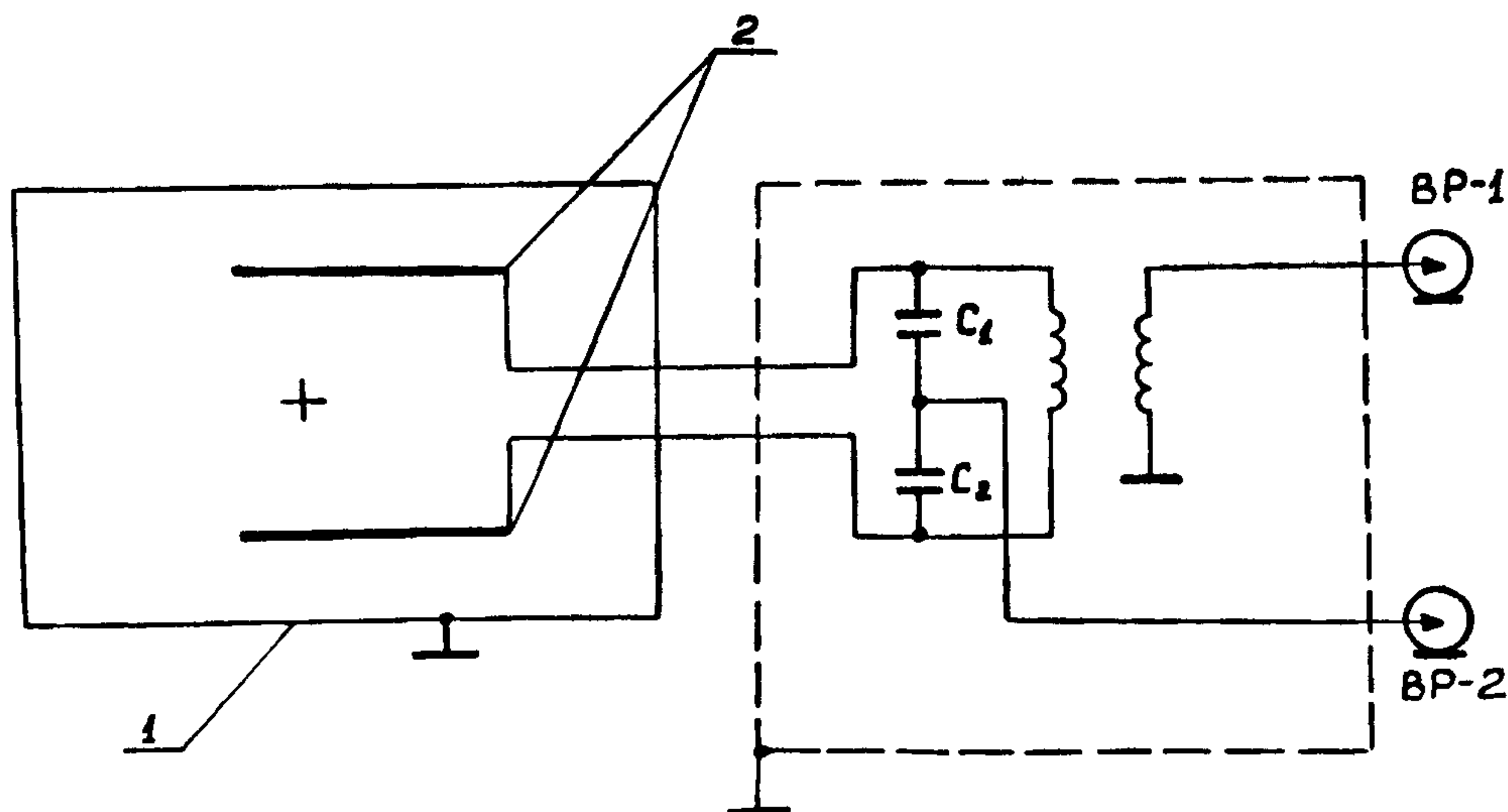


FIG. 6.6. The beam position monitor. 1—vacuum chamber; 2—pick-up electrodes; BP-1—differential output; BP-2—summation output.

chamber. Therefore, the maximum amplification coefficient that did not cause the system self-excitation could not be higher than 500. The optical detection of coherent oscillations of the beam is presumably the most efficient to be used in damping feedback systems, since it enables one substantial suppression of the parasitic feedbacks.

6.6.3. When storing large electron currents it was discovered²⁸ that the damping rate of the coherent betatron oscillations depends on the frequencies of the particle betatron oscillations. Detailed studies have shown that there exist such values ν_z at which the vertical coherent oscillations do not damp for a long time. During the few minutes one can observe the stable oscillations. In this case, practically the whole beam was involved in the coherent motion.

At the revolution frequency 25.125 MHz and $\nu_z=0.836$ the damping was lost within the stop band $\Delta\nu_z=\pm 2 \times 10^{-3}$ (See Fig. 6.9.) Because the betatron frequencies depend on the revolution frequency, the position of resonance region over ν_z was varied. It turned out that the vertical beam-separation plate system could excite the sum-type coherent resonance $\omega_k+\omega_z=\omega_s$.

Each plate of vertical separation was connected with the high-voltage cable to whose other end via the large resistance ($Z \sim$ a few MOhm) the high-voltage rectifier was connected. The length of one of these cables was 25 m which almost matched half of the wavelength (in the cable) corresponding to the frequency $(1-\nu_z)\omega_s$. Thus, one of the separation plates was terminated by the resonant impedance with the maximum the active component at a frequency $(1-\nu_z)\omega_s$. As it was found in § 4.2, such a termination may cause the instability. The instability increment in the thin band of betatron oscillation frequencies turned

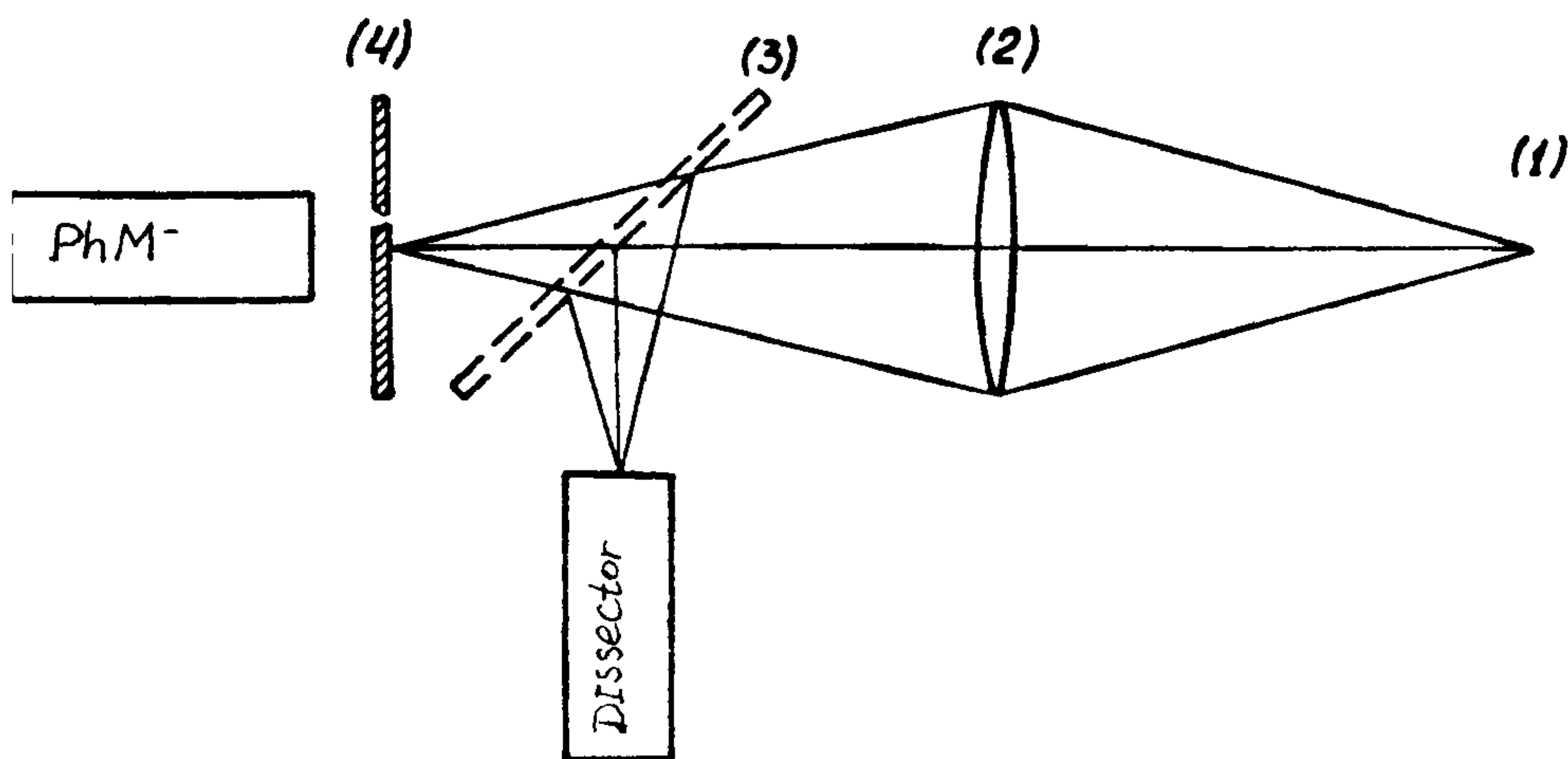


FIG. 6.7. Optical scheme of the beam position monitor. 1—the beam; 2—the objective lens; 3—the half-transparent mirror; 4—the slit

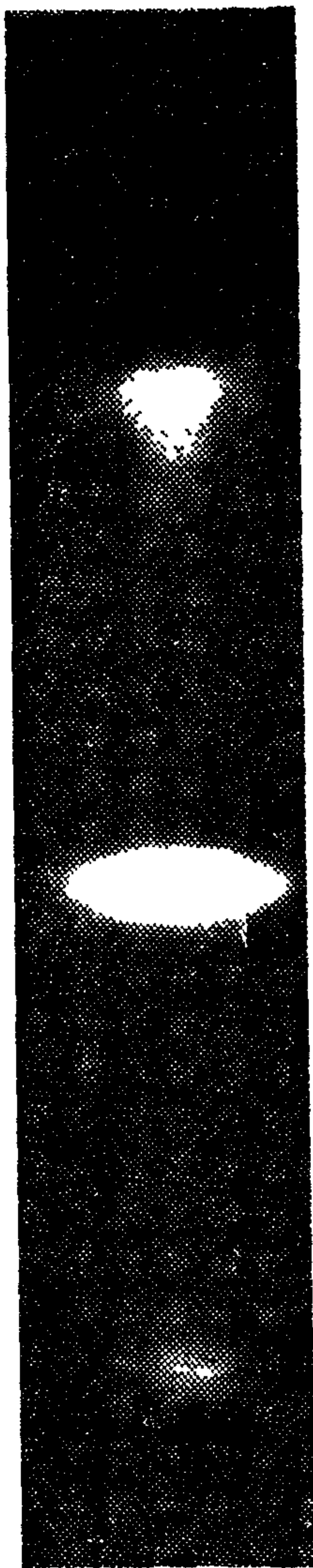
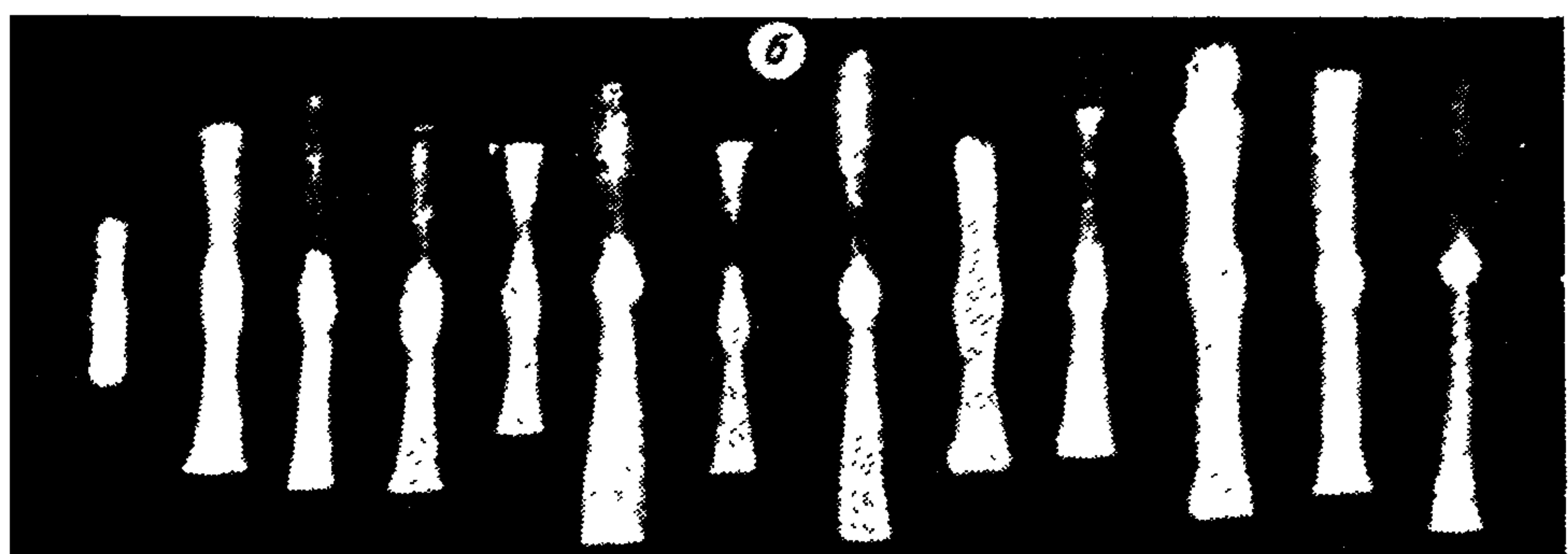
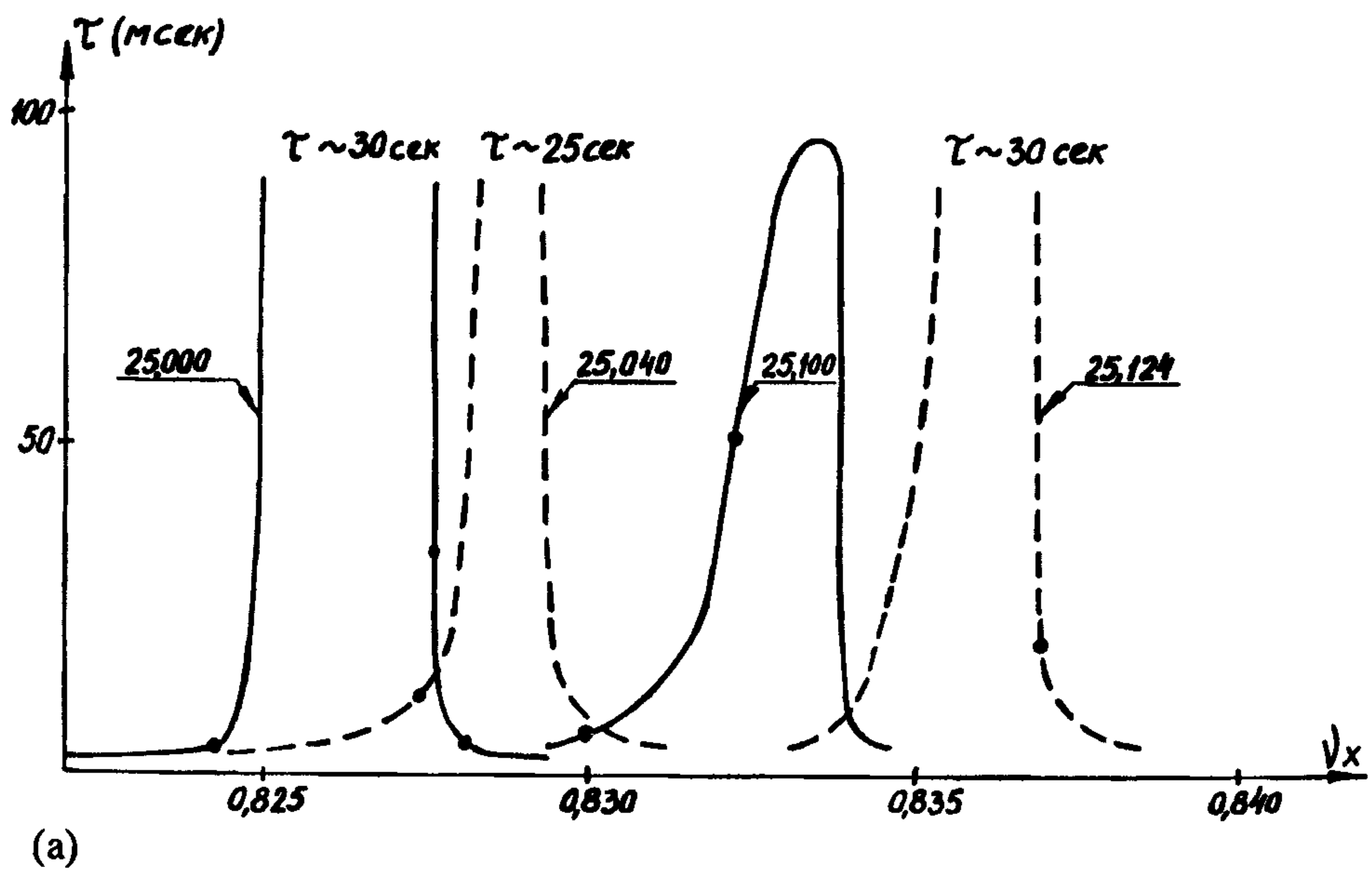


FIG. 6.8. The photo of the beam after the development of its vertical coherent oscillations. The bright spots near the stop-points correspond to the separatrix of the external betatron resonance.

out to be larger than the fast damping decrement; therefore, the coherent instability appeared.

The plate earthing close to the ring (the variation of the resonance frequency ω_k) canceled the instability of vertical oscillations. Further, all the separation plates were loaded with the matched impedance 25 ohm via the large capacitances. As a result, not only did the vertical instability vanish, but the fast damping time became approximately three times less. These measures provided the stability of the betatron coherent oscillations up to beam currents of 4 A.

The threshold current of the electron beam measured in the center of the resonance region was 10 mA. The characteristic feature of the instability was the



(b)

FIG. 6.9. The resonances of the vertical coherent oscillations; $\mathcal{E}=150$ MeV, $I=20$ mA, $f_s=25.00, 25.04$, and 25.14 MHz, 25.126 MHz.

increment dependence on the number of particles in a beam. Usually, the instability appeared after kicking the beam by the inflector while injecting the new portion. The self-excitation of instability was never observed. In the exact resonance the amplitude of coherent oscillations could reach 2 cm. The picture of large stable oscillations looked very similar to that due to the external resonance drive: at large amplitudes two bright spots are seen corresponding to the equilibrium value of amplitude a_s , betatron separatrix and in the center one can see the bright spot corresponding to particles uncaptured to the separatrix.

The long lifetime of coherent oscillation (in these measurements $\tau > 300$ c) means that the frequency spread of betatron oscillations due to the cubic nonlinearity $\partial\nu/\partial a^2 \simeq 10^{-4}/\text{cm}^2$ is small compared to the coherent frequency shift:

$$\Delta\omega_z = 2a_s(\Delta a)_s \frac{\partial\omega_z}{\partial a^2} < \text{Re}\omega.$$

Here $(\Delta a)_s$ is the beam size inside the separatrix of the coherent resonance. It certainly does not equal the size of the nonperturbed beam a_0 , since due to the conservation of the phase space volume $2\pi a_0 = \Delta\psi_b(\Delta a)_s$ the beam size in the separatrix increases.

It is obvious that the particle frequency spread in the separatrix of betatron oscillations cannot exceed the frequencies of small phase oscillations. Thus, one can conclude that the stable oscillations correspond to a large power of coherent resonance and to the strong nonlinearity of betatron oscillations at large amplitudes.

With the stable oscillations sometimes on the signal envelope the excitation of beatings was observed. The frequency of those beatings depended on the detuning from the resonance and on the beam current up to 4 A.

An increase in current from 0.5 to 4 A was accompanied by an increase in both the vertical and horizontal beam sizes, by the excitation of the oscillations of beam sizes, and by the subsequent loss of the whole or of the main part of the beam. This instability was not studied in detail. According to the excitation features this was a multipole instability.

6.6.4. One of the most interesting coherent phenomena found at the storage ring VEPP-2 was the so-called effect of the fast damping of the coherent betatron oscillations.²⁸ Normally, after the inflector's kick the oscillations of the low intensity beam decay with a rate determined either by the decrement of radiation damping $\tau \simeq 1$ sec or by the frequency spread of the beam. In the storage ring VEPP-2 it was found out that, when the beam current exceeds the threshold value 1 mA, decrements of excited coherent oscillations increased with the beam current. Starting from the currents 2–3 mA it was impossible to detect visually the damping of the previously stored part of a beam. Further observation of the coherent motion had to be carried out using the beam position monitor.

The idea of a fast damping mechanism was already discussed in Chapter 5. It consists of the following: the bunch making the oscillations between the matched plates excites the TEM-wave whose energy dissipates in the terminating resis-

tance.⁷⁷ It happened that the plates installed inside the vacuum chamber of VEPP-2 were well matched up to the frequency of 1 GHz. Hence, the waves with frequencies of up to 1 GHz can go out of the chamber without reflections. Such a system provided the dissipation of the coherent motion energy and thereby the fast coherent damping. As shown in Chapter 5, this damping mechanism practically does not depend on the frequencies of betatron oscillation.

In the vacuum chamber of the storage ring seven plates were placed up and down the equilibrium orbit. (See Fig. 6.5.) The wave resistance of each plate was 25 ohm. All plates were matched from one end: three plates of the electron and positron inflectors were matched to the frequency of up to 50 MHz. The nonmatched ends of the inflector plates were connected to the electrode of commutating discharges of the high-voltage generator. The radial plates were connected from one end with a high-voltage rectifier through the high-voltage cable, the other ends were disconnected.

Observations have shown that the fast damping rate did not depend on the bunch length. The variation of high-frequency voltage on the resonator within the range 3–30 kV did not change the damping decrements.

Damping decrements as functions of beam current and particle energy are in good agreement with Eq. (5.4.30) written for betatron oscillations $m_c=0$ under the condition that the machine chromatism is compensated

$$\frac{d\ln\nu_z}{d\ln R_0} \simeq 2.$$

Using Eq. (5.4.30) one can easily obtain that for seven matched plates with $Z_0=25$ ohm and $E_z=0.88U/l_1$ the value τI equals 100 mA · ms [Fig. 6.10 (a), line 2]. The straight lines in this figure show the linear dependence of damping decrement on the number of particles in a bunch (on current). In the same figure it is seen that the decrease in the number of plates down to three increases the damping time (line 1 in Fig. 6.10). The combination of the data in Fig. 6.10 can be described by a simple formula:

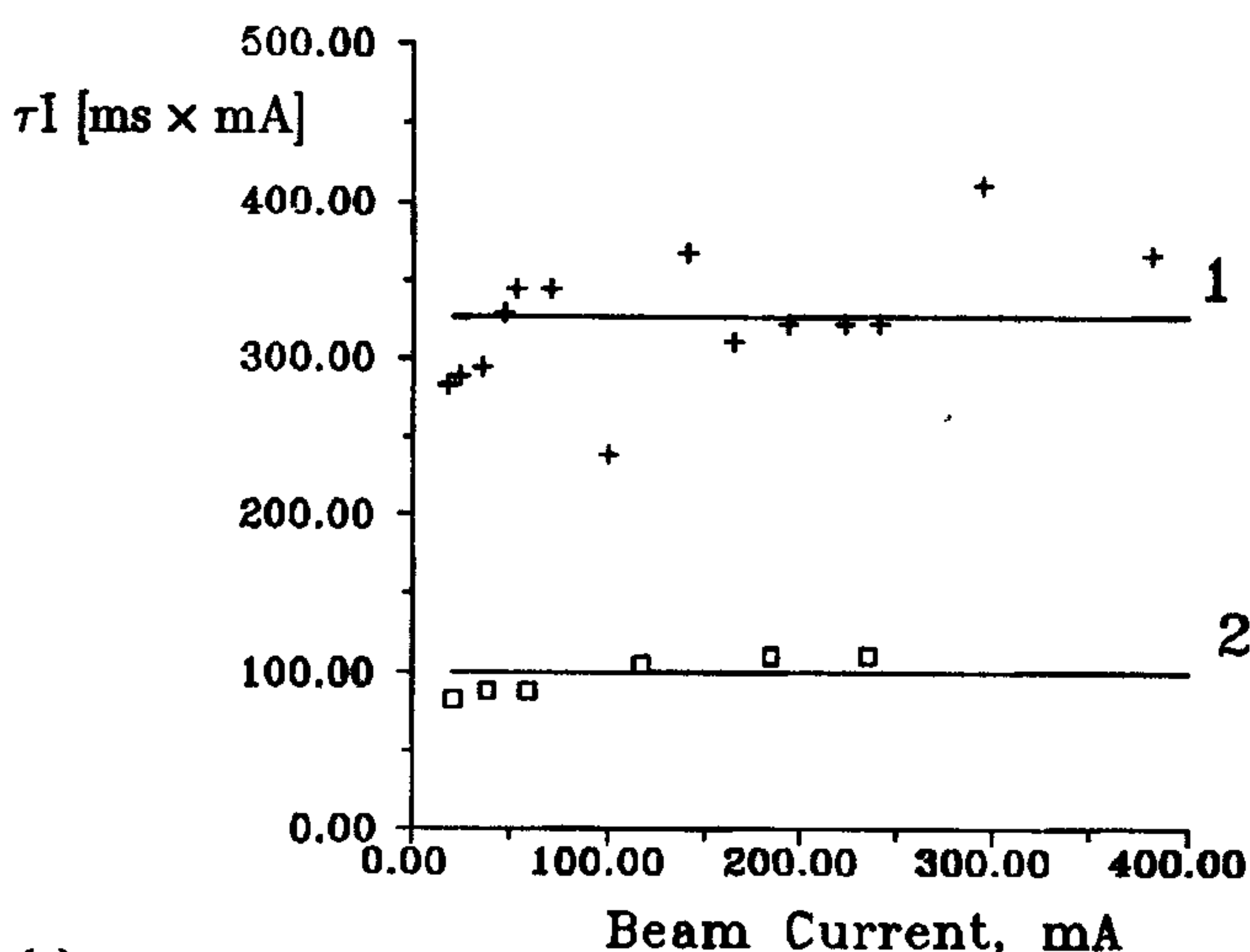
$$\tau I \propto \gamma.$$

The large spread of points on line 1 in Fig. 6.10(a) is due to the appearance of beatings of oscillations, when the beam is tuned close to the coherent resonance. Except for the resonance instability the narrow range $\Delta\nu \simeq 2.5 \times 10^{-3}$ of tunes with resonance damping of oscillations was also found. In the given region τI decreased down to 50 mA · mc that seemed to be related to some differential coherent resonance $\omega_k - \omega_z = n\omega_s$.

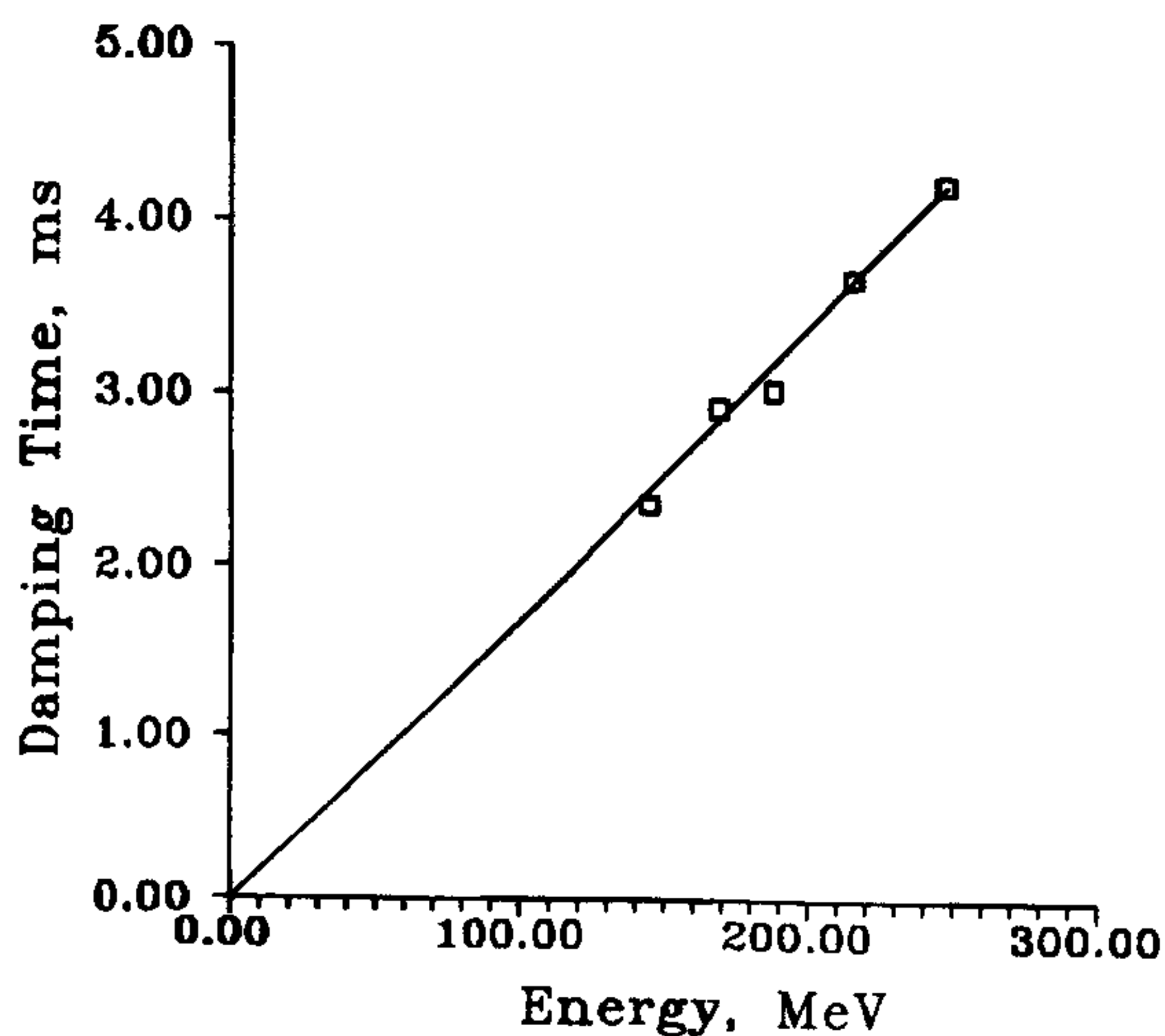
The dependence of decrement on betatron frequency was checked in the wide range $\Delta\nu_z = \pm 0.1$. It was found out that the fast damping disappeared when ν_x , ν_z occurred on the coupling resonance. Prior to matching the plates the same effect was seen on the vertical coherent resonance. The disappearance of damping on the coupling resonance was due to the fact that on this resonance the

dipole momentum of a bunch was equal to zero. But the appearance of vertical instability meant that the instability increment was higher than the fast damping decrement.

6.6.5. The accumulation of large positron currents in the chosen working point of ν_x was limited by the instability of horizontal coherent oscillations. This instability occurred in a quite narrow range of betatron frequencies $\Delta\nu_x = \pm 2.5 \cdot 10^{-3}$ at currents higher than 2 mA. When approaching the dangerous frequency band the self-excitation of oscillations occurred with a



(a)



(b)

FIG. 6.10. Dependence of the damping time of the fast damping on the beam current (a) (1—three matched plates; 2—seven matched plates), and on the energy (b) ($I=50$ mA).

substantial loss of the stored current. The large detuning over ν_x enabled storing positron current of up to 10 mA. However, such a detuning was not favorable to optimize the accumulation rate of positrons.

Further on similar instability was also detected in the electron beam but with the threshold current of 10 mA. The instability increment was proportional to the current value and the value τI was constant and equal to 300 mA · ms.

In the mode of positron storage ($\mathcal{E} = 100$ MeV) the instability appeared at frequencies $\nu_x = 0.714$, $\nu_z = 0.821$ and revolution frequency $f_s = 25.040$ MHz; $\nu_x = 0.701$, $\nu_z = 0.835$ at $f_s = 25.040$ MHz. Unfortunately, the reasonable cavity frequency matching the resonant condition

$$(m_x \nu_x + m_z \nu_z - n) \omega_s = \omega_k$$

was not found. We intentionally give here the general resonance relation since the optical beam position monitor in this case indicated two-dimensional oscillations. Typically the instability excitation occurred after the injection in the ring of a new portion of particles (i.e. after the excitation of vertical coherent oscillations). Before being cooled in the vertical direction the beam blew-up in the radial dimension up to $a_x \simeq 1.5$ cm. The main part of the beam executed coherent oscillations during a few minutes. During this time particles from the excited part came into the central part of the beam. The characteristic feature of this instability was the linear dependence of the instability threshold current on the bunch length. The instability increments were linearly dependent of N .

Similar to the vertical instability the excitation of radial oscillations depends on the conditions for matching radial plates of ion clearing. The matching of radial plates did not guarantee the possibility of the particle drops. The instability suppression was achieved by introducing the feedback system. This enabled the storage of sufficiently large positron currents (of up to 50 mA) without instabilities.

The feedback was performed in the following way: the input signal at frequency $(1 - \nu_x) f_s$ from one of the plates of ion clearing was applied via the amplifier with the amplification coefficient $\mathfrak{K} \simeq 10^4$ (the resonance amplifier with a frequency 7.5 MHz) to a pair of the ion clearing field electrodes. [See Fig. 6.11(a).] Because of parasitic connection between the plates, the system amplification factor was not higher than 700 (due to the self-excitation of the system). Further, the amplifier was replaced by the wideband one on whose input the filter plugs were installed at 4, 10, 21, and 25 MHz as at these frequencies the system with a beam was self-excited.

The problem became more complicated by the fact that simultaneously the stability of the electron and positron beams should be provided. The choice of phases in the feedback system was carried out in the following way: the current was stored and then by varying the signal delay the phase was found where self-excitation occurred. In such a way the dependence of the excitation threshold current on the delay time of a signal in the feedback circuit was found in Fig. 6.11(b).

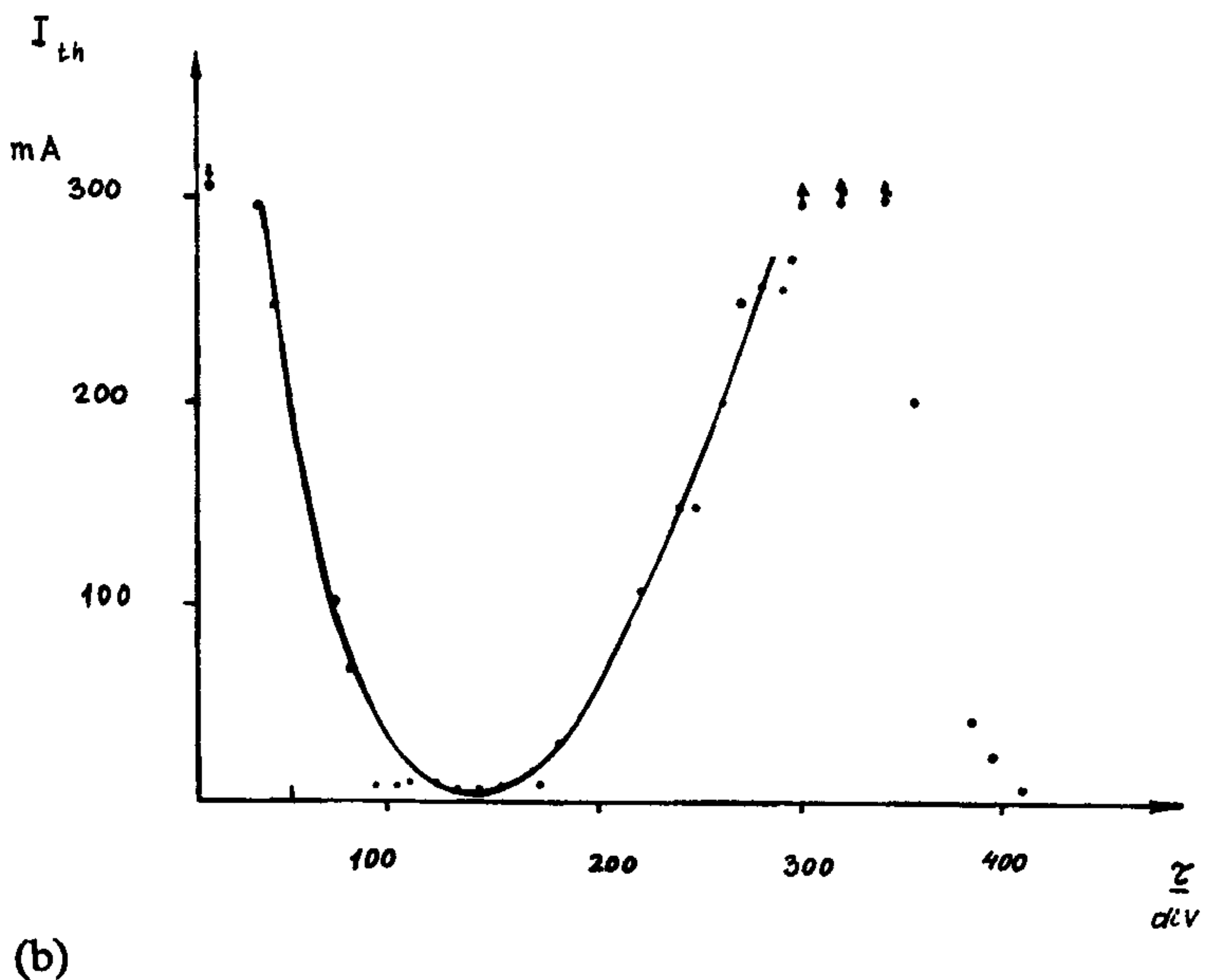
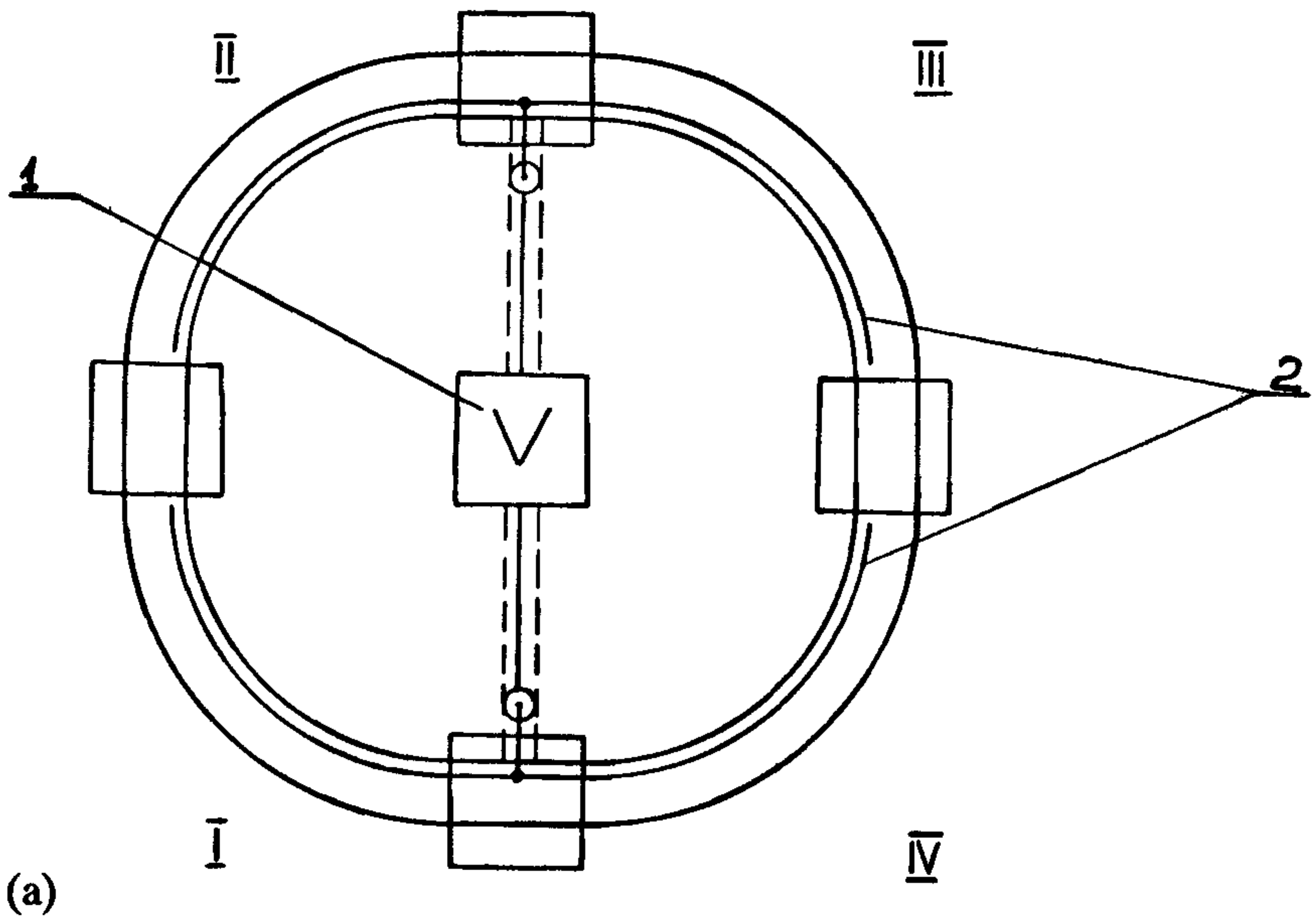


FIG. 6.11. The feedback system to damp radial coherent oscillations in VEPP-2 (a) (1—amplifier, 2—clearing field electrodes) and the stability diagram of radial coherent oscillations (b) (in points marked by $\uparrow$ the accumulation was not limited by radial instability, φ —the operating phase for positron beam).

We would like to draw attention to the convenience of the study of all the functional dependencies of the instability increments with the use of the feedback systems. The fast disconnection of the feedback system enables (synchronously with the disconnection) studying the time behavior of the bunch instabilities at currents much larger than instability threshold. In practice, this behavior cannot be studied at natural self-excitation when the problem arises without the synchronization of signals.

During the reconstruction when taking the frequency characteristics of the ion clearing field electrodes the mode 49.5 MHz was found which could not be canceled by matching the high-voltage oscillations in the unit since it has its minimum voltage at the cable connection point. For suppressing this mode the plates were connected via the resistances (See Fig. 6.12) that enabled the cure of this instability.

6.6.6. The longitudinal phase instability was the main obstacle for obtaining large currents. In the relativistic case, the coherent phase oscillations are unstable if the frequency of accelerating voltage is higher than the resonance frequency of the accelerating resonator.⁷⁸

The physical meaning of the instability is simple. During operation of the storage ring at an energy higher than the critical one, which is always realized for the electron and positron storage rings, the bunch of particles inducing the voltage on resonator, loses energy. In this case, the power of coherent loss is proportional to the square of the number of particles in a bunch. With the excitation of coherent phase oscillations the revolution frequency of bunches is changed. Let us consider the case where the frequency of the RF generator (revolution) is lower than the resonance frequency of the resonator.

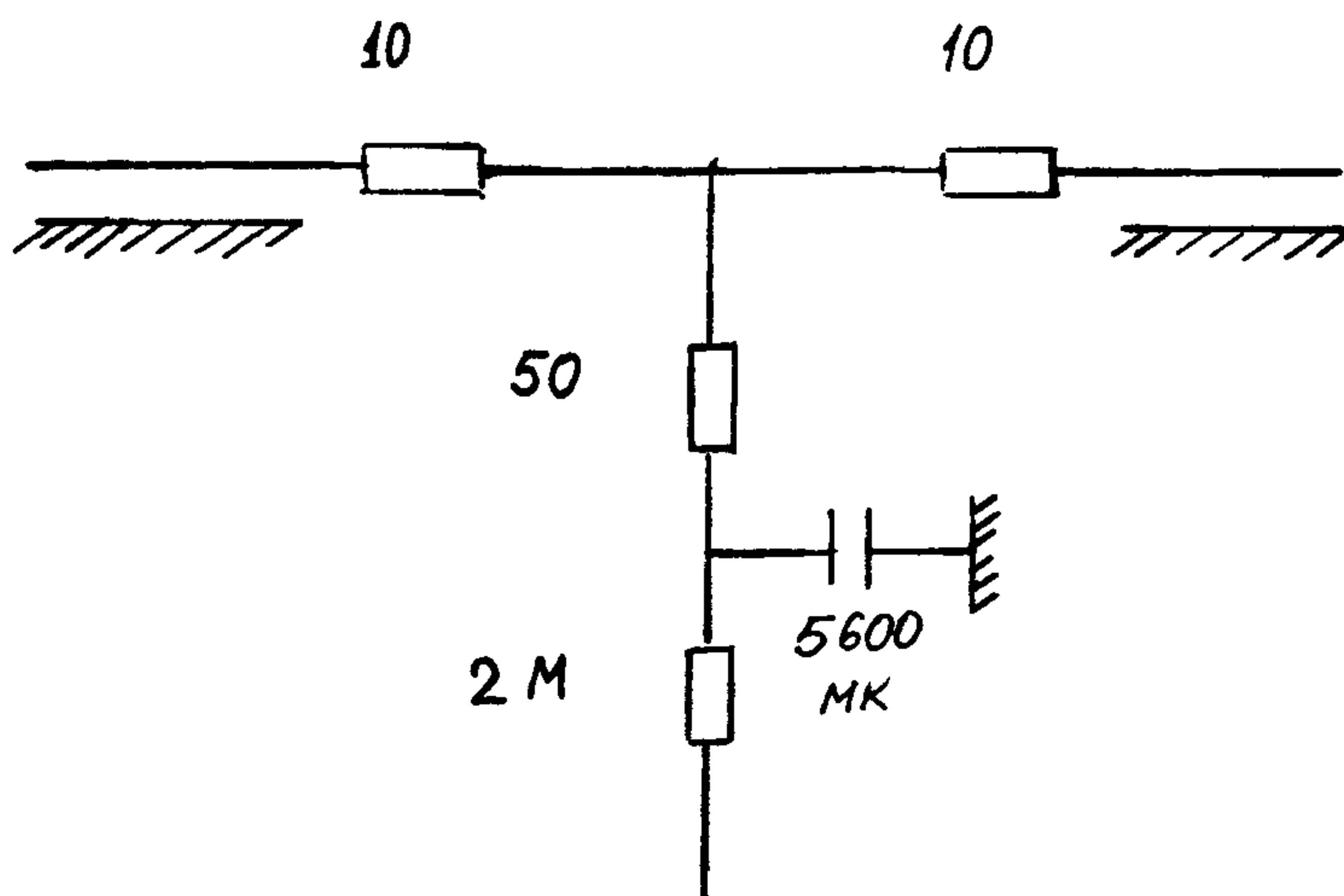


FIG. 6.12. The matching scheme of the ion clearing field electrodes.

If an energy of particles in a bunch is higher than the energy of an equilibrium particle, the revolution frequency and coherent loss decrease since the bunch in frequency is going farther from the resonance frequency of a resonator. Thereby, an increase in energy of oscillations leads to the decrease in energy loss and to the appearance of the antidamping of coherent phase oscillations. For damping this instability one has to tune the resonator to the frequency somewhat lower than the frequency of accelerating high-frequency voltage—lower than the particle revolution frequency.

The results of first experiments on the storage of intense beams have shown that at the “correct” tuning of the resonator, the instability nevertheless occurred at relatively small currents. The cause of the instability turned out to be the parasitic resonator at frequency 452 MHz which is the section of radial line formed by the resonator discs and resonating on the wave of type E_{010} . The Q-factor of the resonator was about 3000 and the shunt resistance was close to 30 kOhms. Thus, the parasitic resonator frequency by 200 kHz exceeded the frequency of the 18th harmonics of revolution frequency. For providing the stability the revolution frequency should be increased by 20–30 kHz to make the frequency of 18th harmonics higher than the frequency of the parasitic resonator. Such a retuning turned out to be possible after correction of magnetic field and the suppression of nonlinear resonances. Upon correction, the stability of oscillations was maintained even during the shift to the unfavorable slope of frequency characteristic of the main resonator. In other words, the damping decrement introduced by the resonator of the 18th harmonics turned out to be larger than the decrement introduced by the main resonator.

The operative tuning of the parasitic resonator was performed with the device that smoothly returned the resonator frequency. With the help of this device the dependencies of the threshold currents on the detuning were removed and currents were obtained of up to 0.5 A. With a certain ratio of currents and some resonator tuning, the phase oscillations did not lead to loss of the current and the stable coherent oscillations occurred with a small modulation of oscillation

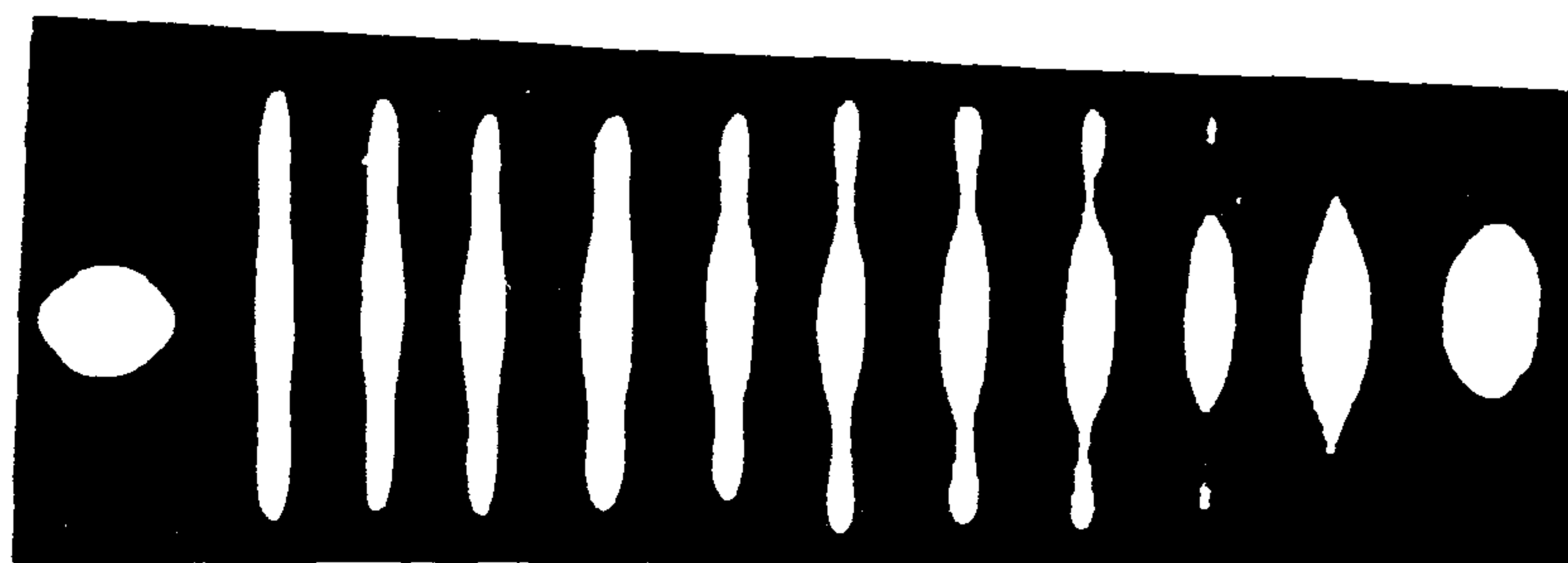


FIG. 6.13. The cine-gram of the phase coherent oscillations. The typical period of oscillations was in the range 1–2 s.

amplitude. (See Fig. 6.13.) Such a picture could be observed during 5–10 min. The lifetime of particles in the common separatrix generated due to resonance interaction particles and parasitic resonator as well as the nonlinearity of synchrotron motion depended on the individual effects pushing the particle out of the common separatrix.

The particles escaping the common separatrix damped to the equilibrium phase for the radiation times. Therefore, with time the central part of a beam became brighter and the coherent oscillations ceased.

To cure the phase instabilities of coherent oscillations as well as to decrease the bunch length, the new two-frequency resonator was manufactured and mounted in the ring during the upgrade shutdown. (See Fig. 6.14.) The resonator of the first harmonic (voltage up to 12 kV) provides the storage of a single bunch, while the resonator of the third harmonic provided the voltage of up to 300 kV. A more detailed description of this cavity is found in. Ref. 14.

The requirements of the stability of coherent oscillations of the beam demanded very precise location of eigenfrequencies of higher parasitic modes of the cavity. (See Fig. 6.15.) As a result of this modernization the stable operation of the whole complex was achieved.

The resonator of the third harmonic had a high shunt resistance 1 M Ω and Q-factor 12 000. Using the ideas of the resonance damping, one of the authors

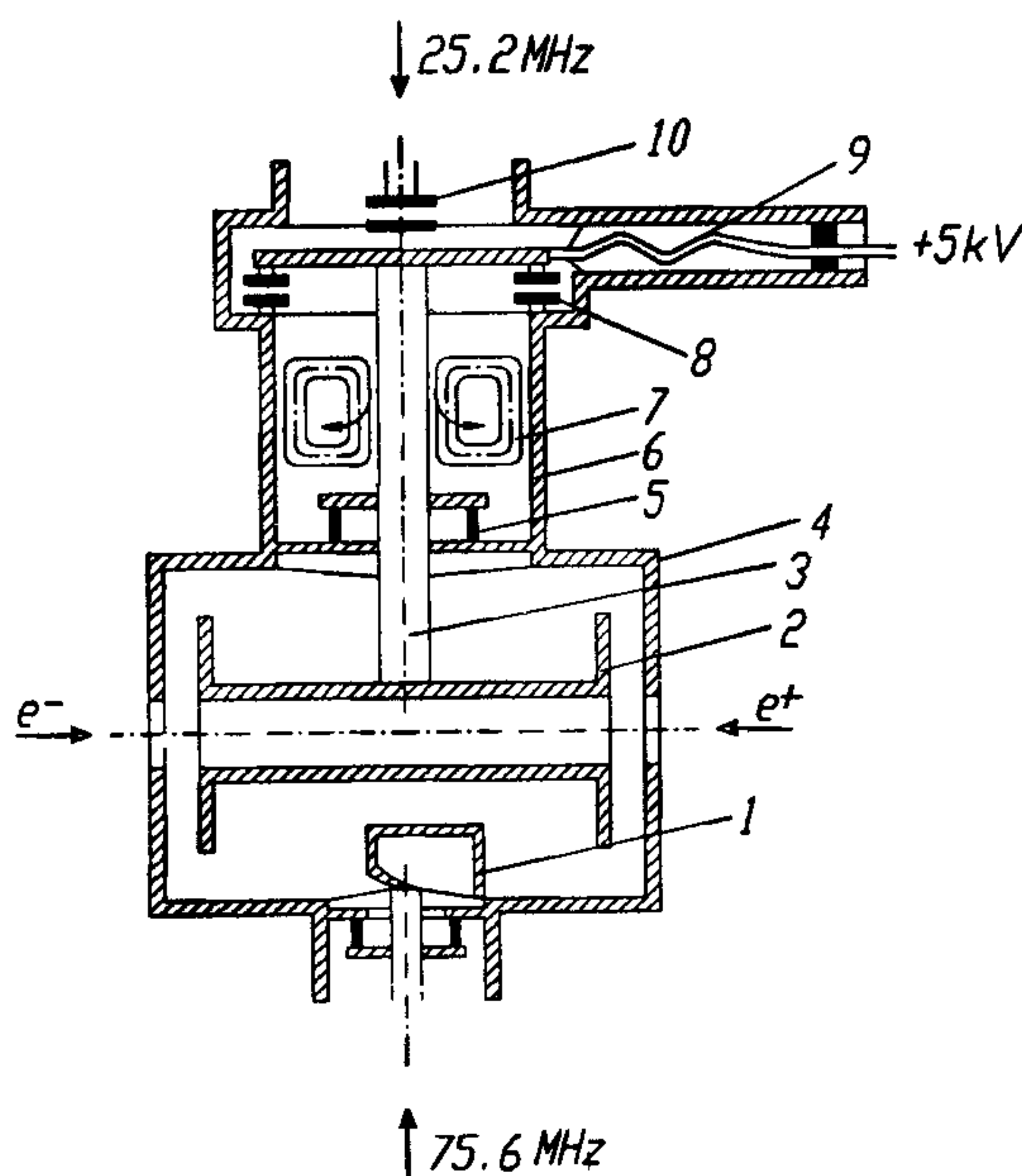


FIG. 6.14. The upgraded two-frequency RF-cavity of VEPP-2. 1—the excitation loop of the 3rd harmonic; 2—the pipe with discs; 3—the stick; 4—the cavity; 5—ceramic insulator; 6—the outer pipe of the individuality of the 1st harmonic; 7—the tuning coils of the 1st harmonic; 8, 10—separating capacities; 9—the choke.

(N. S. Dikansky) achieved an electron current of 4 A at an energy of 200 MeV. To this end, the resonator of the third harmonic was used as a passive one. At the coherent resonance of type $\omega_3 + \omega_c = 3\omega_s$ the stability for only dipole phase oscillations was provided. Note that the sign + in front of frequency of phase oscillations corresponds here to the differential resonance since the synchrotron mass is negative. The increase in current ceased due to the excitation of the quadrupole phase oscillations which was not damped by this resonator.

The time of phase coherent damping at currents ~ 1 A reached a few hundred microseconds. With this tuning the main difficulty was the high (up to 60–70 kV) induced voltage on the resonator of the third harmonic and corresponding change of the phase frequency of synchrotron oscillations. Since the high-frequency system of the first harmonic gave maximum voltage of 12 kV then due to coherent loss on the excitation of the third harmonic resonator the equilibrium phase was approaching $\pi/2$ and the beam dump occurred.

As a result of these experiments the idea appeared of obtaining very short bunches with the use of passive super-high-frequency resonators tuned to the frequency close to the harmonics of revolution frequency. In this case, the compensation for both the radiation and coherent loss should be made by the relatively low-frequency system.

To conclude these four chapters on the stability of coherent oscillations of the beams in storage rings, let us briefly discuss the possibility of using the wide bandwidth passive elements of the vacuum chamber to damp the unstable coherent oscillations of a beam. Those dampers are not so widely used now as,

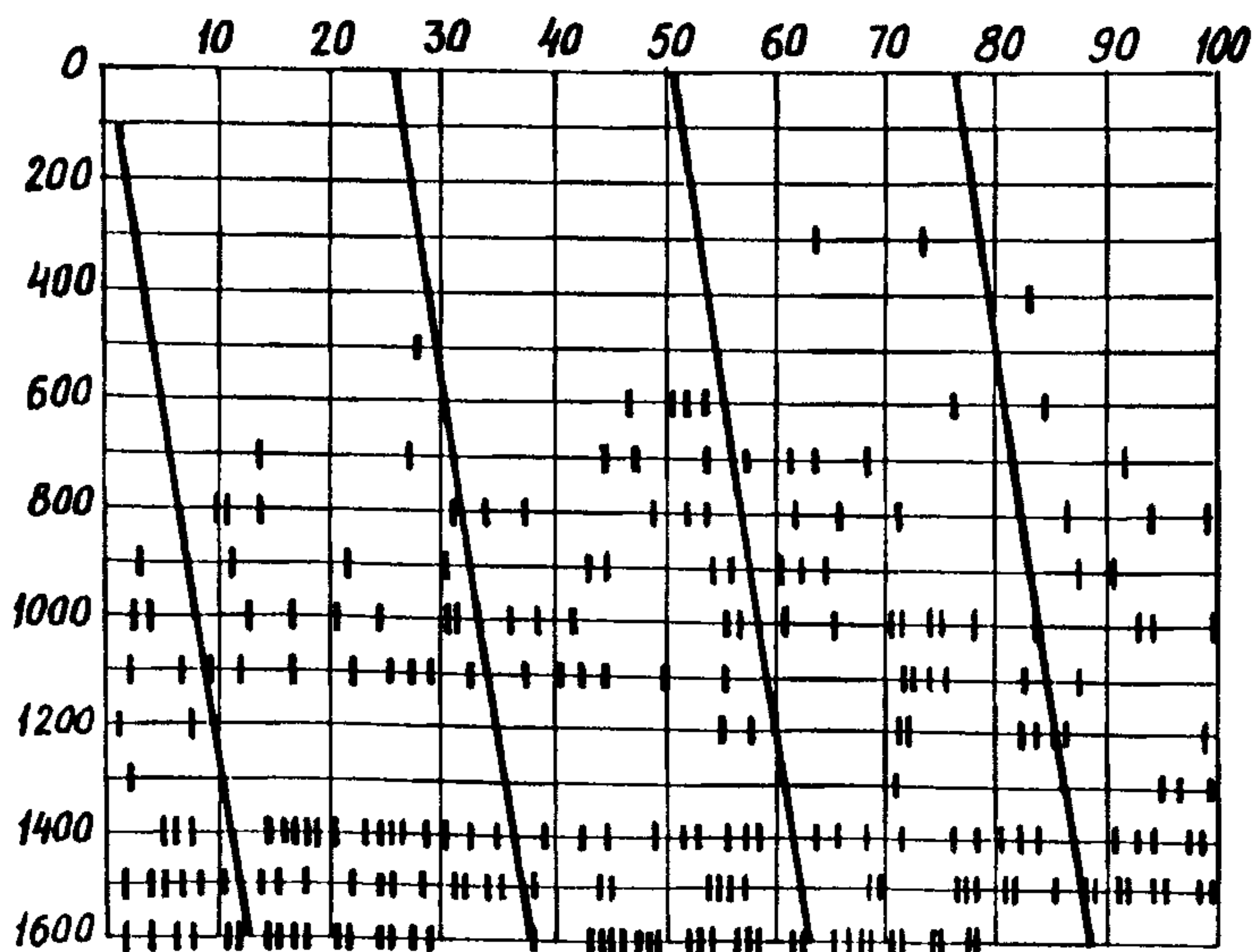


FIG. 6.15. The diagram of the symmetrical modes of the upgraded cavity of the storage ring VEPP-2.

say, feedback systems. But taking the space along the orbit, passive devices enable one to exclude the use of the very precise and the powerful enough electronic systems. In this sense, the passive systems can play the same role for collective motion of the beam as, say, synchrotron radiation for an incoherent one.

According to classification of instabilities proposed in Ref. 28 the most dangerous (by the value of possible increments) are the resonant instabilities due to the interaction of the beam with some high-Q element of vacuum chamber of the storage ring. The characteristic feature of these effects is the resonant dependence of increments of unstable modes on the particle oscillation frequencies. The main method to cure these instabilities is either the suppression or detuning of unstable modes of the cavity or the beam.

The second class of instabilities are the dissipative instabilities due to the interaction of the beam with the low-Q elements. Due to weak, but still important turn-to-turn memory of the beam wakes, the increments of coherent oscillations contain both the single-turn and multiturn contributions. The last generally depend on the position of the working point of the ring. The suppression of individual modes (one or some of them) can be provided with the correct choice of frequencies of particle oscillations, by the enhancement of the contribution of the fast damping in the single-turn part, or by introducing the strong cubic nonlinearity.

The third class comprises of the single-turn effects caused by the interaction of the beam with the elements "without memory." The characteristic feature of these effects is the independence of increments on the particle oscillation frequencies. These instabilities can be damped either by introducing the strong cubic nonlinearity or with the use of the already discussed effect of fast damping of coherent oscillations introduced by the wideband damping system. If one has in mind the passive systems, such a device could consist of the matched plates, low-Q resonators, and in the installations with electron cooling, the cooling electron beam. It is clear that the use of the wideband damping systems fits also for damping the instabilities of the first two classes.

Let us consider in more detail the possibility of efficient damping with such a system not only betatron but also synchrobetatron oscillations. To introduce the decrement into the longitudinal coherent oscillations one should provide the modulation of average energy loss by the particle longitudinal motion.

This can be realized in two ways. First, by the strong frequency modulation of the damping system impedance

$$Z_n(\omega) \simeq \left(\frac{\partial Z_n}{\partial \omega} \right) (\omega - n\omega_s), \quad \omega \simeq n\omega_s.$$

Because the particle revolution frequency depends on energy, the interaction with such an element leads to the fast damping of longitudinal coherent oscillations of a beam.

Another method is based on the use of elements where the energy loss depends on the radial position of a beam. As shown in Chapters 3 and 5, due to the coupling (specific for rings) of the radial and longitudinal degrees of freedom, the interaction with such an element can redistribute the decrement in longitudinal coherent oscillation from radial modes. With such a way of damping, an inevitable decrease in the decrement of radial oscillations should be compensated for by the additional damping system, where the energy losses of a particle do not depend on its radial position $\langle \partial \mathcal{F}_{\parallel} / \partial R_0 \rangle = 0$.

The chromaticity of the storage ring focusing enables one to use the effect of fast damping of betatron coherent oscillations for the simultaneous damping of both the betatron and synchrobetatron modes of a beam of final length. In this case since the redistribution of decrements is caused by the field excitation by longitudinal motion, sufficient betatron oscillation decrements can be provided by an additional damping system located in places with low values of betatron functions.

The given method for damping the betatron and synchrobetatron modes assumes the operation with a certain sign of the machine chromaticity $d\omega_{\perp} / d\omega_s$. In those cases when the operation is desirable at an arbitrary sign of $d\omega_{\perp} / d\omega_s$, the use of the passive systems for the simultaneous damping of betatron and synchrobetatron coherent oscillations requires the enhancement of the fast damping contribution.

Generally, the damping of coherent oscillations by the wideband systems should not exceed the frequency of synchrotron oscillations of particles. In the opposite case, the provision of the stability of coherent oscillations for lengthy beams can require the compensation for the frequency shifts down the levels substantially lower than the decrements introduced by the system. In this case, the oscillation damping itself can be much slower. The latter limitation is of general character and it is valid for both the passive and active damping systems. In particular, this limitation establishes the admissible value of the amplification factor in the wideband active systems.

Schottky Noise and Stochastic Cooling

Let us now study kinetic phenomena in intense, cold beams. In this and the following chapters we shall consider the cases when condition (2.5.1) is (strongly) violated and the calculation of collision integrals should take into account the collective degrees of freedom of the beam. Generally, these degrees of freedom can essentially affect the spectral densities of the beam noise and, hence, the features of kinetic processes.

In this chapter we consider the effects when the beam noise relaxes much faster than the particle distributions. Historically, the interest in this field was significantly stimulated by two problems. The first is the application of nondestructive diagnostics based on the Schottky noise measurement⁹⁷ to the diagnostics of very cold beams.^{90,91} The second is the study of the stochastic cooling method using the wide bandwidth feedback systems invented by S. Van-der-Meer.¹⁰⁰

In both cases, solutions of Eq. (2.4.28) as well as the theory developed in Chapters 3 through 6 become useful for calculations of correlation functions:

$$(\delta X \delta F)_\omega = \lim_{\Delta \rightarrow 0} 2\Delta \langle \delta X^*(\omega + i\Delta) \delta F(\omega + i\Delta) \rangle,$$

describing the basic kinetic processes in intense beams.

7.1. LONGITUDINAL SCHOTTKY NOISE OF AN UNCOOLED, COASTING BEAM

As we saw in Chapter 2, the thermal motion of particles results in the excitation of spontaneous coherent oscillations of the beam. These coherent oscillations have random phases and amplitudes and are seen by pickup electrodes as a noise signal. The measurement and analysis of this so-called Schottky noise signal was recognized in the beginning of the 1970's⁹⁷ as a relevant tool for the nondestructive diagnostic of beams in particle accelerators and storage rings. Due to collective interaction of the beam with its environment an interpretation of the results of Schottky noise measurements generally demands careful study of spectral densities of these fluctuations.

Let us start with the simplest problem—calculating of the spectral density of

fluctuations of the longitudinal field in the uncooled, coasting beams. In this case, Eq.(2.4.28) takes the form

$$\left(\frac{\partial}{\partial t} + \omega_0(\Delta p) \frac{\partial}{\partial \vartheta} \right) \delta F + Ne \delta E(\vartheta, t) \frac{\partial f_0}{\partial \Delta p} = 0. \quad (7.1.1)$$

As previously, we shall assume that the fields induced by the beam only slightly distort the particle motion for times of the order of revolution period. Then the values of coherent frequency shifts due to interaction of particles can be expected to be small values, compared to the frequencies in unperturbed spectrum and, thus, Eq. (7.1.1) can be solved using methods discussed in Chapter 3. In this section we describe the interaction of the beam and surrounding electrodes by the coupling impedance $Z_n(\omega)$, when harmonics of the field and beam density fluctuations are related by

$$\delta E_{n,\omega} = -\frac{e\omega_s Z_n}{\Pi} \delta \rho_{n,\omega} = -\frac{e\omega_s Z_n}{\Pi} \int d\Delta p \delta F_{n,\omega}. \quad (7.1.2)$$

Assuming the time hierarchy is fulfilled, we shall neglect the variation of f_0 in the “coherent” time intervals while solving Eq. (7.1.1). Then the solution of Eq. (7.1.1) can be obtained using the Fourier transformation in azimuth ϑ and time

$$\delta F(\Delta p, \vartheta, t) = \sum_n \delta F_n(\Delta p, t) e^{in\vartheta},$$

$$\delta F_{n,\omega}(\Delta p) = \int_0^\infty dt e^{i\omega t} \delta F_n(\Delta p, t).$$

After these transformations Eq. (7.1.1) can be rewritten in the form

$$\delta F_{n,\omega} = \frac{i\delta F_n^{(0)}}{\omega - \omega_n} + \frac{Ne^2 \omega_s Z_n}{\Pi} \frac{i\partial f_0 / \partial \Delta p}{\omega - \omega_n} \delta \rho_{n,\omega}. \quad (7.1.3)$$

For harmonics of the linear density fluctuations this yields

$$\delta \rho_{n,\omega} = \frac{\delta \rho_{n,\omega}^{(0)}}{\epsilon(n, \omega)},$$

$$\delta \rho_{n,\omega}^{(0)} = i \int_{-\infty}^{\infty} d\Delta p \frac{\delta F_n^{(0)}(\Delta p, t_0)}{\omega - n\omega_0(\Delta p)}, \quad \text{Im} \omega > 0, \quad (7.1.4)$$

where

$$\epsilon(n, \omega) = 1 + \frac{\Omega_n^2}{n\omega_0'} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\omega - \omega_n(\Delta p)}$$

is the longitudinal dielectric constant of the beam, $\omega_n = n\omega_0(\Delta p)$, and

$$\Omega_n^2 = n^2 \frac{Ne^2 \omega_s \omega'_0}{\Pi} \left(\frac{-iZ_n}{n} \right)$$

is the squared longitudinal coherent frequency shift of the beam corresponding to impedance Z_n . In this section, we shall assume the interaction of the beam with low-Q value surroundings, when Z_n depends weakly on the exact value of ω , which therefore can be replaced in the argument of Z_n by $n\omega_s$. When treating Eq. (7.1.3), one has to keep in mind that it relates $\delta F(t)$ with its initial value at the moment t_0 . The reference time t_0 in these equations is quite arbitrary.

Using these expressions we can calculate the spectral density of the beam noise:

$$(\delta\rho_n)_\omega^2 = \frac{(\delta\rho_n^{(0)})_\omega^2}{|\epsilon(n, \omega)|^2}. \quad (7.1.5)$$

For the beam states close to the equilibrium one, $(\delta\rho_n^{(0)})_\omega^2$ is defined by the expression [see Eq. (2.5.23)]

$$(\delta\rho_n^{(0)})_\omega^2 = 2\pi N \int_{-\infty}^{\infty} d\Delta p f_0(\Delta p) \delta[\omega - \omega_n(\Delta p)].$$

Then $(\delta\rho_n)_\omega^2$ reads

$$(\delta\rho_n)_\omega^2 = \frac{2\pi N}{|\epsilon(n, \omega)|^2} \int_{-\infty}^{\infty} d\Delta p f_0(\Delta p) \delta[\omega - \omega_n(\Delta p)]. \quad (7.1.6)$$

The power of such a noise is

$$P_n = \langle |\delta\rho_n|^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\delta\rho_n)_\omega^2 \quad (7.1.7)$$

or after the integration over ω ,

$$P_n = n \int_{-\infty}^{\infty} d\Delta p \frac{f_0(\Delta p)}{|\epsilon(n, \omega)|^2}. \quad (7.1.8)$$

Expressions (7.1.7) and (7.1.8) show that the collective effects can substantially modify the beam noise characteristics when $\epsilon(n, \omega)$ differs noticeably from unity. In its turn, it assumes that the longitudinal temperature $T_{||}$ of the beam is small. If $\Delta\omega_0$ is the frequency spread in a beam, the beam longitudinal temperature reads

$$T_{||} = R_0 \Delta\omega_0^2 / |\omega'_0|.$$

Then in the region

$$(n\Delta\omega_0)^2 \gg |\Omega_n^2| \quad (7.1.9)$$

one has $\epsilon(n, \omega) \simeq 1$ and, correspondingly, the beam noise has the same characteristics as the Schottky noise:

$$(\delta \rho_n)_\omega^2 \simeq \frac{2\pi N}{|n\omega'_0|} f_0 \left(\frac{\Delta \omega_n}{n\omega'_0} \right), \quad P_n = N. \quad (7.1.10)$$

On the contrary, when condition (7.1.9) is violated, the difference of $\epsilon(n, \omega)$ from unity might be quite large. This time, the excitation of collective degrees of freedom can modify substantially both the shape of the spectrum and the power of the noise of the beam. The main features of these changes depend on the character of propagation of coherent oscillations along the beam. Let us initially take $Z_n = iZ''_n$. Then in the region $\Omega_n^2 \gg (n\Delta\omega_0)^2$ one can write

$$\epsilon(n, \omega) \simeq 1 - \frac{\Omega_n^2}{\Delta\omega_n^2} + i\epsilon''(n, \omega),$$

while the dispersion equation $\epsilon(n, \omega) = 0$ results in the decrements of oscillations

$$\delta_n = \Omega_n \epsilon''(\Omega_n) / 2 \ll \Omega_n.$$

In this case, Eq. (7.1.6) yields the expression

$$(\delta \rho_n)_\omega^2 = \frac{2\pi N \int_{-\infty}^{\infty} d\Delta p f_0(\Delta p) \delta[\omega - \omega_n(\Delta p)]}{[1 - (\Omega_n / \Delta\omega_n)^2] + (2\delta_n / \Omega_n)^2},$$

which indicates the splitting of the noise spectrum into two peaks of the width δ_n , shifted from $n\omega_s$ by $\pm \Omega_n$. For beams with the Gaussian momentum distribution

$$f_0 = \frac{1}{(2\pi)^{1/2} \sigma_{\parallel}} \exp\left(-\frac{\Delta p^2}{2\sigma_{\parallel}^2}\right) \quad (7.1.11)$$

Eq. (7.1.6) can be rewritten in the form of the fluctuation dissipative theorem¹⁹

$$(\delta \rho_n)_\omega^2 = N \left(\frac{n\Delta\omega_0}{\Omega_n} \right)^2 \frac{2\epsilon''(n, \omega)}{\Delta\omega_n |\epsilon(n, \omega)|^2}. \quad (7.1.12)$$

Using Eq. (7.1.12) and the dispersion relation for the function

$$w(\omega) = 1 - 1/\epsilon,$$

$$\text{Re} w(\omega) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im} w(\omega')}{\omega - \omega'},$$

where as previously $\mathcal{P}$ means the calculation of the principal value of the integral, one can easily calculate the noise power P_n :

$$P_n = n \left(\frac{n \Delta \omega_0}{\Omega_n} \right)^2 \operatorname{Re} \left(1 - \frac{1}{\epsilon} \right)_{\Delta \omega_n = 0} = \frac{N}{1 + N/N_c}. \quad (7.1.13)$$

Here,

$$N_c = \left(\frac{\sigma_{\parallel}}{p_s} \right)^2 \frac{2\pi p_s v_s}{e^2 \omega_s \alpha_p (Z''/n)}, \quad (7.1.14)$$

which numerically coincides with the threshold of the negative mass instability and indicates the threshold in the behavior of the beam noise. The value P_n in Eq. (7.1.13) has the physical sense of the mean square value of amplitudes of fluctuations $\delta\rho_n$ in the beam. Correspondingly, the probability to find in the beam the density fluctuation with amplitude $|\delta\rho_n|$ reads

$$w(|\delta\rho_n|^2) \propto \exp\left(-\frac{|\delta\rho_n|^2}{2P_n}\right).$$

Below this threshold the noise of the beam exactly coincides with the Schottky noise. Above it the noise of the beam is rather similar to the noise in the dense media, when the noise spectrum consists mainly of collective frequencies and its power is proportional to the media temperature. Eq. (7.1.13) shows explicitly that below transition energy of the ring the propagation along the beam of the long-living coherent oscillations suppresses the noise of the beam. Note also that in the region $N \gg N_c$ Eq. (7.1.12) can be simplified. Within this region one has $|\epsilon'| \gg |\epsilon''|$ everywhere except the roots of the dispersion equation $\epsilon'(\Delta\omega_n) = 0$. Then, using

$$\delta(x) = \frac{1}{\pi} \lim_{\Delta \rightarrow 0} \frac{\Delta}{x^2 + \Delta^2},$$

Eq. (7.1.12) can be rewritten in the form

$$(\delta\rho_n)_\omega^2 = \frac{N_c}{2} [\delta(\Delta\omega_n - \Omega_n) + \delta(\Delta\omega_n + \Omega_n)]. \quad (7.1.15)$$

The calculations given in §3.1 and §3.6 show that for the coasting beam decrements of longitudinal coherent oscillations can substantially decrease in two cases: either the beam longitudinal temperature becomes low enough, or the beam current (N) approaches the threshold of the coherent instability. In the latter case, Taylor expansion of $\epsilon(n, \omega)$ over the powers of $\Delta N/N_{th}$ yields

$$\epsilon(n, \omega) = \Delta N/N_{th} + i\epsilon''(n, \omega), \quad \Delta N = N_{th} - N.$$

Its substitution into Eq. (7.1.6) yields

$$(\delta\rho_n)_\omega^2 = \frac{2\pi^2 N}{1 - N/N_{th}} \delta[\epsilon''_n(\Delta\omega_n)] \int_{-\infty}^{\infty} d\Delta p \delta(\omega - \omega_n) f_0. \quad (7.1.16)$$

The noise spectrum again consists of the peaks whose position is determined by the roots of dispersion equation $\epsilon''(n, \Delta\omega_n) = 0$. But in contrast to the previous case, the amplitudes of these peaks increase with an approach to the instability threshold. For instance, for the negative mass instability one has $N_{th} = N_c$, and hence

$$P_n = \frac{N}{1 - N/N_c}, \quad (7.1.17)$$

which can be compared to P_n from Eq. (7.1.13).

Using Eqs. (7.1.3) and (7.1.4) one can calculate the collision integral, describing kinetic processes in bunches due to interaction of particles via surroundings. In this case, Eq. (10.21) reads

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \Delta p} \sum_n \frac{1}{N} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Re}(e\delta E_n^* \delta F_n)_\omega. \quad (7.1.18)$$

Here the contributing correlation function $(\delta E_{-n} \delta F_n)_\omega$ can be divided into two parts. One comprises the off-diagonal, while the other consists of the diagonal harmonics n and n' . Since the coherent frequency shifts are supposed to be small compared to the revolution frequency ω_s , the collective effects can only affect the diagonal part. The contributions from the off-diagonal terms can be calculated using Eqs. (2.5.26) through (2.5.31). These expressions indicate that, provided the excitation of longitudinal fields predominates, the leading contribution into collision integral gives the interval of harmonics where $|n| \geq n_\omega = \omega_s / \Delta\omega$, while the contribution from lower harmonic numbers is suppressed like $f_0(\omega_s / n \Delta\omega_0)$. Assuming the bandwidth of the electrode system W is small compared to n_ω , below we shall neglect the contribution from these off-diagonal terms in r.h.s of Eq. (7.1.18). The diagonal part of $(\delta E_{-n} \delta F_n)_\omega$ can be obtained using Eqs (7.1.3) and (7.1.4). Repeating the calculation in § 2.5 under the assumption that the distribution function $f_0(\Delta p)$ is close to the equilibrium one results in the kinetic equation

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \Delta p} \left(f_0(\Delta p) \sum_n \frac{e^2 \omega_s Z'_n}{\Pi} \frac{1}{|\epsilon(n, \omega_n)|^2} \right). \quad (7.1.19)$$

The main features of the beam relaxation described by this kinetic equation can be studied using its transformation into simpler Langevin equations. The integration of Eq. (7.1.19) with Δp and $(\Delta p)^2$ yields respectively

$$\begin{aligned}\frac{d}{dt}\langle\Delta p\rangle &= - \int d\Delta p f_0 \sum_n \frac{e^2 \omega_s Z'_n}{\Pi} \frac{1}{|\epsilon(n, \omega_n)|^2}, \\ \frac{d}{dt}\langle(\Delta p)^2\rangle &= -2 \int d\Delta p f_0 \sum_n \frac{e^2 \omega_s}{\Pi} \frac{\Delta p Z'_n}{|\epsilon(n, \omega_n)|^2}.\end{aligned}\quad (7.1.20)$$

Combining these two equations one can get the rate of the beam momentum spread $\sigma_{\parallel}^2 = \langle(\Delta p)^2\rangle - \langle\Delta p\rangle^2$:

$$\frac{d\sigma_{\parallel}^2}{dt} = -2 \int d\Delta p (\Delta p - \langle\Delta p\rangle) f_0 \sum_n \frac{e^2 \omega_s}{\Pi} \frac{Z'_n}{|\epsilon(n, \omega_n)|^2}. \quad (7.1.21)$$

As might be expected in advance, the interaction with the dissipative elements causes general deceleration of the beam. (For normal systems one has $Z'_n > 0$.) The correct modulation of these losses causes the beam cooling.

Closer inspection of Eqs. (7.1.2) and (7.1.4) indicates that the collective reaction of a beam is exhibited in shielding any RF-field affecting the beam. Using this circumstance one can easily write the contribution of electrode noise into Eq. (7.1.19). According to Eq. (2.6.40) we have

$$j_{\parallel}^{(4)} = \pi \sum_n (\delta E_n)^2_{\omega_n} \frac{\partial f_0}{\partial \Delta p} = - \frac{e^2 \omega_s Z'_n}{\Pi v_s} \frac{T_0}{|\epsilon(n, \omega_n)|^2} \frac{\partial f_0}{\partial \Delta p}.$$

The substitution of this equation into Eq. (7.1.19) gives

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \Delta p} \sum_n \frac{e^2 \omega_s Z'_n}{\Pi |\epsilon(n, \omega_n)|^2} \left(f_0 - \frac{T_0}{v_s} \frac{\partial f_0}{\partial \Delta p} \right). \quad (7.1.22)$$

It is seen that if the mean loss is compensated for by the accelerating field (not necessarily RF)

$$\sum_n \frac{e^2 \omega_s}{\Pi} \left(\frac{Z'_n}{|\epsilon_n(\omega_n)|^2} \right)_s$$

the beam reaches its stationary state with momentum distribution function

$$f_0 \propto \exp\left(-\frac{\Delta p^2}{2\sigma_{st}^2}\right)$$

with the momentum spread

$$\sigma_{st}^2 = \frac{T_0}{v_s} \left\{ \frac{d}{d\Delta p} \ln \left(\sum_n \frac{Z'_n(\Delta p)}{|\epsilon(n, \omega_n)|^2} \right) \right\}_s^{-1}. \quad (7.1.23)$$

In this case, both the relaxation time

$$\tau_r^{-1} = \frac{e\omega_s}{\Pi} \frac{d}{d\Delta p} \left(\sum_n \frac{Z'_n(\Delta p)}{|\epsilon(n, \omega_n)|^2} \right),$$

and the beam momentum spread σ_{st} generally depend on the beam current.

Remember that kinetic equation (7.1.22) was obtained assuming that the interaction with surrounding electrodes does not cause any coherent instability of the beam. This means that the value Z'_n cannot be too large since Ω_n^2 should not occur close to the boundary of the stability diagram. For devices with the nonzeroth impedance on the orbit this region can be estimated by Z/n criterion

$$(n\Delta\omega_0)^2 > |\Omega_n^2|. \quad (7.1.24)$$

Within this region $1/\epsilon(n, \omega)$ is quite a smooth function. Therefore, one may expect that the collective reaction of the beam will only slightly change the kinetic coefficients. Moreover, the condition (7.1.24) means also that the friction force due to interaction of particles with a dissipative system of electrodes will also be limited and, thus, cannot reach very high values. In fact, from Eq. (7.1.24) one can write

$$\frac{e^2\omega_s}{\Pi} |Z'| \ll \frac{n\Delta\omega_0^2}{N|\omega'_0|}, \quad \epsilon(n, \omega) \simeq 1,$$

which estimates the value of the friction force $\mathcal{F}_{fr}$ by the expression

$$|\mathcal{F}_{fr}| = \sum_n \frac{e^2\omega_s}{\Pi} |Z'_n| \ll \frac{W^2\Delta\omega_0}{n} \sigma_{\parallel}. \quad (7.1.25)$$

Here, W is the maximum harmonic number where condition (7.1.24) could be violated.

Similarly, using the results of § 3.1 one could calculate the spectral densities of correlators and collision integrals for the interaction of a general kind [see Eq. (13.6)], when the Green's function of the system can be expanded in series over the multipoles. One can find the relevant calculations in Ref. 18 and 101. We will not repeat them here. Note only that according to Eq. (1.3) without Landau damping the transverse coherent oscillations of the coasting beam may become unstable for modes $|n| > |\mathbf{m}_1 \mathbf{v}_1|$. These modes will be stabilized by the Landau damping provided the condition similar to that in Eq. (7.1.24) is valid:

$$\Delta\omega_{mn} \gg |\Omega_{mn}|,$$

where Ω_{mn} and $\Delta\omega_{mn}$ are the coherent frequency shift and frequency spread of the oscillation mode with $\omega \simeq \mathbf{m}_1 \omega_{\perp} + n\omega_s$. Analogous to Eq. (7.1.24) this condition means that collective effects result only in the slight modification of kinetic coefficients and, thus, the kinetic process itself cannot be very fast. The corresponding kinetic equation can easily be obtained from Eqs. (2.5.26)–(2.5.31) by the “renormalization” of the diagonal terms according to the rule

$$e^2 \rightarrow \frac{e^2}{|\epsilon_{mn}(\omega_{mn})|^2}$$

where

$$\epsilon_{mn}(\omega) = 1 + Ne^2 \int d\Gamma \frac{m \partial f_0 / \partial \mathbf{I}}{\omega - \omega_{mn}} G_{mn}(1, 1, \omega_{mn})$$

is the beam dielectric penetrability relevant to this case. The structure of such a kinetic equation is quite similar to the Lenard-Balesku equation in plasma theory.

7.2. LONGITUDINAL SCHOTTKY NOISE OF COOLED BEAMS

Since basic properties of the noise in the normal beam were discussed in the previous section, below we shall concentrate on the analysis of the specific features of the Schottky noise in the beam with strong cooling of particles. As background we shall keep in mind the application of the results to the case of the momentum electron cooling of the coasting beam. (See Chapter 8.)

The influence of strong cooling on properties of the equilibrium Schottky noise spectra can be calculated using the Fokker-Planck equation instead of Eq. (7.1.1) (see also §3.6):

$$\frac{\partial \delta F}{\partial t} + \omega_0(p) \frac{\partial \delta F}{\partial \vartheta} + e \delta E(\vartheta, t) \frac{\partial f_0}{\partial \Delta p} = \frac{\partial}{\partial \Delta p} \left[\lambda \Delta p \delta F + d \frac{\partial \delta F}{\partial \Delta p} \right]. \quad (7.2.1)$$

To calculate the values referring to the equilibrium state of the beam we can say that both λ and the diffusion coefficient d are constants, while $F_0(\Delta p)$ is a Gaussian distribution with $\sigma_{\parallel} = d/2\lambda$. Before making the particular calculations we have to remember that indeed Eq. (7.2.1) is not the usual kinetic equation because the function $\delta F(\Delta p, \vartheta, t)$ entering there is not the conventional distribution function, but is only its fluctuational part averaged over collisions. Generally this function should be calculated from the microscopic equation

$$\frac{\partial}{\partial t} \delta F + \omega_0(p) \frac{\partial}{\partial \vartheta} \delta F + e \delta E(\vartheta, t) \frac{\partial f_0}{\partial \Delta p} = - \frac{\partial}{\partial \Delta p} \{ [-\lambda \Delta p + D(t)] \delta F \},$$

where $D(t)$ is the random function describing collisions. Therefore, the use of Eq. (7.2.1) is based on the assumption that the correlation time of external noise is much shorter than that of beam fluctuations.

Simple calculations with Eq. (7.2.1) yield for the harmonics of fluctuations $\delta \rho_n(\omega)$ the same equation as in Eq. (7.1.4) (see in §3.6 for details),

$$\delta \rho_n(\omega) = \delta \rho_n^{(0)}(\Delta \omega_n) / \epsilon_n(\Delta \omega_n),$$

and Eq. (7.1.5) for the Schottky spectra. Since $\epsilon_n(\omega)$ was calculated in §3.6, here we have to calculate only the spectrum of the beam with low intensity $(\delta\rho_n^{(0)})_\omega^2$. That can easily be done using the definition of $(\delta\rho_n^{(0)})_\omega^2$ as the Fourier harmonic of the correlation function:

$$K_n^{(0)}(\tau) = \left\langle \sum_{a=1}^N \overline{\exp(-in[\vartheta_a(t+\tau) - \vartheta_a(t)])} \right\rangle \quad (7.2.2)$$

$$(\delta\rho_n^{(0)})_\omega^2 = \int_0^\infty d\tau e^{i\omega\tau} K_n^{(0)}(\tau). \quad (7.2.3)$$

The right-hand side in Eq. (7.2.2) can be calculated directly. Using the equations of motion of a particle,

$$\dot{\omega}_a = -\lambda\omega_a + d(t), \quad \omega_a = \omega'_0 \Delta p_a, \quad \dot{\vartheta}_a = \omega_a \quad (7.2.4)$$

where collisions are described by the stationary white noise $D(t)$,

$$\overline{D(t)D(t')} = \overline{D^2} \delta(t-t'), \quad \overline{D^2} = 2\lambda\Delta\omega^2, \\ \overline{D(t_1) \dots D(t_{2k+1})} = 0,$$

$$\overline{D(t_1) \dots D(t_{2k})} = \sum_P \overline{D(t_1)D(t_2) \dots D(t_{2k-1})D(t_{2k})}, \quad (7.2.5)$$

and P means the summation over all combinations of couples t_i, t_j . The simple calculations yield

$$K_n^{(0)}(|\tau|) = \exp\{-q\Psi(\lambda|\tau|)\} \left\langle \sum_{a=1}^N \exp\left\{\frac{in\omega_a}{\lambda}(1-e^{-\lambda|\tau|})\right\} \right\rangle,$$

$$\Psi(x) = x - 2(1-e^{-x}) + (1-e^{-2x})/2, \quad q = (n\Delta\omega/\lambda)^2, \quad (7.2.6)$$

or, using Eq. (2.4.3),

$$K_n^{(0)}(|\tau|) = \exp\{-q\Psi(\lambda|\tau|)\} \int_{-\infty}^{\infty} d\Delta p f_0 \exp\left\{\frac{in\omega'_0 \Delta p}{\lambda}(1-e^{-\lambda|\tau|})\right\}. \quad (7.2.7)$$

In the region $\lambda\tau \gg 1$ and, hence, $\Psi(\lambda\tau) \simeq \lambda\tau$, this function decays with the time constant (see also §3.6)

$$\tau_n = \frac{\lambda}{(n\Delta\omega)^2} \gg (n\Delta\omega)^{-1}.$$

For the Gaussian distribution function Eq. (7.2.7) can be rewritten in the form

$$K_n^{(0)}(n\Delta\omega) = \frac{2N}{\lambda} e^q \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \frac{q+l}{(\Delta\omega_n/\lambda)^2 + (q+l)^2}. \quad (7.2.8)$$

The power of this spectrum coincides ($P_n=N$) with that for the normal, low-intensity beam, but the widths of Schottky bands are much smaller provided the cooling is strong:

$$K_n^{(0)}(n\Delta\omega) = 2N \frac{\lambda q}{\Delta\omega_n^2 + (q\lambda)^2}, \quad q \ll 1. \quad (7.2.9)$$

The Schottky noise spectrum for the cooled beam can be obtained by the substitution of $\epsilon_n(\omega)$ from Eqs. (3.7.17) and (7.2.8) into Eq. (7.1.5). As the comparison of Eqs. (3.6.17) and (7.2.8) gives

$$K_n^{(0)}(\Delta\omega_n) = \frac{2N}{\Delta\omega_n} \operatorname{Im} \left[\frac{\epsilon_n - 1}{\xi} \right], \quad (7.2.10)$$

one can write $(\delta\rho_n)_\omega^2$ using only one function, $\epsilon_n(\omega)$:

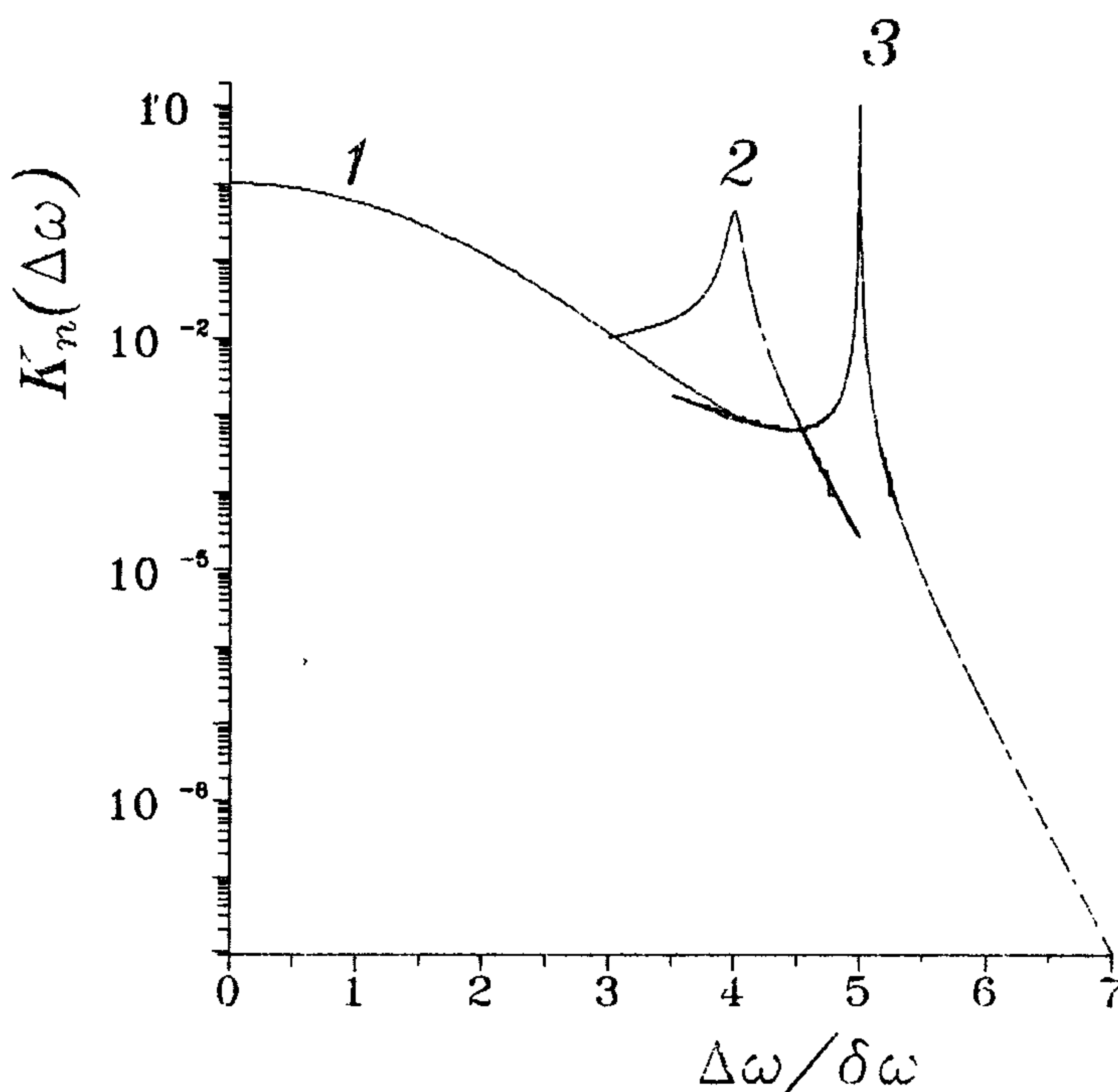


FIG. 7.1. Dependence of the shape of the Schottky spectrum on the beam intensity; $q=100$; $\xi''=0$; 1— $\xi'=16$, 2— $\xi'=16$, 3— $\xi'=25$.

$$(\delta\rho_n)_\omega^2 = \frac{2N}{\Delta\omega_n |\epsilon_n(\Delta\omega_n)|^2} \text{Im} \left[\frac{\epsilon_n - 1}{\xi} \right], \quad \xi = \Omega_n^2 / (n\Delta\omega_0)^2. \quad (7.2.11)$$

It is easy to verify that for normal beams with Gaussian momentum distribution Eq. (7.1.5) coincides with Eq. (7.2.11), which thus gives the universal representation for Schottky spectra of both normal and cooled beams.

As seen from Figs. 7.1 and 7.2, with the increase in the beam current the Schottky spectra may get rather sharp spikes when

$$|\Omega_n^2| \gg (n\Delta\omega)^2, \quad |\Omega_n^2| \gg \lambda^2.$$

However, close inspection of these figures shows significant difference in the widths of the spikes on these spectra for high-frequency ($q \gg 1$, Fig. 7.1) and low-frequency ($q \ll 1$, Fig. 7.2) modes. The dissipation in the beam environment causes the asymmetry of the high-frequency and low-frequency spikes. (See Fig 7.3.)

First, the Schottky noise measurements with deeply cooled proton beam have been carried out on the storage ring NAP-M.^{99,102} The main goal of these experiments was the measurement of the beam longitudinal temperature after electron cooling as well as the study of the influence of the cooling beam on the characteristics of the beam noise.

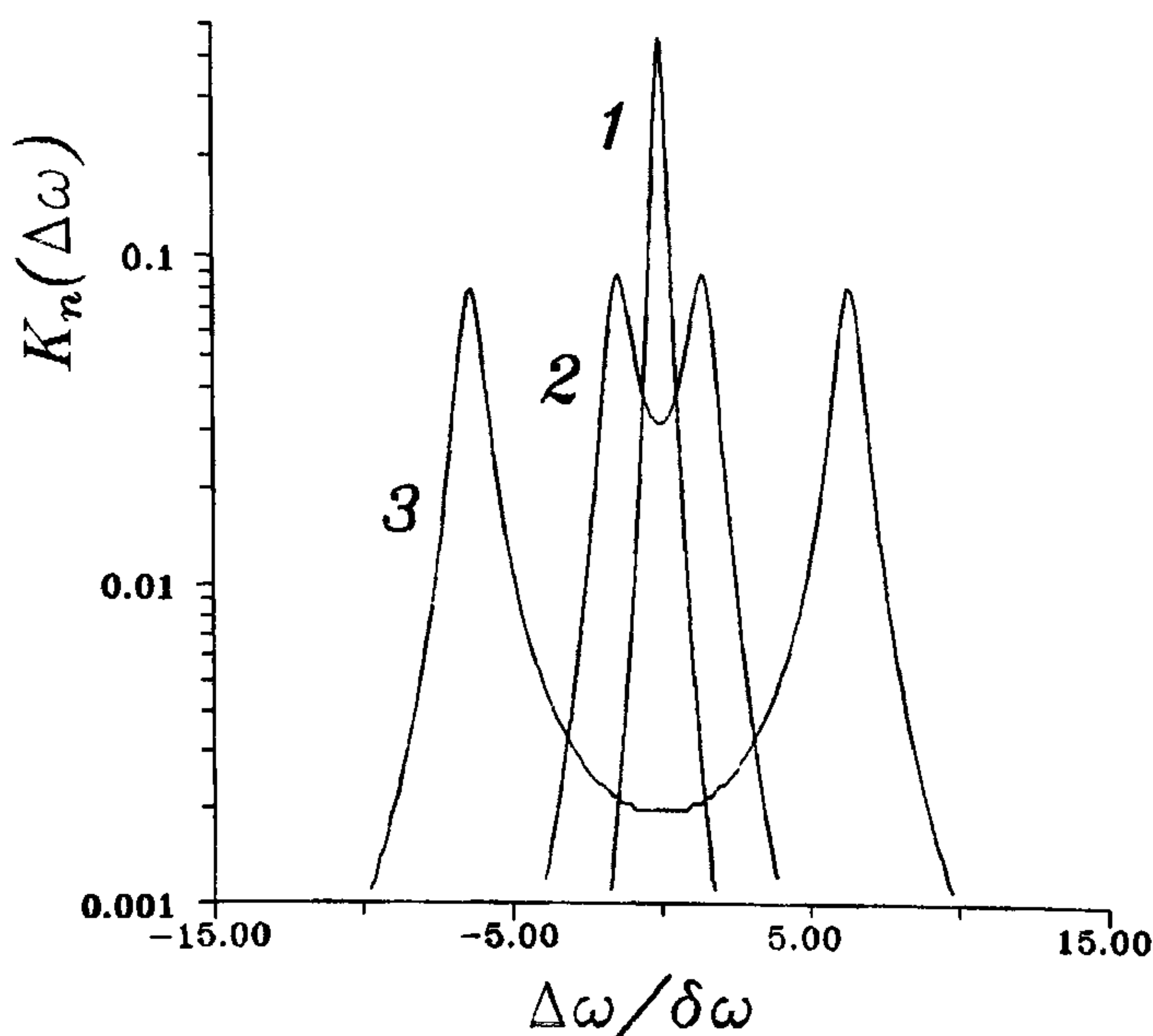


FIG. 7.2. Dependence of the shape of the Schottky spectrum on the beam intensity; $q=0.01$; $\xi''=0$; 1— $\xi'=16$, 2— $\xi'=256$, 3— $\xi'=4096$.

The storage ring NAP-M (see Fig. 7.4) was designed initially as the prototype of the antiproton accumulator NAP and then was constructed especially to study the electron cooling. This was a proton synchrotron with the following parameters:

Energy	1.5–150 MeV
Bending radius	3 m
Straight sections	4 × 7.1 m
Betatron tunes:	
ν_x	1.34
ν_z	1.24
Aperture of the vacuum chamber:	
in dipoles ($h \times w$)	± 7 times 11 cm
in straight sections (ϕ)	7 cm
η in straight sections	6 m
Transition energy	110 MeV
Average vacuum	5×10^{-10} torr

Typically, the measurements have been carried out after the acceleration of the proton beam until the energy 65 MeV and its subsequent debunching. The beam temperature could be varied by the manipulations with the cooling device. The beam noise spectra were measured using the integral pickup electrode on the fifth and eighth harmonics of the revolution frequency. The block diagram of the

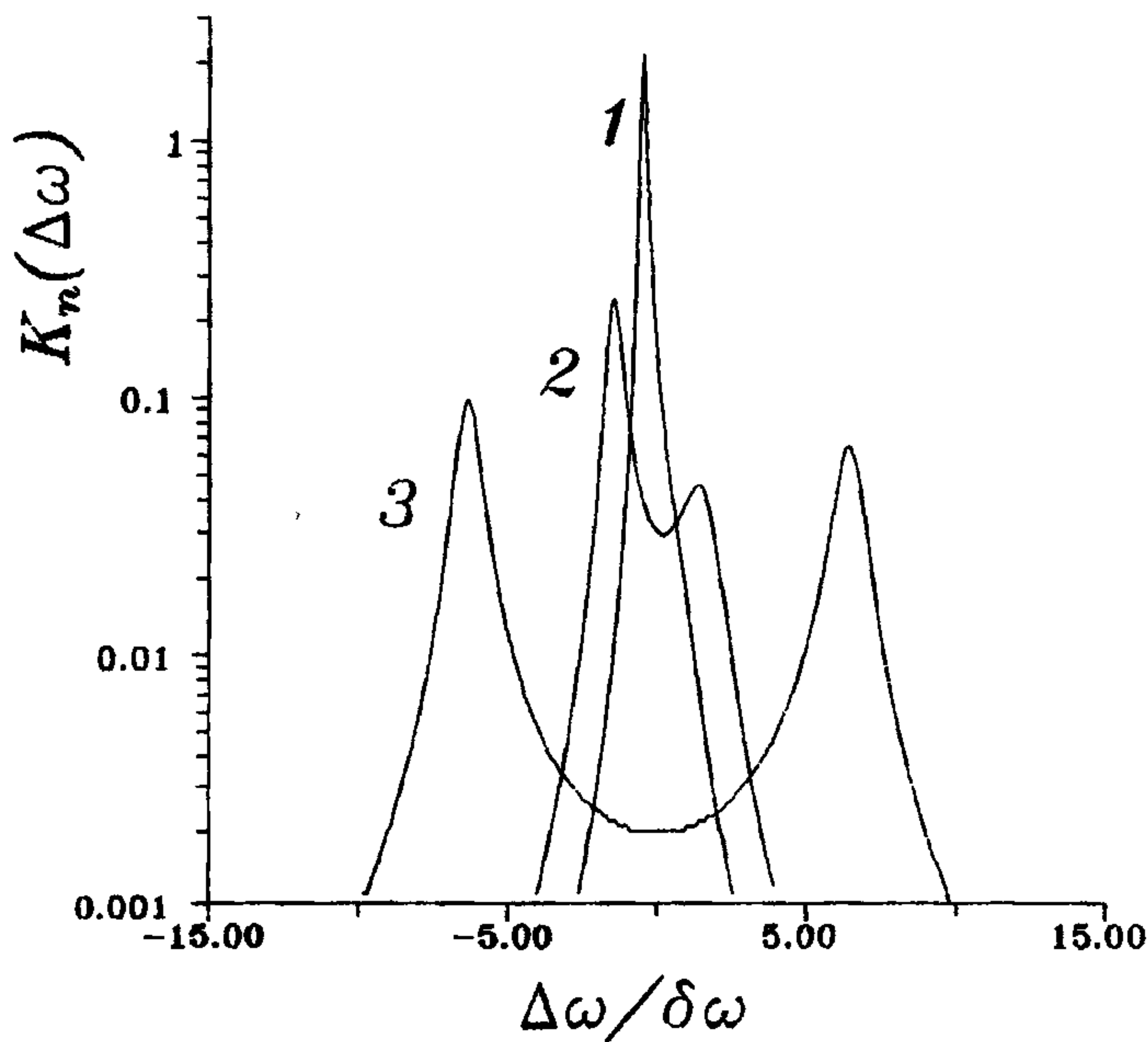


FIG. 7.3. Same as in Fig. 7.3, but $\xi'' = -64$.

eighth harmonics channel is shown in Fig. 7.5. To enhance the sensitivity of measurements, the capacity of the pickup C_d and the inductance L were chosen to form the resonance circuit tuned to the eighth harmonic of the revolution frequency (17.6 MHz). The circuit Q-factor was $Q = 500$. The shunt impedance of the circuit was 40 KOhms. Therefore, with the connection of the field tetrode the measurement system noises were practically determined only by the heat noises of the circuit and were 10^{-18} W within the frequency band of ± 1 KHz. (The amplifier noises were lower than 12 db.) The input cascade of the amplifier was performed according to the cascode scheme. In the following schemes the double transform of the frequency was used. The output signal of the second mixer (synchronous detector) after the transformation into digital code was transmitted to a computer. The frequency band of the measured signal on the ADC input was 1 KHz, the clock frequency of ADC was 2 KHz, and the conversion error was 0.1%.

The measured data was processed using the FFT code. The transformation used was a 512-word-long array comprising the results of measurements performed for the 0.25-s period. In the selected method with the parallel Fourier

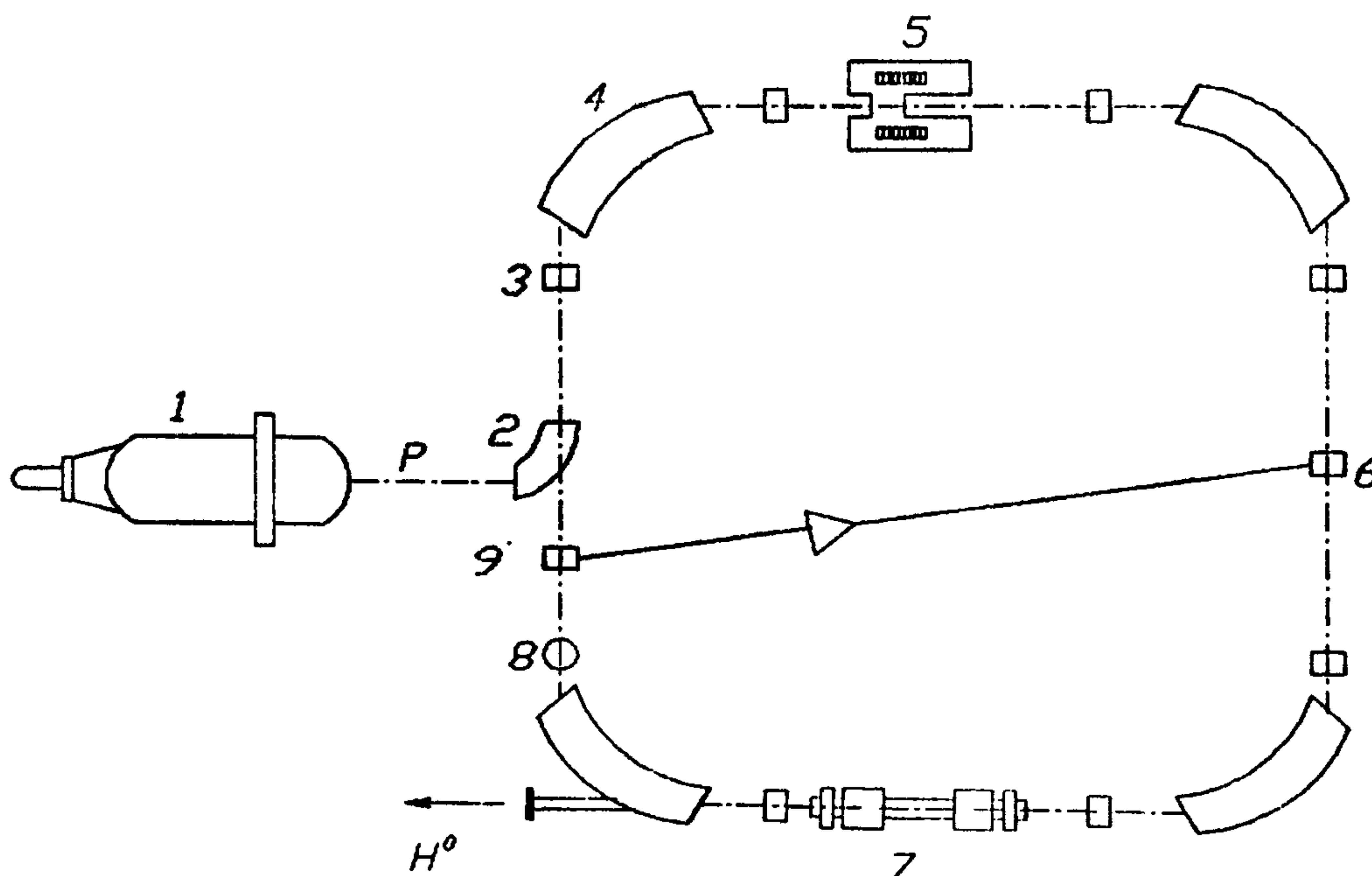


FIG. 7.4. Schematic layout of the cooling ring NAP-M; 1—RF cavity; 2—stochastic cooling pick-up; 3—electron cooling device; 4—Schottky pick-up; 5—stochastic cooling pick-up.

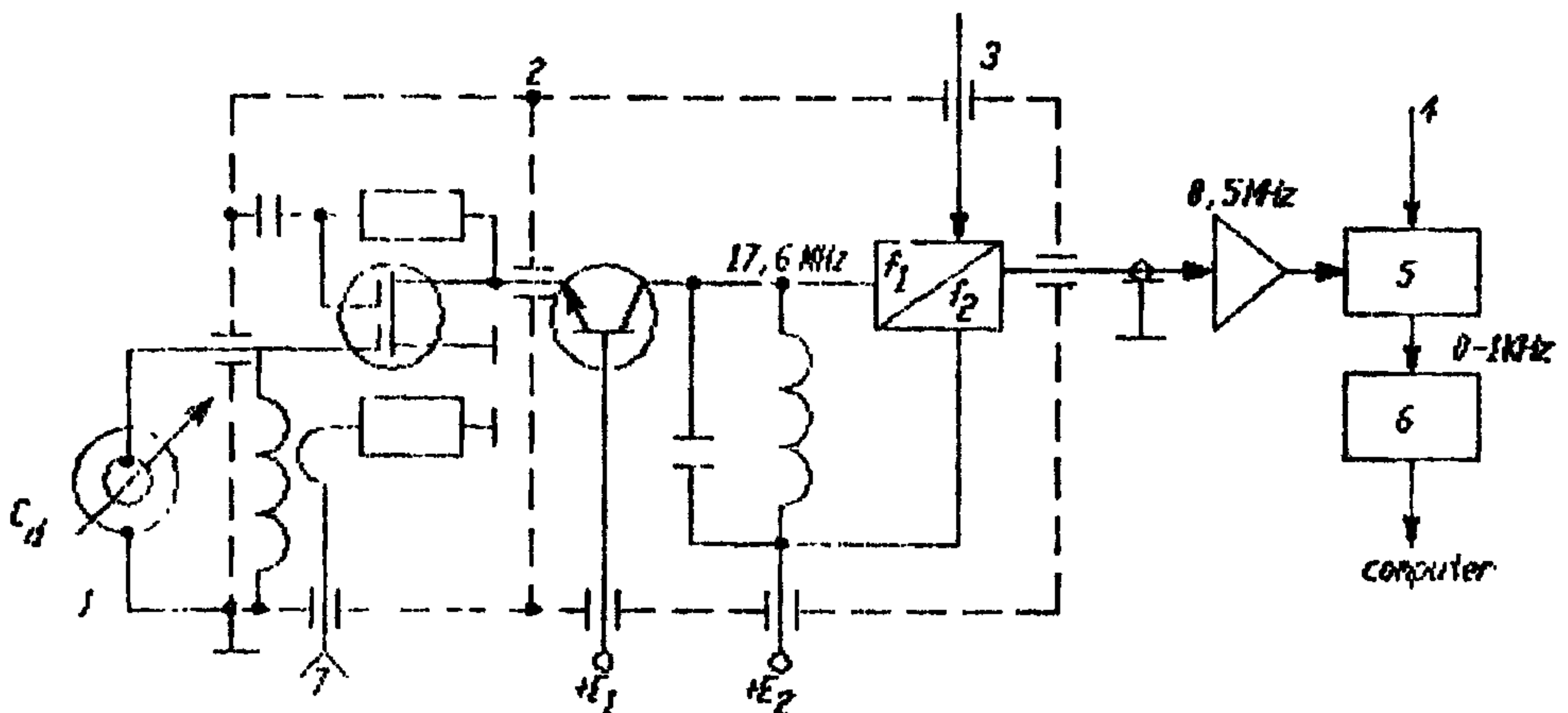


FIG. 7.5. The block-diagram of the channel to measure Schottky noise signal at the 8th harmonic of the revolution frequency; 1—the beam; 2—pick-up station; 3—the 1st tuner 18.1 MHz; 4—2nd tuner 0.5 MHz; 5—synchronous detector; 6—analogue to digital converter; 7—calibration input.

analysis the time for measuring the spectrum of $\Delta\Omega$ width with the given resolution $\Delta\omega$ was $\Delta\Omega/\Delta\omega$ times lower than that for the method of subsequent sweeping of the spectrum. This is especially important for measurements of nonstationary processes as it enables one to decrease substantially the influence of the ripples of the guiding magnetic field of the ring and, correspondingly, the beam revolution frequencies on the results of measurements.

The main effect discovered even in the first measurements¹⁰² was strong Schottky noise signal suppression after the beam cooling. The subsequent improvement of the measuring device enabled the reliable measurements of quite low noise signals of the cooled beam. These more precise measurements of the dependence of the signal power on the beam current (see Fig. 7.6) showed that due to beam cooling the noise power decreases approximately by two orders of magnitude and within the current range 0.5–10 mA the noise power of a cooled beam does not depend on its current I_p . The same figure indicates that the noise power of the uncooled beam linearly grows with an increase in I_p .

According to the diagram in Fig. 7.5, the relation between the voltage on the pickup electrode U_n and an amplitude of azimuthal harmonics of the beam density $\delta\rho_n$ is defined by the equation

$$\langle |U_n|^2 \rangle = 2 \left(\frac{eQ l_d}{C_d \Pi} \right)^2 \langle |\delta\rho_n|^2 \rangle. \quad (7.2.12)$$

Here, Q is the quality factor of the input circuit, while l_d and C_d are the length and capacity of the pick-up electrode respectively. This expression was used for the calculation of amplitudes of harmonics $|\delta\rho_n|$ in Fig. 7.6. An increase in the

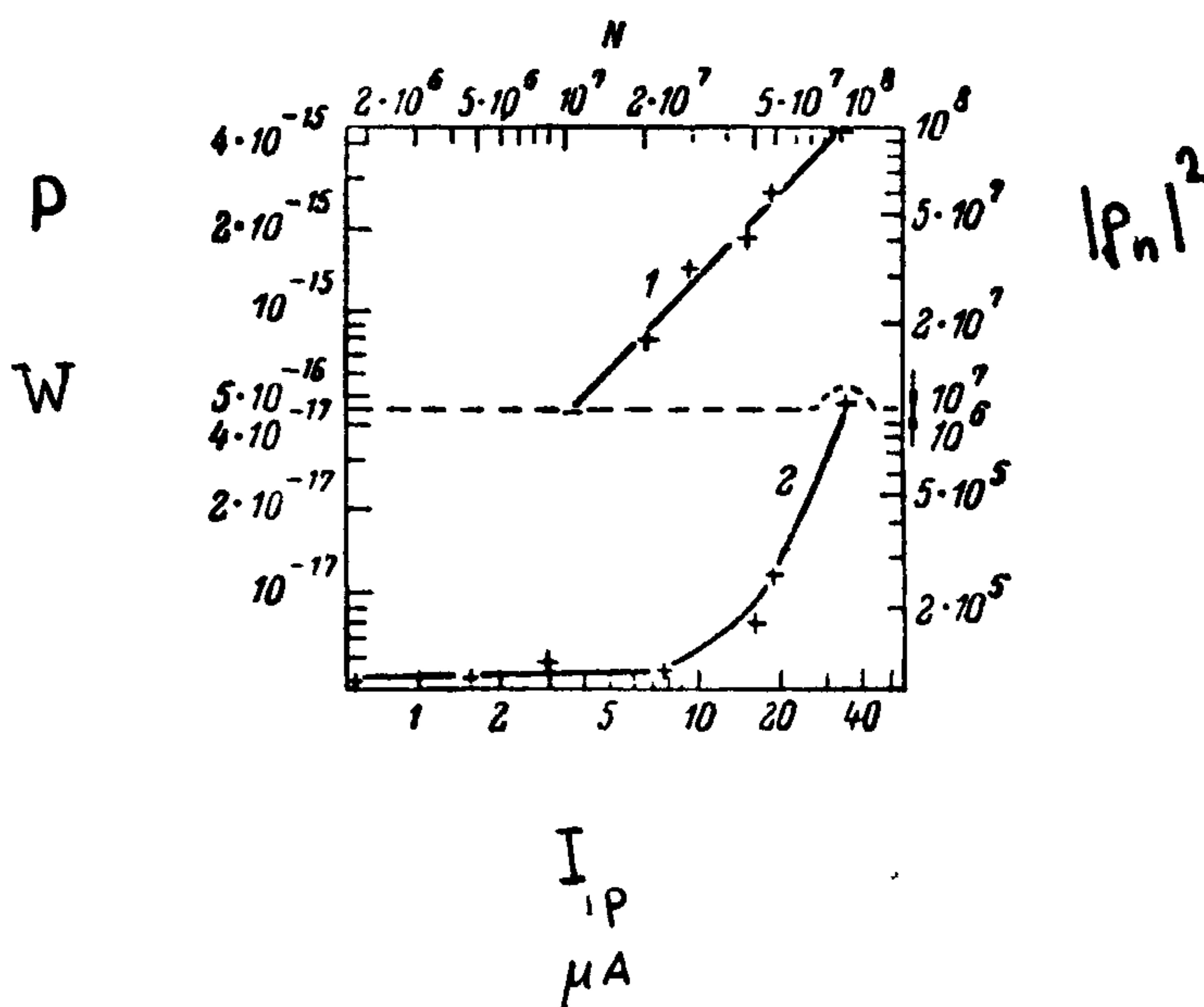


FIG. 7.6. The dependence of the Schottky noise power on the beam current measured at the 8th harmonic of the revolution frequency; 1—uncooled beam, 2—cooled beam

noise power of the cooled beam after the current $I_p = 10$ mA (curve 2 in Fig. 7.6) is apparently caused by the IBS.*

At large beam currents two well pronounced peaks appear in the noise spectrum of the cooled beam. (See Fig. 7.7.) As was already mentioned, this feature is related to the propagation of longitudinal waves upstream and downstream the beam. According to Eq. (7.1.15) the distance between these peaks in the spectrum is equal to $2\Omega_n$. With the decrease in the beam current the spectrum maxima are becoming closer and at a small beam current [Fig. 7.7 (c), $I_p = 0.5$ mA] become beyond the resolution of the monitor.

The dependence of the relative coherent frequency shift Ω_n/n on the proton beam current has been calculated using the data of measurements of the distance between the spectrum maxima for the fifth and eighth harmonics of revolution frequency. (See Fig. 7.8.) One can see that Ω_n/n is proportional to $I_p^{1/2}$. Using these data and Eq. (3.2.21), one can recalculate the longitudinal coupling impedance of the vacuum chamber:

$$\frac{Z_n}{n} = \left(\frac{\Omega_n}{n\omega_s} \right) \frac{p_s v_s}{e I_p \alpha_p}, \quad I_p = \frac{Ne\omega_s}{2\pi}. \quad (7.2.13)$$

*Strictly speaking this statement needs additional analysis of IBS in extremely cold beams. (See Chapter 8 for more details.) Another reason for such an increase is the approaching of the beam to the threshold of some longitudinal instability.

For parameters of the storage ring NAP-M this equation and data from Fig. 7.8 give $Z/n = 557 \pm 50$ Ohm, which is in good agreement with the impedance of the ideally conducting chamber with radius $l_\perp = 3.5$ cm, containing the beam of the radius $a_0 = 0.01$ cm.

The results of the noise spectrum measurements can also be used for the calculation of the momentum spread of the beam. For the hot, low-intensity beams ($n_c \gg n$) the value $\Delta p/p$ is related to the noise spectrum width by the expression

$$\frac{\Delta p}{p} = \frac{1}{\alpha_p} \left(\frac{\Delta \omega_n}{n \omega_s} \right). \quad (7.2.14)$$

For the uncooled beam (see Fig. 7.7) the half-width of the spectrum is 250 Hz on the eighth harmonics of the revolution frequency ($\omega_s/2\pi = 2.2 \times 10^6$ Hz). This gives the value for the momentum spread $\Delta p/p = 1.8 \times 10^{-4}$. In contrast to this, after cooling down $N_c \ll N$ the width of the spectrum is determined by the beam intensity, while the noise power P_n is determined by the momentum spread. Rewriting Eq. (7.1.13) in the form

$$\frac{P_n}{(P_n)_0} = \frac{\langle |\delta \rho_n|^2 \rangle}{N} = \left(\frac{n \Delta \omega_0}{\Omega_n} \right)^2,$$

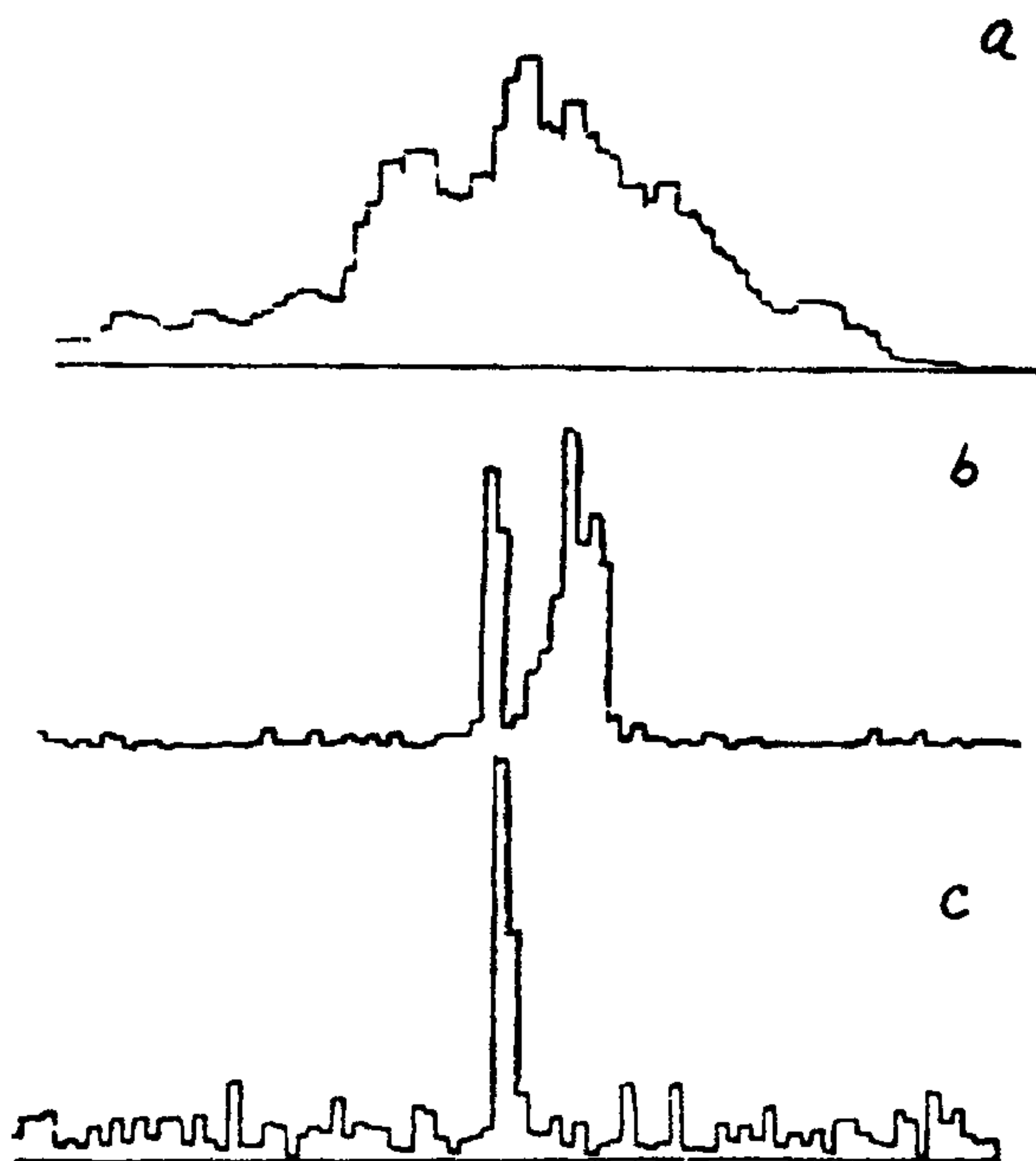


FIG. 7.7. Schottky noise spectra of the uncooled (a), and of the cooled proton beam (b,c).

where $(P_n)_0$ is the noise power of the beam of the same intensity but with larger spread over momenta. (See Fig. 7.6, curve 1.) Using the data given in Figs. 7.6 and 7.8 for the cooled beam at current $I_p \leq 10$ mA we obtain

$$\frac{\Delta p}{p} = \frac{1}{\alpha_p} \frac{\Omega_n}{n\omega_s} \left(\frac{P_n}{(P_n)_0} \right)^{1/2} \simeq 1.4 \times 10^{-6}.$$

This momentum spread corresponds to an extraordinarily small longitudinal temperature of the beam $T_{\parallel} = 1$ K.

Using Eqs. (7.1.13) and (7.1.14) one can estimate the region of harmonic numbers where the noise of a beam with the given longitudinal temperature will exactly be the Schottky noise. The value Z''/n in Eq. (7.1.14) is approximately constant up to harmonic numbers of the order of $|n| = n_0 = R_0/l_{\perp}$. ($l_{\perp}$ is the transverse pipe aperture.) After $n = n_0$ the ratio Z''/n decreases approximately like n^{-2} . Then the condition $N_c \gg N$ will be satisfied in the region

$$n^2 \gg n_D^2 = (\Omega_n / \Delta\omega_0)^2 |_{n \gg n_0}, \quad (7.2.15)$$

where $r_D = R_0/n_D$ is the Debye radius corresponding to the longitudinal temperature of a beam. This condition is very clear—fluctuations of density harmonics

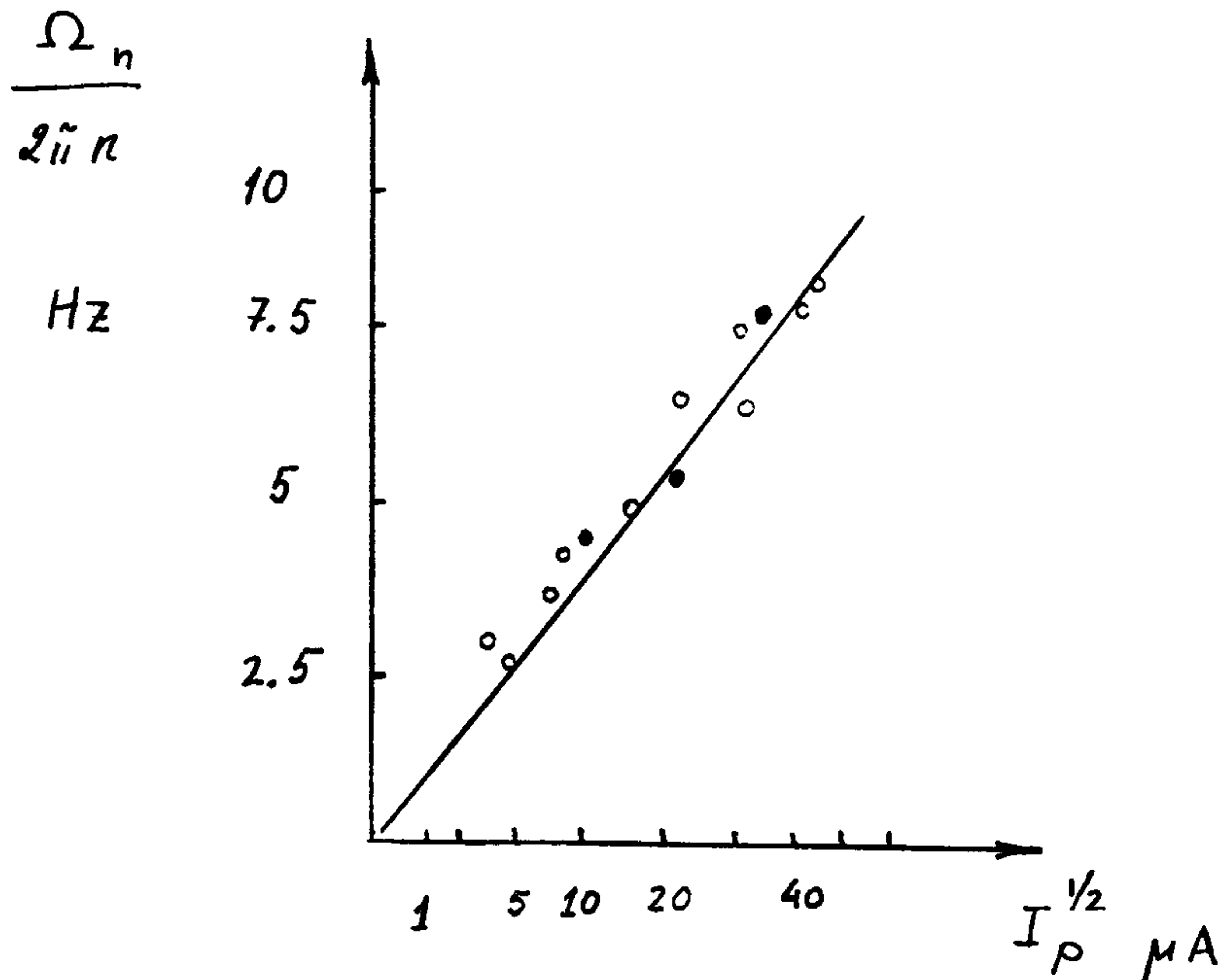


FIG. 7.8. Dependence of the coherent frequency shift measured by the splitting in the Schottky noise spectra on the beam current; $\circ$ data measured at the 5th and $\bullet$ at the 8th harmonics of the revolution frequency.

beyond the Debye scale are statistically independent (if only $2\pi n_D \ll N$).

The growth of the power and spectral densities of the beam noise near the thresholds of coherent instabilities can be useful for the study of parameters of these instabilities from the subthreshold region. In this case, the distance to the threshold can be regulated, for example, varying the beam frequency spread (say, due to beam cooling). The peak-up electrodes should have symmetry relevant to these measurements.

The results of the analysis performed show that the data obtained in measuring the beam noise spectrum have information on the number of particles in a beam, on the revolution frequency spread, and on the values of damping decrements. But, as seen from the data obtained, the interaction of particles and friction can distort substantially the spectrum shape, which should be taken into account while calculating the beam parameters.

7.3. STOCHASTIC COOLING OF LOW-INTENSITY BEAMS

By now, one of the most developed methods for cooling relatively low intensity beams of heavy particles is the method of stochastic cooling proposed by Van-der-Meer in 1972.¹⁰⁰ (Reference 103 provides a wide bibliography on the subject.) In this method, the signal from the pickup electrode proportional to the particle deviation from the equilibrium orbit through the wideband feedback is applied to the element correcting the measured deviation (kicker). With the correct selection of the phase advance in the feedback circuit, the interaction of particles with such a system leads to the cooling of a beam.

Let us estimate the damping decrements for betatron oscillation of a beam cooled by the simplest system in Fig. 7.9. (See also Ref. 104.) Let the pickup electrode and kicker consist of two pairs of plates terminated by their characteristic impedances. We shall assume also that their apertures l_1 and lengths l coincide and correspond to the bandwidth of the cooling system Wf_s :

$$l_1 \propto l \simeq \frac{v_s T_s}{W} = \frac{\Pi}{W} \quad (7.3.1)$$

Passing the pickup electrode, a particle induces the voltage

$$U \simeq \frac{e\omega_s W Z_0}{2\pi} \frac{x(t)}{l_1}. \quad (7.3.2)$$

Here, Z_0 is the characteristic impedance of plates. After its wideband amplification the voltage U is applied to the kicker simultaneously with the arrival of a particle. As a result, the momentum of the particle is changed by the value

$$\Delta p_x = \frac{\kappa e U(t-\tau)}{l_1} \frac{T_s}{W} = \kappa e^2 Z_0 \frac{x(t-\tau)}{l_1^2}, \quad (7.3.3)$$

where κ is an amplification factor, and τ is the delay time for the signal in the feedback circuit. The variation of an amplitude of betatron oscillations $a_x = \sqrt{\epsilon \beta_x}$, which corresponds to Eq. (7.1.3), is equal to

$$\Delta \epsilon_x = \kappa \frac{2e^2 Z}{p_s l_1^2} x(t - \tau) \frac{\partial x}{\partial \psi_x}.$$

Dividing this expression by the revolution period T_s and averaging both parts over ϵ_x and ψ_x , we obtain the average variation rate for the beam emittance

$$\begin{aligned} \frac{\partial \epsilon_x}{\partial t} &= \left\langle \frac{\Delta \epsilon_x}{T_s} \right\rangle = -\lambda_x \epsilon_x \\ \lambda_x &= \kappa \sin \Delta \mu_x \frac{e^2 \omega_s Z_0}{2\pi p_s} \frac{\sqrt{\beta_{p-u} \beta_k}}{l_1^2}. \end{aligned} \quad (7.3.4)$$

Here, β_{p-u} and β_k are the values of β -functions in pickup electrode and kicker, and $\Delta \mu_x$ is a phase advance of betatron oscillations between the pickup electrode and the kicker. From Eq. (7.3.4) it is seen that the interaction of a particle with the cooling system will lead to a decrease in the beam emittance if

$$\kappa \sin \Delta \mu_x > 0. \quad (7.3.5)$$

If $\kappa > 0$, it means that the pickup electrode and kicker should be located along the orbit in such a way that

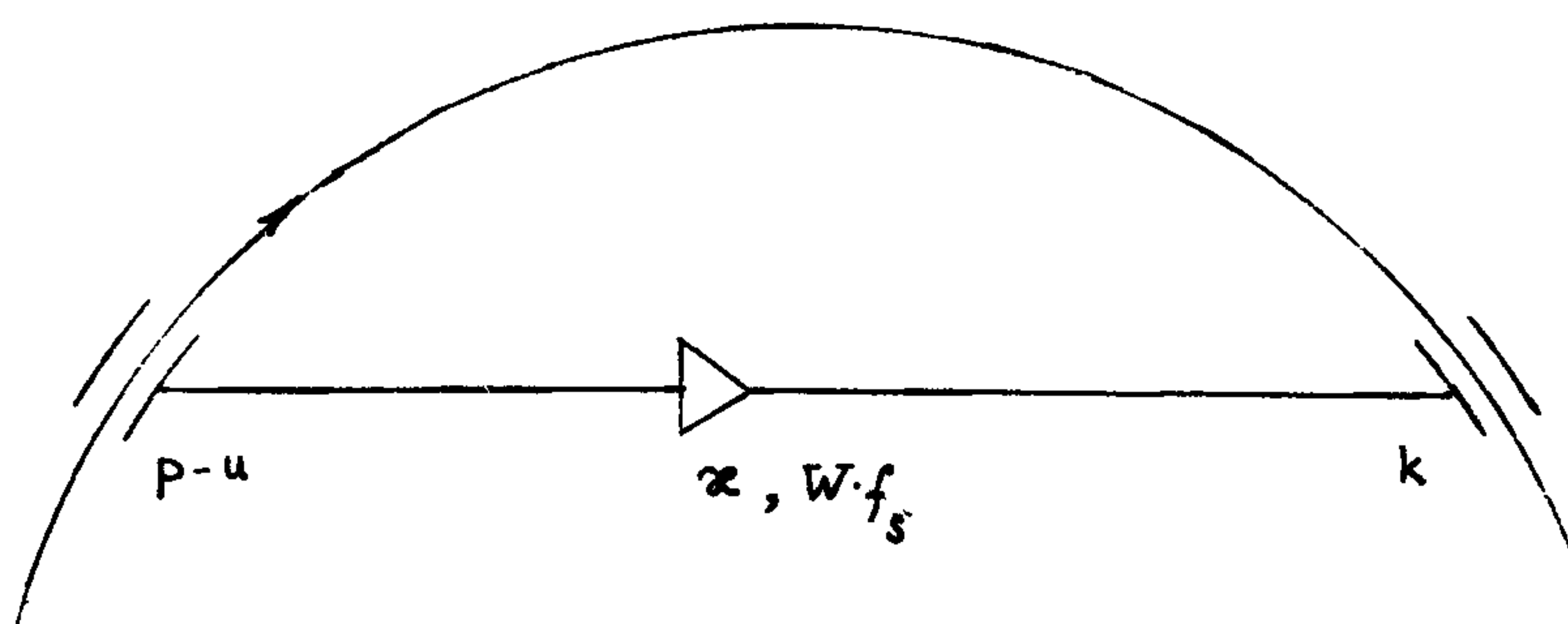


FIG. 7.9. The simplest example of the betatron stochastic cooling system; the distance between the pick-up and kicker is integer number of the quarters wave-lengths of betatron oscillations.

$$\Delta\mu_x = \frac{\pi}{2} + 2\pi k, \quad k=0, 1, 2, \dots$$

Actually, in the real schemes the pickup electrode measures the dipole momentum of the beam sample which has an azimuthal length $\Delta\vartheta \simeq 1/W$. The latter is stochastically varied in time. This fact has resulted in some misunderstandings in explanations of the stochastic cooling mechanism. (See, for instance Refs. 101 and 105–107.) The physics of the process was revealed in Ref. 18, which pointed out that cooling is caused by signals produced by individual particles in the feedback circuit and by the dissipation of particle energy in the circuit. In this sense, the method of stochastic cooling differs from the other methods only by the way of introducing the dissipative forces onto the orbit of the storage ring. From the physical point of view, like in the other methods, the cooling is connected with the action of the radiation reaction force on the particle. [See Eq. (2.5.41)].

As the stochastic cooling scheme consists typically of the relatively low frequency elements, the description of its kinetics is complicated because we must take into account the interaction between cooling particles. The signal induced by the particle given exists on the kicker during the time $\sim T_s/W$, determined by the bandwidth of the cooling system $W\omega_s$. Therefore, the particles located at a distance $\Delta\vartheta < 1/W$ along the azimuth are interacting with each other via the cooling system. As the beam is cooled down, the time of the interaction for the given couple of particles increases. Therefore, generally speaking, the influence of collective effects and the intraparticle interactions on the cooling process become stronger.

All these effects and limitations caused by them will be considered later. In this section, we confine ourselves to the consideration of the simplest kinetic effects, which can be described with the collision integral with a current, given by Eq. (2.5.25). For the sake of simplicity let us make the calculations for the coasting beam. By many reasons this case turns out to be the most important now.

The elements measuring and correcting the particle motion in the method of stochastic cooling can have such a symmetry when damping is performed on one of the degrees of freedom. The fields in these elements are increasing as they near the electrodes and therefore can be expanded in series over the multipoles. In this case, if the transverse size of the beam a is small compared to the aperture of the vacuum chamber l_1 , one can only take into account the leading terms of these expansions.

Taking into account these remarks, let us consider the cooling of the energy spread of a coasting beam using two schemes: the scheme with the notchfilter proposed by L. Torndal [Fig. 7.10(a)] and the so-called Palmer scheme, where for the measurement of momentum Δp deviation from its equilibrium value the differential pickup electrode is used [Fig. 7.10(b)].

The calculation of the cooling decrements and temperature of a beam can be

simplified if the kinetic equation (2.5.15) is replaced by the Langevin equation. Integrating Eq. (2.5.15) with Δp^2 , we obtain

$$\frac{d}{dt} \langle (\Delta p)^2 \rangle = \int d\Gamma 2\Delta p j_{\parallel}(\mathbf{I}, t). \quad (7.3.6)$$

Taking into account that for the coasting beam $m_{\parallel} = n$ and $\omega_{\parallel} = \omega'_0 \Delta p$ we get, for instance, that the contribution $j^{(1)}$ into Eq. (7.3.6) is defined by the expression $\sigma_{\parallel}^2 = \langle \Delta p^2 \rangle$

$$\left(\frac{d\sigma_{\parallel}^2}{dt} \right)_1 = -2 \sum_n e^2 n \int d\Gamma \Delta p f_0 \text{Im} G_{nn}(1, 1, \omega_n), \quad (7.3.7)$$

$$\omega_n = n\omega_0(\Delta p).$$

As the trajectory of a particle does not vary inside either the pickup electrode or kicker, the Green function included in the expression $G_{nn}(1, 1, \omega)$ can be replaced

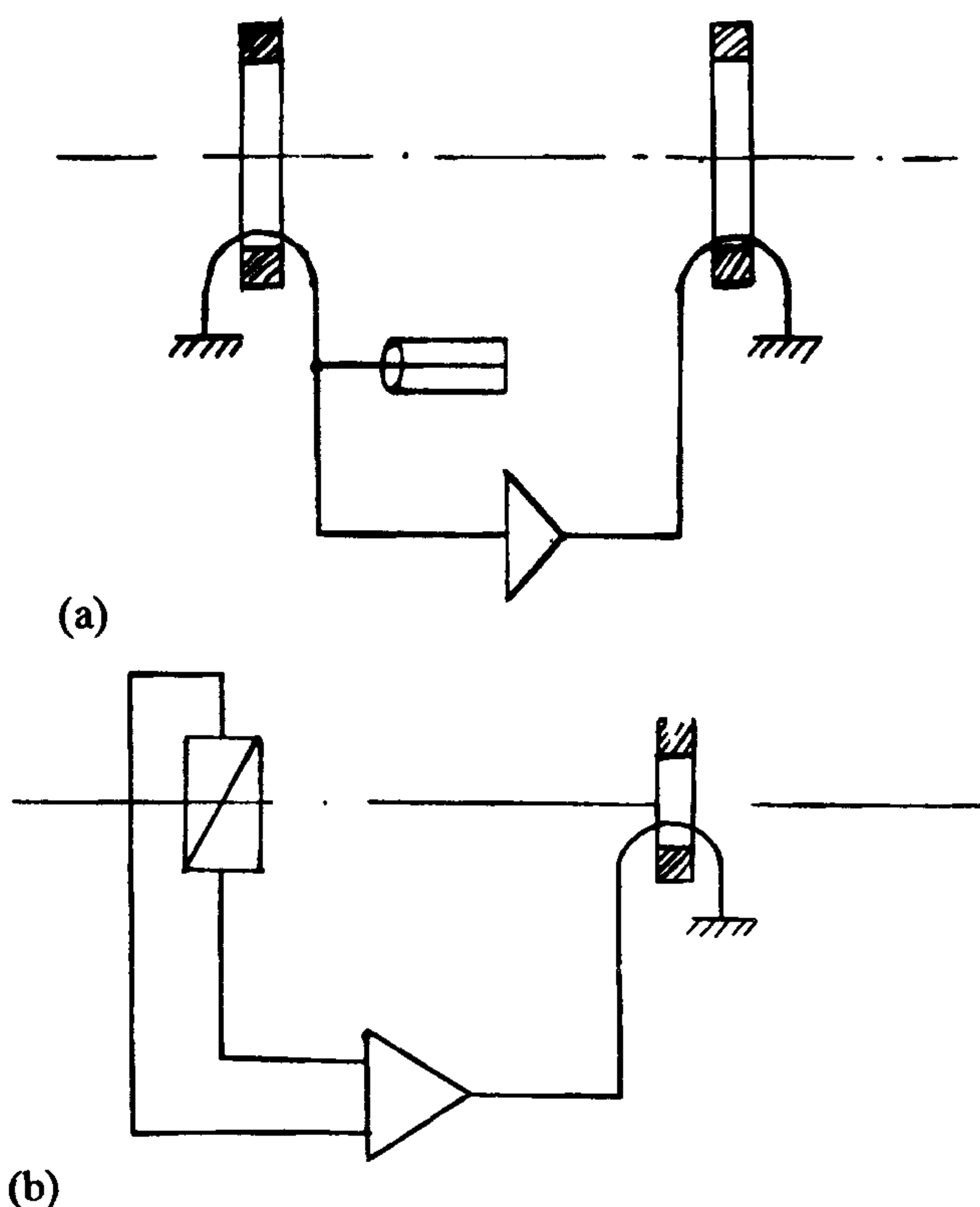


FIG. 7.10. The simplest scheme of the momentum stochastic cooling with the notch-filter (a) and with differential pick-up electrodes (b).

by the impedance of the feedback circuit

$$G_{nn}(1,1,\omega) = \frac{\omega_s - iZ_n(\omega)}{\Pi n}. \quad (7.3.8)$$

Denoting $Z_n = Z'_n + iZ''_n$, rewrite Eq. (7.3.7) in the form

$$\left(\frac{d\sigma_{\parallel}^2}{dt} \right)_1 = -\frac{2e^2\omega_s}{\Pi} \sum_n \langle \Delta p Z'_n(\Delta p, \omega_n) \rangle. \quad (7.3.9)$$

The latter expression indicates that in the general case, the cooling of an energy spread is achieved by either the frequency or radial modulation of the particle energy loss.

Let us use these formulas in the study of beam cooling with the system shown in Fig. 7.10(a). In such a scheme the particle passing the pickup electrode induces a voltage

$$U_k(t) = \frac{ie\omega_s}{2\pi} \sum_n \frac{n\omega_s Z_0 L_k}{Z_0 - in\omega_s L_k} e^{-in\omega_0 t}, \quad (7.3.10)$$

where Z_0 is the characteristic impedance of the cable, and L_k is the pickup electrode inductance. In this scheme, the presence of the short-circuited cable results in the subtraction of the instantaneous and delayed signals induced by the particle in the system

$$U(t) = U_k(t) - U_k(t - \tau_0).$$

If the delay time τ_0 is chosen equal to the revolution period of the equilibrium particle T_s , the total impedance of the feedback circuit can be written in the form

$$Z_n(n\omega_0) = i\kappa_n \frac{n\omega_s Z_0 L_k}{Z_0 - in\omega_s L_k} (1 - e^{-in\Delta\omega_0 T_s}). \quad (7.3.11)$$

Here, $\Delta\omega = \omega'_0 \Delta p$, and κ_n is the amplification coefficient of the system. If harmonic numbers in Eq. (7.3.11) are not too high, one can use the expansion

$$1 - \exp(-in\Delta\omega_0 T_s) \simeq in\Delta\omega_0 T_s,$$

and thus rewrite Eq. (7.3.11) in the form

$$Z_n = 2\pi\kappa_n \frac{n\omega_s Z_0 L_k}{Z_0 - in\omega_s L_k} \omega'_0 \Delta p. \quad (7.3.12)$$

The substitution of this expression into Eq. (7.3.9) yields

$$\left(\frac{d\sigma_{\parallel}^2}{dt}\right)_1 = -2\lambda_{\parallel} \sigma_{\parallel}^2,$$

$$\lambda_{\parallel} = 2\alpha_p \omega_s \left(\frac{e^2 \omega_s Z_0}{p_s v_s}\right) n_0 \sum_{n=1}^{\infty} \frac{\kappa'_n n^2}{n_0^2 + n^2}, \quad n_0 = Z_0 / (\omega_s L_k). \quad (7.3.13)$$

These equations clearly indicate that the interaction with the feedback loop will cool the beam down $\lambda_{\parallel} \geq 0$, provided $\alpha_p \kappa'_n \geq 0$. Assuming κ' to be constant within the interval $|n| \leq W$, one can rewrite $\lambda_{\parallel}$ in Eq. (7.3.13) in the form

$$\lambda_{\parallel} \simeq 2\alpha_p \kappa_n \omega_s \left(\frac{e^2 \omega_s Z_0}{p_s v_s}\right) W n_0 \left(1 - \frac{n_0}{W} \arctan \frac{n_0}{W}\right). \quad (7.3.14)$$

This equation shows that with an increase in the bandwidth $W \gg n_0$ the decrement increases proportionally to W . Moreover, in the region $n_0 \leq W/2$ the decrement in Eq. (7.3.14) grows linearly with an increase in n_0 . Therefore, in general, for accelerating the cooling process an inductance of pickup electrode should not be large.

The contribution of the amplifier noise into $d\sigma_{\parallel}^2/dt$ can be calculated in a similar way. As the specific feature of this scheme is that Z_n turns to zero at $\omega = n\omega_s$, the noise causes the parametric heating of a beam

$$\int d\Gamma \Delta p j^{(4)} = 4T_0 Z_0 W \left(\frac{\kappa_n e^2 \omega_s \alpha_p W}{p_s}\right) \sigma_{\parallel}^2. \quad (7.3.15)$$

The substitution of Eqs. (7.3.13) and (7.3.15) into Eq. (7.3.6) yields

$$\left(\frac{d\sigma_{\parallel}^2}{dt}\right) = -2\lambda_{\parallel} \left(1 - \frac{\kappa}{\kappa_{\max}}\right) \sigma_{\parallel}^2, \quad (7.3.16)$$

$$\kappa_{\max} = \frac{pc}{T_0} \frac{n_0}{4\pi W^2}, \quad (7.3.17)$$

is the maximum admissible value of the amplification coefficient.

As a numerical example let us estimate the cooling decrements, the required amplification coefficients, and the powers of amplifiers of the feedback system for cooling the beam of the intensity $N \simeq 2.5 \times 10^7$. Take the storage ring with the following parameters: $R_0 = 15$ m, $\alpha_p = 0.1$, $\omega_s = 2 \times 10^7$ s⁻¹, $\gamma = 2$. In such a ring by the order of value $\eta \simeq R_0/\alpha_p$ and thus, to cool the beam down the momentum spread $(\sigma_{\parallel}/p_s) = 10^{-2}$, the bandwidth of the cooling system must be of the order $W/T_s \simeq 5 \times 10^8$ Hz. For higher decrements the inductance of the pickup electrode should be small. The estimates show that with the aperture l_1 corresponding to $(\sigma_{\parallel}/p_s) = 10^{-2}$, the minimum accessible inductance of pickup electrode is 10 H. With these parameters and typical value $Z_0 = 50$ ohm, one has $W = 157$, $n_0 = Z_0/L_k = 25$. From Eq. (7.3.13) it follows that, for instance, to get the damping time 2s the required amplification factor is

$\kappa = 1.35 \times 10^9$. The power of such an amplifier must be unreasonably high:

$$P = \kappa^2 T_0 W \omega_s / 2\pi = 0.75 \times 10^7 \text{ W.}$$

This difficulty is eliminated splitting the cooling system on the reasonably high amount of parallel channels. If, for instance, the cooling system consists of $\mathcal{M}$ identical cells, a desired cooling time is obtained for $\mathcal{M}$ times smaller amplification coefficient. This reduces the required power of the amplifiers in $\mathcal{M}^2$ times.

The value of the equilibrium energy spread of a cooled beam in the scheme considered can be determined by two factors: the losses in filter and the energy exchange between longitudinal and transverse degrees of freedom. We do not give here the relevant estimations, but shift to an analysis of cooling kinetics in the Palmer scheme.

In the Palmer scheme [Fig. 7.10(b)] the energy deviation of a particle is measured using the differential pickup electrode, which is located at the place with nonzeroth dispersion function. As earlier let us say that the pickup electrode is composed of two plates of azimuthal length ϑ_0 loaded by the wave resistance Z_0 .

When passing such a pickup electrode the particle induces the signal

$$U(t) = \frac{e\omega_s Z_0}{4\pi} \frac{x}{l_1} \sum_n e^{-in\omega_s t} (1 - e^{-in(1+\beta)\vartheta_0}), \quad \beta = v/c \simeq 1. \quad (7.3.18)$$

Replacing here x by $\eta \Delta p / p_s$ one can write Z_n in the form

$$Z_n = \frac{Z_0}{2p_s} \frac{\eta \Delta p}{l_1} \kappa (1 - e^{-in(1+\beta)\vartheta_0}). \quad (7.3.19)$$

The substitution of this expression into Eq. (7.3.9) yields

$$\left(\frac{d\sigma_{\parallel}^2}{dt} \right)_1 = -2\lambda_{\parallel} \sigma_{\parallel}^2, \quad \lambda_{\parallel} = \frac{e^2 \omega_s Z_0}{\Pi p_s} \frac{\eta}{l_1} \sum_n \kappa'_n \sin^2 \frac{n\vartheta_0}{2} (1 + \beta). \quad (7.3.20)$$

Taking κ'_n as a constant within the bandwidth of the cooling system $|n| \leq W$, and $W\vartheta_0 \gg 1$, we can rewrite Eq. (7.3.20) in the form

$$\lambda_{\parallel} \simeq \frac{e^2 \omega_s Z_0}{2\pi p_s v_s} \frac{\eta}{l_1} \kappa W. \quad (7.3.21)$$

For such a scheme the current $j^{(4)}$ reads

$$j^{(4)} = -\frac{e^2 T_0 Z_0}{\Pi R_0} \frac{\partial f_0}{\partial \Delta p} \sum_n |\kappa_n|^2. \quad (7.3.22)$$

Assuming in the region $|n| \leq W$ the ratio $|\kappa''_n| < |\kappa'_n|$ and slow dependence of κ'_n on n , one can write, instead of Eq. (7.3.16)

$$\frac{d\sigma_{\parallel}^2}{dt} = -2\lambda_{\parallel} \sigma_{\parallel}^2 + \frac{2e^2 T_s Z_0 W}{\pi R_0^2} \kappa^2. \quad (7.3.23)$$

This equation indicates that in the Palmer scheme the beam can be cooled to the equilibrium momentum spread:

$$\frac{\sigma_{\parallel}^2}{p_s^2} = \kappa \frac{l_{\perp}}{\eta_0} \frac{T_0}{p_s v_s}. \quad (7.3.24)$$

Therefore, to cool the beam until it reaches the given value of the equilibrium spread, the coefficient factor κ must be small. Thus, cooling decrement is limited by the value

$$\lambda_0 = W \frac{e^2 \omega_s^2 Z_0}{2\pi T_0} \left(\frac{\eta_0}{l_{\perp}} \frac{\sigma_{\parallel}}{p_s} \right)^2. \quad (7.3.25)$$

For the numerical example discussed above, to reach the equilibrium temperature of the beam, corresponding to $\eta \sigma_{\parallel} / p_s l_{\perp} = 10^{-2}$ at the room temperature of the cooling system λ_0 in Eq. (7.3.25) must be limited by the value $\lambda_0 \leq 1.6 \times 10^{-3} \text{ s}^{-1}$.

The comparison of Eqs (7.3.14) and (7.3.21) shows that in the equivalent conditions the ratio of the cooling decrements in the schemes with the differential pickup electrodes λ_d and with the notch filter λ_f can be estimated

$$\frac{\lambda_d}{\lambda_f} \approx \frac{\eta_0}{\Pi |\alpha_p|} \frac{R_0}{2n_0 l_{\perp}}. \quad (7.3.26)$$

Therefore, for beams with relativistic energy $\gamma \gg \gamma_{tr}$ moving in the ring without large modulation of the dispersion function $|\eta| \approx |\alpha_p| \Pi$ the cooling scheme with notch filters gives some gain in the cooling rate if

$$W \gg n_0 \gg R_0 / 2l_{\perp}.$$

To conclude this section, let us emphasize that, as seen from the expressions obtained, the amplifier noise in some way or other poses the limit for the efficiency of stochastic cooling. Another class of limitations is related to the mutual influence of particles in the cooled beam. As already mentioned, such an influence can be either collective or collisional. The method of stochastic cooling is especially sensitive to these effects because of the macroscopically large radius of forces acting between the particles. The study of features of the beam cooling kinetics connected with the collective effects requires us to pass over the range of the limit in Eq. (2.5.1). These effects will be considered later.

7.4. COLLECTIVE EFFECTS IN THE STOCHASTIC COOLING METHOD

As §7.1 showed, the collective reaction of a beam leads to the shielding of fields induced by individual particles and thereby influences the kinetic processes in the beam. This influence should be especially strong in the method of stochastic cooling, first, due to the macroscopically large radius of forces acting between the particles, and, second, since this kind of cooling system causes the fast damping of coherent oscillations of the beam. Below we shall see that the introduction of fast damping into the collective motion changes in general the collision integral and cooling rate for the intense beam.

Prior to considering these questions, let us try to apply equations from §7.1 to calculate the kinetic coefficients of the stochastic cooling of an intense beam. In the most favorable case, $|Z'| \gg |Z''|$, the collective interaction of a beam with the feedback loop can be described in terms of the dielectric constant

$$\epsilon(n, \omega) = 1 - \frac{i\delta_n \Delta\omega_n}{n\omega'_0} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\Delta\omega_n - n\omega'_0 \Delta p}, \quad (7.4.1)$$

where

$$\delta_n = \frac{Ne^2 \omega_s}{\Pi} \left(\frac{dZ'_n}{d\Delta p} \right)_s$$

are the decrements of fast damping of coherent oscillations of the beam due to its interaction with the cooling system. As we are interested in studying the kinetics of intense beams, below we shall assume that

$$\delta_n \gg |n\Delta\omega_0|. \quad (7.4.2)$$

According to Eqs. (7.1.6) and (7.1.8) the spectral density and power of the beam stationary noise are determined by the behavior of $\epsilon(n, \omega)$ on the real axis of the complex variable ω . In the region defined by Eq. (7.4.2), Eq. (7.4.1) can be simplified:

$$\epsilon(n, \omega) \simeq \begin{cases} 1 - i \frac{\delta_n \Delta\omega_n}{(n\Delta\omega_0)^2}, & |\Delta\omega_n| < |n\Delta\omega_0|, \\ 1 + i \frac{\delta_n}{\Delta\omega_n}, & |\Delta\omega_n| \gg |n\Delta\omega_0|. \end{cases} \quad (7.4.3)$$

Then for momentum distribution functions close to the Gaussian kind, the noise spectrum

$$(\delta\rho_n)_\omega^2 = \frac{2\pi N}{|n\omega'_0|} \frac{f_0(\omega)}{1 + \left(\frac{\delta_n \Delta\omega_n}{n^2 \Delta\omega_0^2}\right)^2} \quad (7.4.4)$$

is concentrated in the narrow band of frequencies

$$|\Delta\omega_n| < \frac{(n\Delta\omega_0)^2}{\delta_n} \ll |n\Delta\omega_0|. \quad (7.4.5)$$

The integration of Eq. (7.4.4) over ω yields

$$P_n = \frac{N}{|n\omega'_0|} \int_{-\infty}^{\infty} d\Delta\omega_n \frac{f_0(\Delta\omega_n)}{1 + \left(\frac{\delta_n \Delta\omega_n}{n^2 \Delta\omega_0^2}\right)^2} \simeq N \frac{|n\Delta\omega_0|}{\delta_n}. \quad (7.4.6)$$

It is seen that similar to the case with the nondissipative interaction between particles, the power of the noise of the cool beam [in terms of condition (7.4.2)] is suppressed by its collective reaction. Such a suppression is caused by the propagation along the beam of the long-living coherent oscillations with decrements

$$\Delta\omega_n = -i \frac{(n\Delta\omega_0)^2}{\delta_n} \ll -i |n\Delta\omega_0|.$$

Note that with a special distribution δ_n inside the bandwidth of the system $\delta_n \propto n$, the power of the beam noise does not depend on the harmonic number n . One can also see that the fast damping effect of coherent oscillation does not directly contribute into Eq. (7.1.8).

The rate of the variation of the beam temperature is defined by Eq. (7.1.21). The substitution of Eq. (7.4.1) into Eq. (7.1.21) shows that the contribution of the long-living and short-living oscillations into the beam cooling

$$d\sigma_{\parallel}^2 / dt = -\sigma_{\parallel}^2 \sum_n \frac{1}{n} \frac{(n\Delta\omega_0)^2}{\delta_n}$$

are of the same order of magnitude and, provided $\delta_n \propto |n|$, can be estimated by

$$\lambda_{\parallel} = -\frac{d\ln\sigma_{\parallel}^2}{dt} = \frac{1}{N} \frac{(W\Delta\omega_0)^2}{(\delta_n/n)}. \quad (7.4.7)$$

Here, W/T_s is the cooling system bandwidth. Eq. (7.4.7) indicates that when the beam phase space density exceeds the value defined by Eq. (7.4.2), the cooling decrement decreases as $(N^2\kappa)^{-1}$. In fact, such a result has been obtained in Refs. 18 and 105 through 107. The comparison of Eqs. (7.4.7) and (7.3.9) enables the conclusion on the existence of the optimal amplification coefficient κ_{opt} . The value κ_{opt} can be estimated by the comparison of δ_n and $|\Delta\omega_0|$. This

yields the well-known expression for the cooling decrement¹⁰⁰

$$\lambda_{opt} = \frac{W^2 \Delta \omega_0}{N}. \quad (7.4.8)$$

The simple model¹⁰⁸ shows that generally speaking it is not true. Let the beam be divided into groups (samples) of azimuthal length ϑ_0 . An average number of particles per group is $N^* = N\vartheta_0/2\pi$. According to the general idea of the method let us say that the particles of the given group are perturbed by the force whose value averaged over the revolution period is proportional to its dipole momentum

$$\frac{d\Delta p_a}{dt} = \frac{\delta \Delta p_a}{T_s} = -\lambda_0 N^* \mathcal{P}(t),$$

$$\dot{\varphi}_a = \omega'_0 \Delta p_a, \quad \mathcal{P}(t) = \sum_{a=1}^{N^*} \Delta p_a / N^*. \quad (7.4.9)$$

This choice of the cooling force corresponds to such a delay in the feedback circuit when the self-action of a particle reaches its maximum. The radius of the action of the force in this model is ϑ_0 . Using this value we can define the bandwidth of the system as $W = 2\pi/\vartheta_0$.

Eq. (7.4.9) shows that until no one particle has left the interval ϑ_0 , the dipole momentum $\mathcal{P}$ exponentially decays with the damping decrement $\delta = N^* \lambda_0$. In this case, the motion of a particle in the sample is described by

$$\Delta p_a(t) = \Delta p_a^{(0)} - \mathcal{P}^{(0)}(1 - e^{-\delta t}), \quad \varphi_a(t) = \varphi_a^{(0)} + \omega'_0 \Delta p_a^{(0)} t. \quad (7.4.10)$$

Here, $\Delta p_a^{(0)}$ and $\varphi_a^{(0)}$ are the initial values of the momentum and phase of a particle, while $\mathcal{P}^{(0)}$ is that of the dipole momentum of the sample. Note that such a motion conserves the momentum spread of the beam:

$$\sum_a (\Delta p_a^{(0)} - \mathcal{P}^{(0)})^2 = \sum_a (\Delta p_a - \mathcal{P})^2.$$

From Eq. (7.4.10) it is evident that in addition to damping a decrease in the dipole momentum can be caused by the deflection of the group as a whole, which induces the signal compensating for the initial value of the dipole momentum. As a result, in time of the order of δ^{-1} the force acting on the particles in a group will decay practically to zero. In other words, the field induced, for example, by particle 1 is shielded by the remaining particles of the group in time $1/\delta$. Note that in contrast to the problems with the nondissipative interaction between particles the rise time of the shielding depends here not on the momentum spread but rather on the intensity of the beam.

The transitions of particles from one group to another cause the change in the initial conditions in Eq. (7.4.9). The mean time for a particle being in the given

group can be estimated by $\tau_0 = \vartheta_0 / \Delta\omega_0$. Therefore, the mean frequency of a change in initial conditions in Eq. (7.4.9) is of order τ_0^{-1} .

Let us calculate the rate of the variation for the momentum spread in a beam. Using Eq. (7.4.1) we write

$$\sum_{a=1}^N \delta(\Delta p_a)^2 = \sum_{\alpha=1}^{n_g} \sum_{a=1}^{N^*} \delta(\Delta p_a)^2 = - \sum_{\alpha=1}^{n_g} N^* (\mathcal{P}_\alpha^{(0)})^2 (1 - e^{-2\delta t}).$$

Performing here the statistical averaging

$$\sum_a \delta(\Delta p_a)^2 = N \sigma_{\parallel}^2, \quad \langle \mathcal{P}_\alpha^{(0)} \rangle = 0,$$

$$\langle (\mathcal{P}_\alpha^{(0)})^2 \rangle = \frac{\sigma_{\parallel}^2}{N},$$

and dividing the result by τ_0 we obtain

$$\frac{d\sigma_{\parallel}^2}{dt} = - \frac{\sigma_{\parallel}^2}{N^* \tau_0} [1 - \exp(-2\delta \tau_0)].$$

This yields for the cooling decrement

$$\lambda_{\parallel} = - \frac{1}{2} \frac{d \ln \sigma_{\parallel}^2}{dt} = \lambda_0 \frac{1 - e^{-2\delta \tau_0}}{2\delta \tau_0}$$

or

$$\lambda_{\parallel} = \frac{\pi \Delta \omega_0}{N \vartheta_0^2} \left[1 - \exp \left(- \frac{N \vartheta_0 \lambda_0}{\pi \Delta \omega_0} \right) \right]. \quad (7.4.11)$$

This expression shows that the influence of the collective reaction of the beam on the cooling is negligible, so $\lambda_{\parallel} \simeq \lambda_0$ if time interval τ_0 is substantially smaller than the time of collective reaction $1/\delta$. On the contrary, in the asymptotical region $\tau_0 \delta \gg 1$ the cooling decrement is limited by the value

$$\lambda_{\max} = \frac{\pi \Delta \omega_0}{N \vartheta_0^2} = \frac{1}{2} \frac{W^2}{N} \left(\frac{\Delta \omega_0}{2\pi} \right). \quad (7.4.12)$$

In the problem considered the value τ_0 determines the correlation time due to interaction of particles through the cooling system (or so-called collision time). Therefore, the obtained equations can be formulated in the following way: the influence of the collective reaction of the beam on the cooling is negligible unless the time of the collective reaction is larger than the collision time.

The asymptotic limitation of decrements described by Eq. (7.4.12) could be foreseen. On any interval τ_0 the effective time when a particle is seen by the cooling system is equal to $1/\delta$. In the region $\tau_0 \delta \gg 1$ this decreases of the cooling

decrement by the factor $\tau_0\delta$:

$$\lambda_{\parallel} = \lambda_0/(\tau_0\delta) = \lambda_{\max}.$$

Let us now find the kinetic equation that takes into account the contribution from such a nonstationary beam noise. An important element in constructing the collision integral in Eq. (2.4.12) is the averaging procedure. As § 2.4 said, the averaging in Eqs. (10.8) through (10.11) is adopted as the averaging over a long enough time interval

$$\langle \dots \rangle = \int_0^T \frac{dt}{T} (\dots). \quad (7.4.13)$$

This time interval for averaging should be selected in accordance with the time hierarchy of the problem. Using the Maxwell equations the contribution into Eq. (2.5.15) correlation functions can be expressed in terms of the nonsimultaneous correlation function $\langle \delta F(\Delta p, \vartheta, t) \delta F(\Delta p, \vartheta_1, t_1) \rangle$. It is clear that the latter is contributed only from particles spaced by distances no larger than the radius of action of forces ϑ_0 . The mean frequency of the generation of such fluctuations in a beam is determined by the mean time taken for a particle to leave the interaction region $\tau_0 \simeq \vartheta_0/\Delta\omega_0$. On the time intervals $\Delta t \simeq \tau_0$ the development of the fluctuations is described by Eq. (7.1.1) with an initial condition $\delta F^{(0)} = \delta F(\Delta p, \vartheta, t_0)$, but outside τ_0 the value of $\delta F^{(0)}$ can be changed due to the thermal motion of particles. To this end, the average in Eq. (7.4.13) can be rewritten in the form

$$\int_0^T \frac{dt}{T} (\dots) = \int_0^{\tau_0} \frac{dt}{\tau_0} \ll \dots \gg, \quad (7.4.14)$$

where $\ll \dots \gg$ means the averaging over the initial fluctuations $\delta F^{(0)}$. Taking this into account, one can rewrite Eq. (7.1.18) in the form

$$\frac{\partial f_0}{\partial t} = - \frac{\partial}{\partial \Delta p} \int_0^{\tau_0} \frac{dt}{\tau_0} \ll e \delta E(\vartheta, t) \delta F(\Delta p, \vartheta, t) / N \gg. \quad (7.4.15)$$

An important difference between Eq. (7.4.15) and equations in § 7.1 is that in the collision integral appear some additional terms due to the fluctuations, which are nonstationary on time intervals $\Delta t \simeq \tau_0$. The contributions from these terms become essential if the collective motion can develop for $\tau_k \leq \tau_0$. For the stochastic cooling systems in the analogy to other problems, when the beam interacts with surrounding electrodes, this condition is easier to meet since the interaction region radius ϑ_0 is macroscopically large. On the contrary, in the problems with a microscopic-size interaction region ϑ_0 nearly always one meets $\tau_k \gg \tau_0$. In the last case when integrating in Eq. (7.4.15) over t one can actually take $\tau_0 \rightarrow 0$, and the kinetic equation can be transformed into the well-known kinetic equation of the Lenard-Balesku type.^{18,101,106,107}

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \Delta p} \frac{1}{N} \ll e \delta E(\vartheta, t) \delta F(\Delta p, \vartheta, t) \gg.$$

Equation (7.4.15) itself is as complicated a nonlinear kinetic equation as those discussed in §7.1. But the dependence of cooling decrements on the number of particles in a beam N can be studied using more simpler Langevin equations. Multiplying Eq. (7.4.15) by Δp and integrating it over momenta of particles we obtain

$$\frac{d\sigma_{\parallel}^2}{dt} = \frac{2e}{N} \int_0^{\tau_0} \frac{dt}{\tau_0} \sum_n \left\langle \delta E_n^*(t) \int_{-\infty}^{\infty} d\Delta p \Delta p \delta F_n(\Delta p, t) \right\rangle. \quad (7.4.16)$$

Substituting in Eq. (7.1.2)

$$Z_n(\Delta p) = -\frac{\Pi \delta_n}{N e^2 \omega_s} \Delta p,$$

and using Eqs. (7.1.3) and (7.1.4) we can write

$$\frac{d\sigma_{\parallel}^2}{dt} = -\frac{2}{N} \sum_n \frac{\delta_n}{N} \int_0^{\tau_0} \frac{dt}{\tau_0} \ll |\delta \mathcal{P}_n(t)|^2 \gg.$$

$$\delta \mathcal{P}_n(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \delta \mathcal{P}_{n,\omega} e^{-i\omega t},$$

$$\delta \mathcal{P}_{n,\omega} = \frac{1}{\epsilon(n, \omega)} \int_{-\infty}^{\infty} d\Delta p \frac{\Delta p \delta F_n^{(0)}(\Delta p)}{\omega - \omega_n(\Delta p)}, \quad \omega_n = n\omega_0(\Delta p) \quad (7.4.17)$$

and $\epsilon(n, \omega)$ is defined by Eq. (7.4.1). These equations show that specific features of the dependence of the dipole momentum $\delta \mathcal{P}_n$ on the time is determined by the ratio of the revolution frequency spread $\Delta \omega_0$ and the decrements of coherent oscillations δ_n due to interaction of the beam with the cooling system. If the revolution frequency spread in the beam is large, $|n\Delta \omega_0| \gg \delta_n$, one can take $\epsilon(n, \omega) \simeq 1$. Then $\delta \mathcal{P}_n(t)$ reads

$$\delta \mathcal{P}_n(t) \simeq \int_{-\infty}^{\infty} d\Delta p \Delta p \delta F_n^{(0)}(\Delta p) \exp[-i\omega_n t].$$

On the contrary, in the region $|n\Delta \omega_0| \ll \delta_n$ one has

$$(\omega - \omega_n) \epsilon(n, \omega) \simeq \omega - \omega_n + i\delta_n$$

and correspondingly

$$\delta \mathcal{P}_n(t) \simeq e^{-\delta_n t} \int_{-\infty}^{\infty} d\Delta p \Delta p \delta F_n^{(0)}(\Delta p).$$

Substitution of these expressions into Eq. (7.4.17) yields

$$\frac{d\sigma_{\parallel}^2}{dt} = -2\lambda_{\parallel} \sigma_{\parallel}^2, \quad (7.4.18)$$

with

$$\lambda_{\parallel} \sum_n \frac{\delta_n}{N} = \begin{cases} 1, & \delta_n \ll |n\Delta\omega_0| \\ \frac{1 - \exp(-2\delta_n\tau_0)}{2\delta_n\tau_0}, & \delta_n \gg |n\Delta\omega_0|, \end{cases}$$

or ($\vartheta_0 = 2\pi/W$)

$$\lambda_{\parallel} = \begin{cases} \lambda_0, & \frac{N\lambda_0\vartheta^2}{2\pi} \ll \Delta\omega_0, \\ \frac{\pi\Delta\omega_0}{N\vartheta_0^2}, & \frac{N\lambda_0\vartheta^2}{2\pi} \gg \Delta\omega_0. \end{cases} \quad (7.4.19)$$

Equation (7.3.19) comprises the main terms of the expansion of the cooling decrement in the powers of parameter $(\tau_0\delta_n)^{-1}$. More detailed calculation of the right-hand side in Eq. (7.4.17) yields the correction defined by Eq. (7.4.7).

Thus, instead of the limitation due to the additional heating, the given modification of the collision integral predicts the asymptotical limitation of the decrement during the beam cooling. Such a behavior of decrements with the beam intensity (or temperature) is essentially related to the fast damping of the collective motion of the beam due to its interaction with the cooling system and to a large radius of the particle interaction. To this end the cooling of a cold (or intense) beam is only effective during the damping times of its coherent fluctuations.

The dependence of cooling time on the beam phase space density was experimentally studied in Ref. 109. (See in Fig. 7.11.) In the same figure the solid line presents the fitting of these data by Eq. (7.4.11):

$$\tau = \frac{\tau_{\max}}{1 - \exp(-\tau_{\max}/\tau^0)}, \quad \tau_{\max} = 1.2 \frac{2\pi N}{W^2 \Delta\omega_0}, \quad \tau^0 = 3.38 \text{ min.}$$

One can see that the calculations are in reasonable agreement with the results of measurements.

Coming back to Eq. (7.4.19) we have to note that the replacement of the linear dependence of the force on the particle momentum in Eq. (7.4.16) by the more complex expression

$$\delta E_{n,\omega} = -\frac{e\omega_s}{\Pi} \int_{-\infty}^{\infty} d\Delta p Z'_n(\Delta p) \delta F_n(\Delta p, t)$$

does not change Eq. (7.4.19) in the asymptote region. Such a replacement simultaneously changes the single-particle cooling decrement λ_0 and the damping decrement of coherent oscillations δ_n , whereas $\lambda_{\max} \propto \lambda_0/\delta_n$ remains unaltered. On the essence, this is the corollary of §3.4's theorem on the sum of decrements of coherent modes.

To conclude this section let us discuss how much noise power will be detected by the pickup electrodes from the beam under stochastic cooling. The stationary level of the noise is defined by Eq. (7.4.6). In this case, the contribution of the nonstationary level of the beam noise is not taken into account and therefore it should be calculated additionally. The total noise power of the beam reads

$$P_n = \int_0^T \frac{dt}{T} |\delta \rho_n(t)|^2. \quad (7.4.20)$$

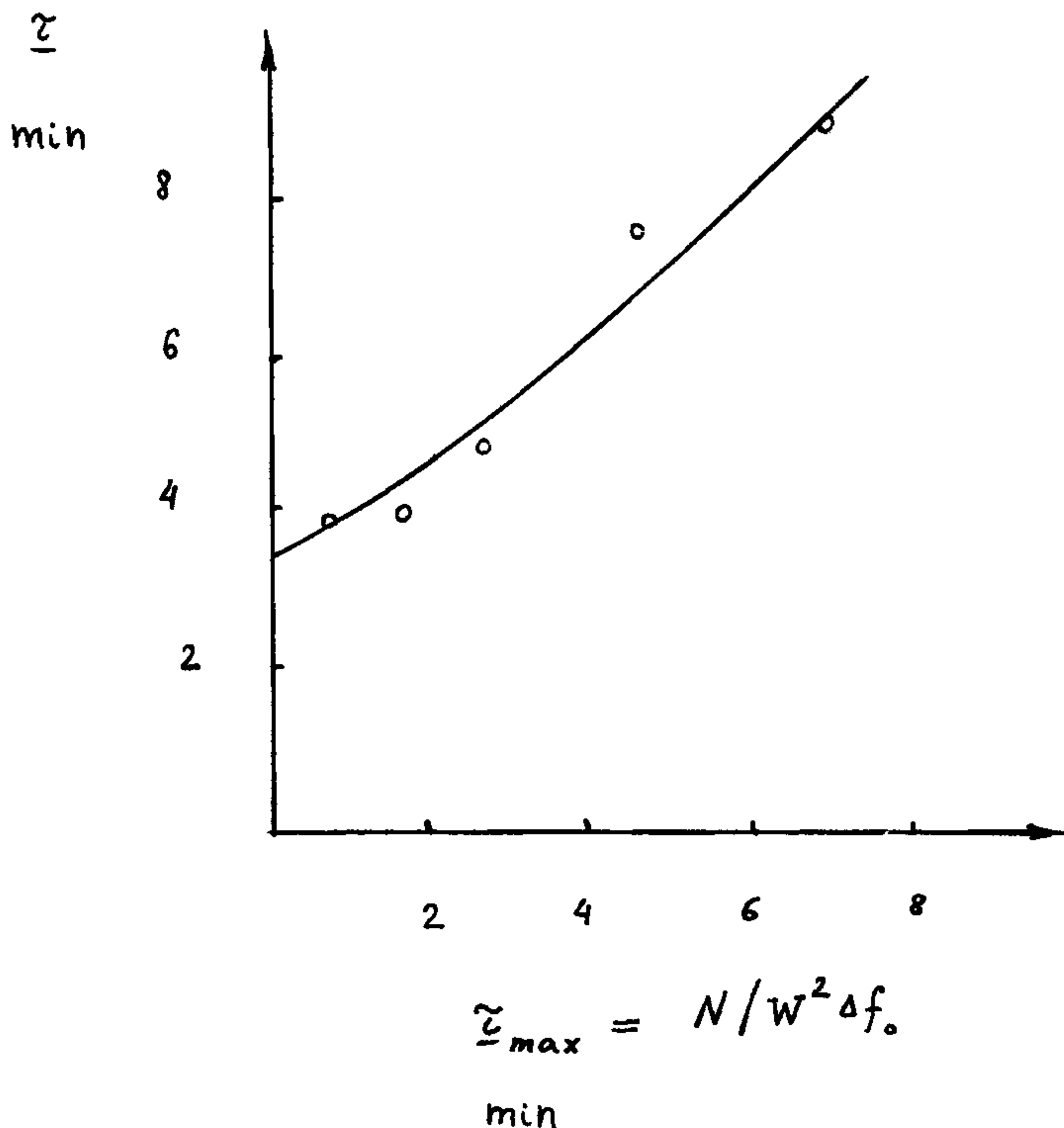


FIG. 7.11. Dependence of the stochastic momentum cooling time on the beam phase density ($N/W^2 \Delta f_0$) (o-measurements from [109]).

The substitution here of the solutions of Eq. (7.1.1) after averaging yields

$$P_n = \int_0^{\tau_0} \frac{dt}{\tau_0} \ll \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\epsilon(n, \omega)} \int_{-\infty}^{\infty} d\Delta p \frac{\delta F_n^{(0)}(\Delta p)}{\omega - \omega_n} \right|^2 \gg. \quad (7.4.21)$$

The integral over ω is determined by the residues in the roots of the dispersion equation $\epsilon(n, \omega) = 0$. The subsequent expression for P_n can be rewritten as

$$P_n = P_n^{\text{st}} + \delta P_n,$$

where P_n^{st} is defined by Eq. (7.4.6) and δP_n gives the contribution to P_n from nonstationary fluctuations. The calculation of δP_n in the limit $\tau_0 \delta_n \gg 1$ results in

$$\delta P_n \simeq \frac{N}{2\delta_n \tau_0} [1 - \exp(-2\delta_n \tau_0)] \simeq \frac{\pi \Delta \omega_0}{\lambda_0 \vartheta_0^2}. \quad (7.4.22)$$

One can see that the contributions of P_n^{st} and δP_n have the same order of magnitude: from the nonstationary level of the noise $\delta P_n = N$ the pickup electrode measures the fraction $N/(\delta_n \tau_0)$. Summing the contributions from Eqs. (7.4.6) and (7.4.22) we obtain

$$P_n = N \frac{|n \Delta \omega_0|}{\delta_n}. \quad (7.4.23)$$

These results show that, if collective oscillations of the beam decay faster than particles leave the interaction region, the kinetic equation generally cannot be reduced to the Lenard-Balescu type equation. The difference between these equations increases with an increase in the length of the interaction region and with a decrease in the frequency spread of the beam. The satisfaction of these conditions in the method of stochastic cooling leads to the cooling decrements becoming asymptotically limited by the value defined by Eq. (7.4.12). The complication of the feedback circuit can somewhat remove the asymptote region but does not change the asymptotic behavior of decrements.

7.5. ACCUMULATION OF PARTICLES USING STOCHASTIC COOLING

Let us now discuss the basic features of the accumulation of intense beams using stochastic cooling. As was already mentioned, the influence on the beam of the amplifier noise as well as collective shielding of particles generally limit the cooling decrements and thereby decrease the efficiency of the method. One more class of limitations is related to the requirement of coherent stability of the stored beam. For certainty, here we imply the schemes using the so-called momentum stacking when the injected particles are transported to the center of storage area due to their acceleration or deceleration.¹⁰⁵ In such schemes the stored beam

typically is not bunched. For this reason, below we focus on the stability of coherent oscillations of a coasting beam.

If the values of the coherent frequency shifts due to interaction with the cooling system are smaller than the revolution frequency ω_s , the collective properties of the stored beam are described by the dielectric constant in Eq. (7.4.1). This expression indicates that if $f(\Delta p)$ is close to the Gaussian distribution whose center coincides with zeros $Z_n(\Delta p)$, the interaction with the damping system causes the fast damping of coherent oscillations.

In this case, the possible instabilities of coherent oscillations are due to a strong difference of distribution $f(\Delta p)$ from that of thermodynamic equilibrium. In addition, the instability of coherent oscillations can be caused by the displacement of a beam in the pickup electrode.¹⁰⁹ The last instability is due to the fact that at large displacement of the beam in the pickup electrode the modulation of $Z_n(\Delta p)$ becomes less important and, indeed, the beam interacts with the element, which has a nonzero, dissipative impedance on the orbit. Therefore, we can expect that with an increase in the displacement of the beam in the pickup the stability diagram will take its standard form. The described instability of coherent longitudinal oscillations has been observed in experiments on stochastic cooling at the cooling ring NAP-M.¹⁰⁹

In these experiments both the electron and stochastic cooling were available for the beam manipulations. Upon the initiation of the instability either by additional displacement in the pickup or by additional electron cooling, the coherent signal from the integral pickup electrode could be observed during a few seconds before it disappeared. This was accompanied by the blow-up of the beam noise power and by sharp shrinking of the observed noise spectrum, which was interpreted as the beam signal monochromatization. (See Fig. 7.12). During the instability P_n decreased while the width of the spectrum $\Delta\omega$ increased up to the values corresponding to the new stationary state. The fitting of the experimental data agrees with the blow-up of the frequency spread due to this instability until the value

$$\Delta\omega_{st} = AN^{1/3}.$$

A decrease in the amplification coefficient or the beam current resulted in the instability vanishing. Though the described instability does not directly limit the accumulation rate, it determines some tolerances on the matching of the beam center with the position of equilibrium orbit in the pickup electrodes.

One more—and probably more important—class of coherent instabilities in the rings with stochastic cooling is related to the possibility of the overcorrection of coherent oscillations of the beam by the cooling system.¹⁰⁸ Qualitatively it is obvious that the number of particles participating simultaneously in interaction with the system, which has bandwidth W , is equal to $N^* = N/W$. Therefore, one can expect the decrements of coherent oscillations N^* times larger than the particle cooling decrements. Under these conditions an increase in cooling rate can lead to excessive corrections for coherent oscillations and, thus,

their excitation instead of damping. This is equivalent to such high values of δ_n in Eq. (7.4.1) that formally the calculated damping times for coherent oscillations of the beam become lower than T_s . This means, for instance, that such an instability cannot be stabilized by the Landau damping.

To avoid the discussion of some particular scheme let us consider the model, when the field induced by the beam is defined by the expression

$$E_\omega(\vartheta) = -g(\vartheta - \vartheta_k)\kappa(\omega) \int d\Delta p' d\vartheta' g(\vartheta') \Delta p f(\Delta p, \vartheta), \quad (7.5.1)$$

$$g(\vartheta) = \begin{cases} 1, & |\vartheta| \leq \vartheta_0, \\ 0, & |\vartheta| > \vartheta_0. \end{cases}$$

$\kappa(\omega)$ is the transition coefficient of the feedback circuit, and ϑ_k is an azimuthal distance between the centers of the kicker and the pickup electrode. If the delay

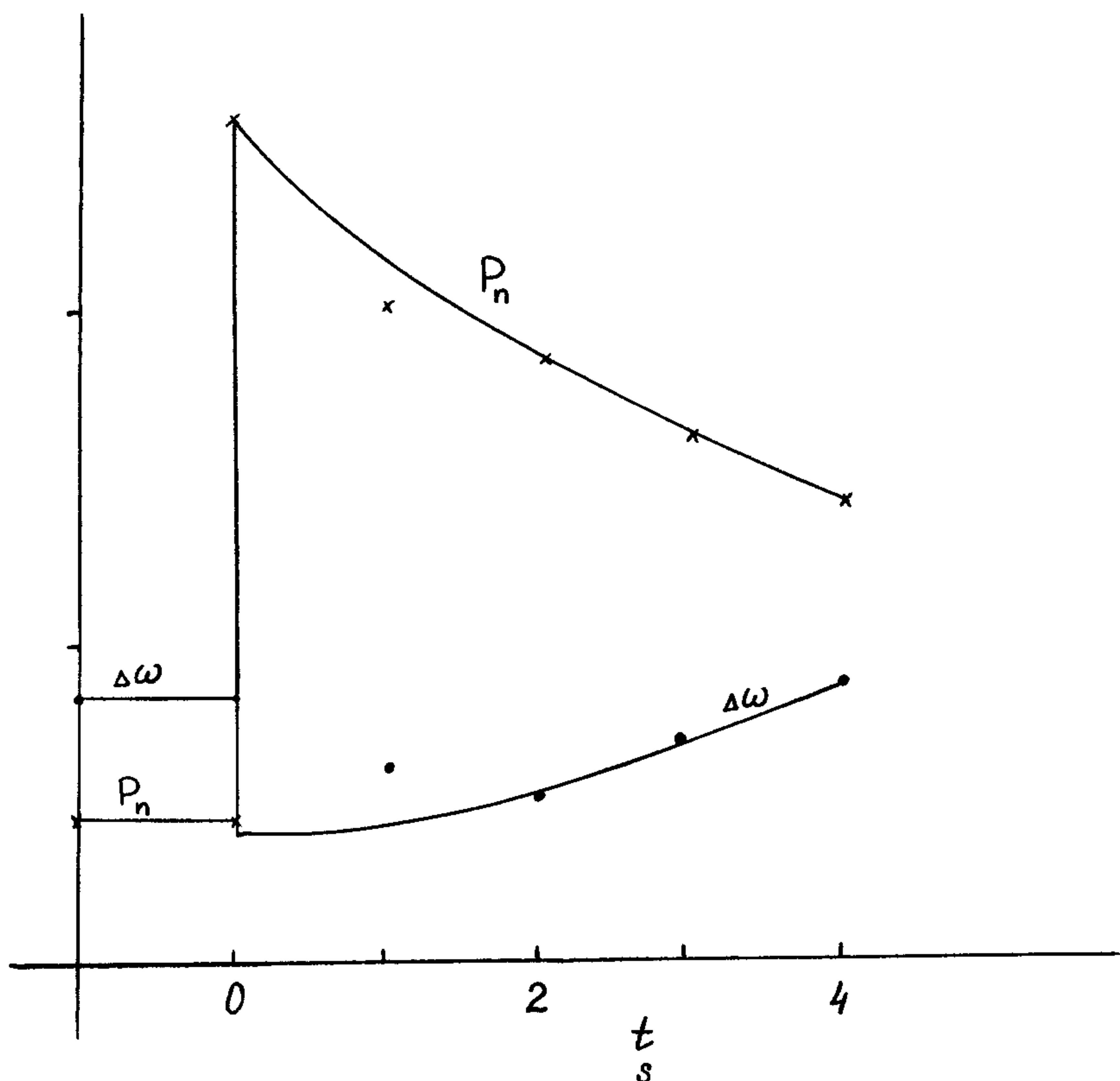


FIG. 7.12. Dependence of the Schottky noise power P_n and of the beam frequency spread $\Delta\omega$ on time in the beam with unstable longitudinal coherent oscillations.

time for the signal in the circuit is τ , then $\vartheta_k = \omega_s \tau$. Using the Fourier expansions for eigensolutions

$$f(\Delta p, \vartheta, t) = e^{-i\omega t} \sum_n f_{n,\omega} e^{-in\vartheta},$$

of the linearized Vlasov equation we write

$$f_{n,\omega} = \frac{iNg_n f'_0}{\omega - \omega_n} \sum_{n'} \kappa(n'\omega_s) g_{n'} e^{i(\omega - \omega_n)\tau} \int_{-\infty}^{\infty} d\Delta p' \Delta p' f_{n',\omega}(\Delta p'). \quad (7.5.2)$$

Since we expect that the values of coherent frequency shifts will not be small compared to ω_s , we cannot neglect in Eq. (7.5.2) the coupling between harmonics with different n . The integration of Eq. (7.5.2) with Δp yields the dispersion equation:

$$1 = iN \sum_{n=-\infty}^{\infty} \kappa_n |g_n|^2 e^{i(\omega - n\omega_s)\tau} \int_{-\infty}^{\infty} d\Delta p \frac{\Delta p f'_0}{\omega - n\omega_0(\Delta p)}, \quad (7.5.3)$$

which determines the spectrum of coherent oscillations in this case. The strength of the interaction in this equation is determined by the coefficient κ_n . If the interaction is not strong, Eq. (7.5.3) leads to ordinary solutions

$$\omega = n\omega_s + \Delta\omega_n, \quad \Delta\omega_n = -iN |g_n|^2 \kappa_n,$$

$$|\Delta\omega_n| \ll \omega_s, \quad \lambda_{||} = \lambda_0 = \sum_n \kappa_n |g_n|^2.$$

But with the enhancement of cooling due to increase of κ_n , the beam can enter the region

$$N |g_n|^2 \kappa_n \gg \omega_s$$

In this region one can neglect in Eq. (7.5.3) the influence of the frequency spread on the roots. Moreover, with $\vartheta_0 |\omega| \leq \omega_s$ and $|\omega| \leq W\omega_s$, which corresponds to typically rather short pickups and kickers used in the cooling system, one can omit in the sum in Eq. (7.5.3) the dependencies of κ_n and g_n on n . This results in

$$1 = \frac{Ng^2 \kappa}{\omega_s} \sum_{n=-\infty}^{\infty} \frac{e^{i(\nu - n)\vartheta_k}}{i(\nu - n)}, \quad \nu = \omega/\omega_s, \quad (7.5.4)$$

with g^2 and κ being the values of $|g_n|^2$ and κ_n at some specific harmonic $|n| \leq W$. Replacing here

$$\frac{e^{i\alpha\vartheta_k}}{i\alpha} = - \int_{\vartheta_k}^{\infty} ds e^{i\alpha s}$$

and using the formula

$$\sum_{n=-\infty}^{\infty} e^{-ins} = 2\pi \sum_{k=-\infty}^{\infty} \delta(s-2\pi k),$$

we can rewrite Eq.(7.5.4) in the form

$$1 = -\frac{2\pi N g^2 \kappa}{\omega_s} \sum_{k=-\infty}^{\infty} \int_{\vartheta_k}^{\infty} dse^{isv} \delta(s-2\pi k) = \frac{2\pi N g^2 \kappa}{\omega_s} \frac{1}{1-\Lambda},$$

$$\Lambda = \exp(-2\pi i v), \quad \text{Im} v \geq 0,$$

or

$$\Lambda = 1 - \frac{2\pi N g^2 \kappa}{\omega_s}.$$

The eigenvalue Λ defines the amplification of the harmonic f_n per turn. Coherent oscillations will be stable if $|\Lambda| \leq 1$. This yields the limitation

$$|e^{-2\pi i v}| = \left| 1 - \frac{2\pi N g^2 \kappa}{\omega_s} \right| \leq 1. \quad (7.5.5)$$

Under the accepted assumptions $W = \pi/\vartheta_0$, and

$$N g^2 \kappa \simeq \frac{N \lambda_{\parallel}}{W} \equiv \frac{N}{W \tau_{\parallel}},$$

where $\tau_{\parallel}$ is the cooling time of a single particle. The substitution of these expressions into Eq. (7.5.5) enables one to rewrite the stability condition in the form¹⁰⁸

$$N \leq 2W \frac{\tau_{\parallel}}{T_s}. \quad (7.5.6)$$

With the given $\tau_{\parallel}$ both Eq. (7.5.6) and Eq. (7.4.19), calculated for the “optimum” bandwidth $W \simeq \omega_s/\Delta\omega$, limit the number of particles in the cooled beam approximately by the same value. But since the dynamic instability described above cannot be stabilized by Landau damping, it definitely will be accompanied by the losses of particles and, thus, will give more strong limitation of the intensity of the cooled beam. In fact, this limitation determines the accessible value for the accumulation rate in storage rings with stochastic cooling. For accumulation one has to work in the region of parameters where the newly injected particles will cool for the time between the successive injections τ . Hence, the amplification factor should satisfy the condition $\tau \geq \tau_{\parallel}$. Otherwise, after the accumulation of $2W\tau_{\parallel}/T_s$ particles the beam will be unstable and new particles will be lost. To continue the accumulation the amplification factor of the feedback system should be decreased with the corresponding increase of

$\tau_{\parallel}$ and τ . This means that the accumulation rate becomes dependent on the number of particles that have been already stored:

$$\frac{dN}{dt} = \frac{\Delta N}{\tau} = \frac{\Delta N}{N} \frac{2W}{T_s}, \quad (7.5.7)$$

with ΔN the number of injected particles. The number of particles stored for time t can be obtained by integrating Eq. (7.5.7):

$$N(t) = 2 \sqrt{\Delta N W t / T_s} \quad (7.5.8)$$

If $1/\tau_1$ is the repetition rate of the injector, the fraction of injected particles that can be used for the accumulation

$$k = \frac{N(t)}{N t} = 2 \tau_1 \left(\frac{W}{\Delta N T_s t} \right)^{1/2}$$

decreases when the accumulation time increases.

Let us also discuss briefly one more suggestion for how to overcome the limitations due to mutual influence of the stored and injected particles, which was suggested¹¹⁰ especially for stochastic stacking schemes.* It is based on such a decrease in the amplification coefficient of the feedback circuit across the stack [see in Eq. (7.1.22), $Z'_n \rightarrow \kappa_n(\Delta p) Z_n(\Delta p)$] that will satisfy the condition**

$$Z'_n(\Delta p) f_0(\Delta p) = C,$$

where C is related to the flux of injected particles $C \propto j_{\parallel}(\Delta p_{inj})$. From the requirement

$$|\epsilon_n(\omega_n(\Delta p))|^2 \simeq 1$$

one can conclude that in this case the value of the flux $j_{\parallel}(\Delta p_{inj})$ cannot be high:

$$\left| \frac{e^2 \omega_s C}{\Pi} \int_{-\infty}^{\infty} d\Delta p \frac{\partial \ln f_0 / \partial \Delta p}{\omega - \omega_n(\Delta p)} \right| \ll 1.$$

In addition, this modification does not cancel the limitation due to the overcorrection of coherent oscillations of the beam by the cooling system.

*The schemes where the particles are accumulated due to their acceleration or deceleration from the injection region.

**Generally, this requirement is based on the assumption that the beam kinetic in the stack-core region, where particles are accumulated, and in the stack-tail region, which particles pass during accumulation, can be described separately. In the tail region, since injected deposits can be supposed to be far from the equilibrium, we can neglect in Eq. (7.1.22) the contribution from the amplifier noise and, assuming low intensity of the beam in the tail region, the collective shielding of particles. Then the requirement of the constant flux across the stack yields the mentioned condition. When particles approach the core region both assumptions are violated and, hence, the shape of the energy gain obviously must be changed.

To evaluate this and other possibilities in more detail, one can use, for instance, the Monte-Carlo numerical simulation of stochastic cooling and stacking.¹⁰⁸ The reason in such an analysis is additionally related to the fact that the kinetic equations obtained above generally are very complex, nonlinear equations. They are useful for estimating the role of some particular effect, if it is a leading one, but they are quite inconvenient for describing any real process. The kinetic equations were obtained essentially based on the assumption that the beam state is close to the thermodynamic equilibrium state. Such an assumption is automatically violated for the newly injected particles. As to the final stage of the relaxation, it can be violated for central particles in the distribution $|\Delta p| \leq \sigma_{||}$, for which the friction force depends nonlinearly on Δp . The Monte-Carlo analysis of the cooling process based on tracing the motion of individual particles is free from such an assumption and therefore enables one to make certain conclusions on the validity of predictions using some kinetic equations.

Below we shall follow Ref. 108, which for simulations used a model close to that described in §7.4. The coasting beam was divided into samples with the azimuthal length $\vartheta_0 = 2\pi/W$, where W is the number of samples. Within the sample the position of a particle in the phase space was corrected using the following equations

$$\begin{aligned} \Delta\omega_a^m &= \Delta\omega_a^{(m-1)} + \sum_{b=1}^{N^*} \alpha\{\Delta\omega_b^{(m-1)}\} \Delta\omega_b^{(m-1)} \\ &\quad + \left\langle \sum_{b=1}^{N^*} \alpha\{\Delta\omega_b^{(m-1)}\} \Delta\omega_b^{(m-1)} \right\rangle_W, \\ \Delta\vartheta_a^m &= \Delta\vartheta_a^{(m-1)} + T_s \Delta\omega_a^{(m-1)}. \end{aligned} \quad (7.5.9)$$

Here, m marks the steps (turns), N^* is the number of particles in the sample, and the brackets $\langle \dots \rangle_W$ mean the averaging over the samples. The subtraction of the mean value of the kick in Eq. (7.5.9) describes the possibility for the pickup electrode to measure only ac components of beam current. (See Fig. 7.13.)

After each turn the particles were resorted among the samples according to their azimuthal position. This simulated the traveling of particles between samples as their momenta change. The initial conditions for particles were prepared using the random number generator. The initial conditions for ϑ were distributed within the interval $0-2\pi$, and the initial values of frequencies were distributed within the desirable region.

During the calculations the dependence on time has been traced for the momentum distribution and for the number of particles in the beam. Particles getting the deviation of the frequency $|\Delta\omega| \geq \Delta\omega_{\max}$ were excluded from subsequent calculations. With new particle deposit the particles having momenta in the injection region also were excluded from calculations. That simulated the particle losses.

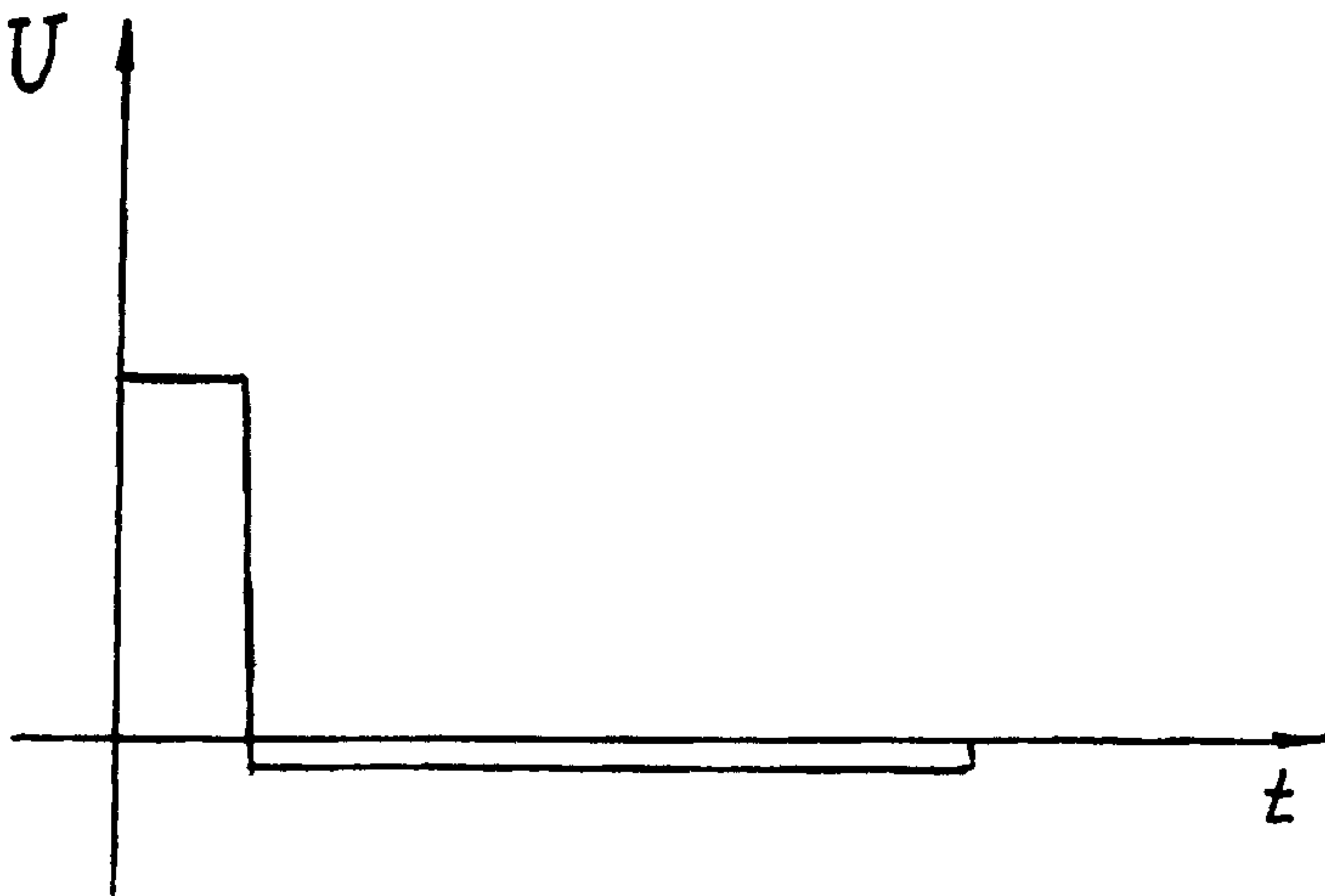


FIG. 7.13. Schematic plot of the accelerating wake used in Eqs. (7.5.9).

To verify the analytic estimations of the influence of the shielding, diffusion, and coherent stability on the process of stochastic cooling, these were simulated assuming the linear dependence of the friction force on $\Delta\omega$, which corresponds to the constant factors α in Eq. (7.5.9).

The calculated dependences of the revolution frequency spread on the number of revolution turns (i.e., on the time; see Fig. 7.14) definitely show that with

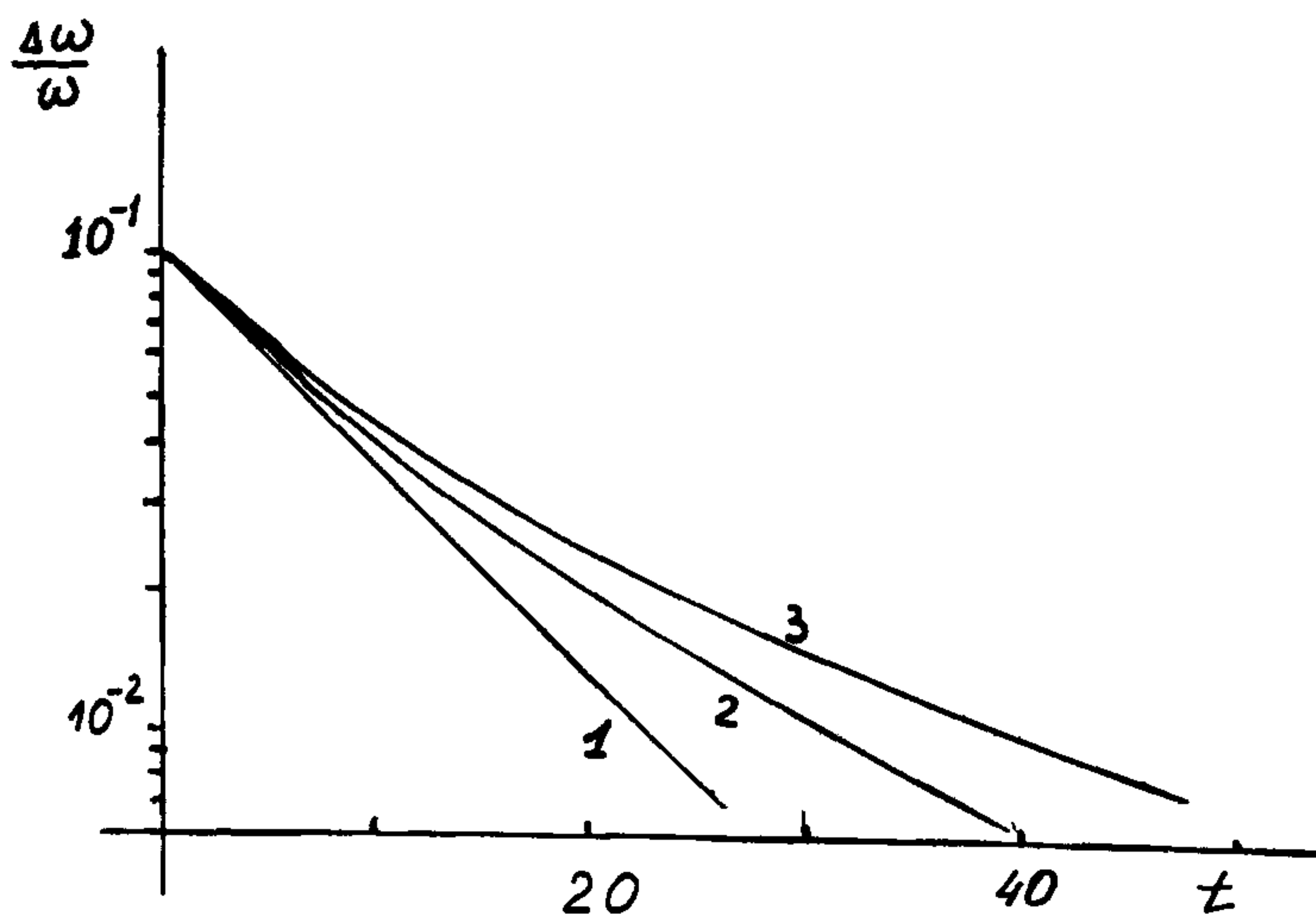


FIG. 7.14. The dependence of the frequency spread on the number of turns; $\alpha=0.1$; 1— $N=1$, 2— $N=50$, 3— $N=100$.

growth of beam intensity its cooling time increases. In these calculations, the value α was used, which corresponds to the damping of the energy deviation of a single particle by a factor e in ten turns. Such a rough value of α enables one to simulate the main effects with a relatively small number of particles. Using these data the local cooling decrements of particles have been calculated. It has been found that the obtained dependences of $\lambda_{\parallel}$ on beam intensity and the frequency spread ($\Delta\omega_0$) (see Fig. 7.15) are fitted well by the formula

$$\lambda_{\parallel} \simeq \begin{cases} \lambda_0, & \Delta\omega_0 \gg N\lambda_0/W^2, \\ 1.25 \frac{W^2}{N} \frac{\Delta\omega_0}{2\pi}, & \Delta\omega_0 \ll N\lambda_0/W^2. \end{cases} \quad (7.5.10)$$

As before, λ_0 is the cooling decrement of a single particle. This result agrees with Eq. (7.4.19). It also agrees with results of simulations obtained in Ref. 111. The numerical coefficient 1.25 in Eq. (7.5.10) is determined by the shape of the wake used in these simulations.

The tracking of the dipole oscillations of the beam showed that it decays, when N is small, but blows up (see Fig. 7.16) in the region

$$N \geq N_{th} = W/\alpha.$$

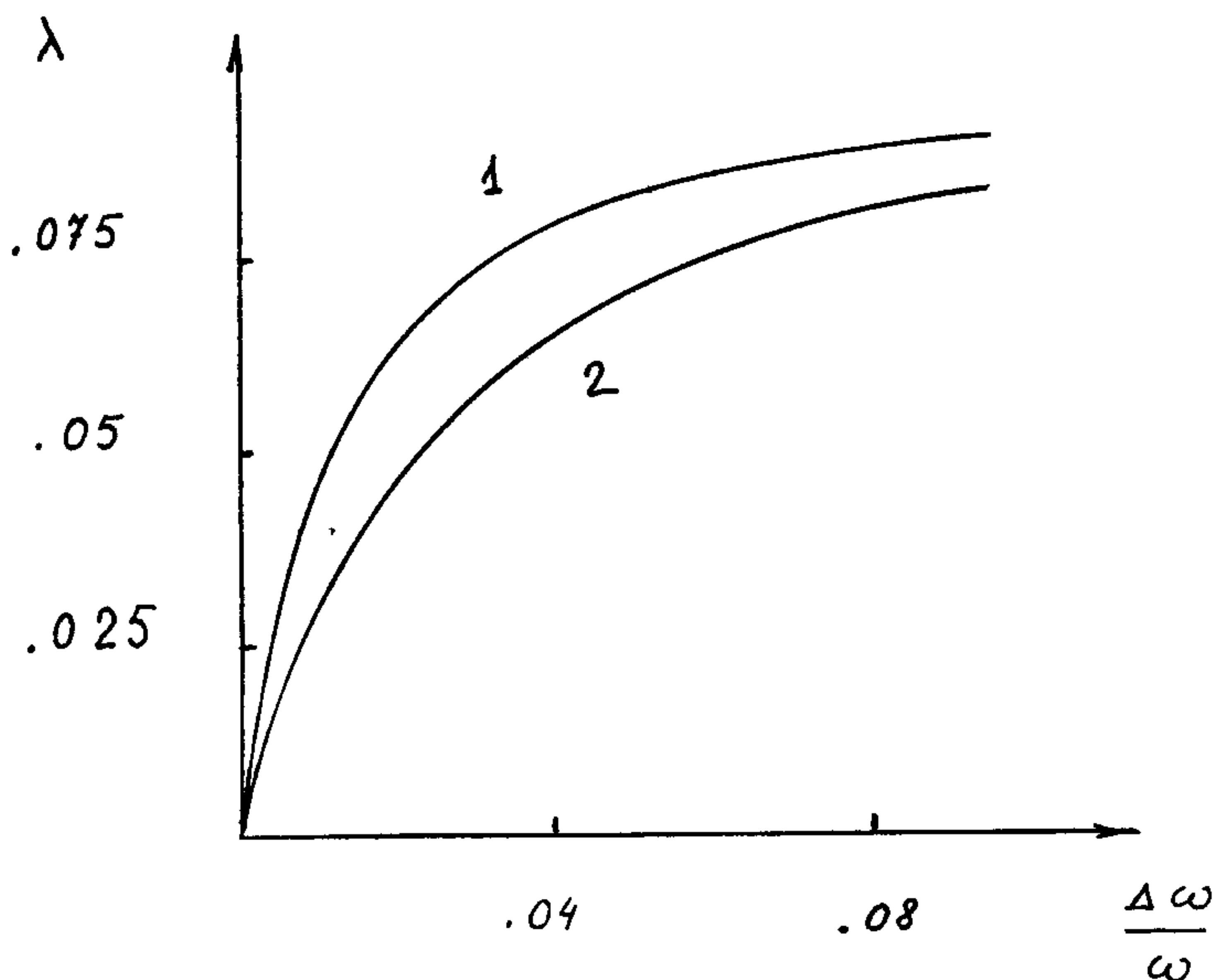


FIG. 7.15. The dependence of the cooling decrement on the frequency spread in the beam $\alpha=0.1$; 1— $N=50$, $N=100$.

In Ref. 108 stochastic stacking was simulated for two dependences of α on $\Delta\omega$: $\alpha = \text{const}$ and

$$\alpha(\Delta\omega) = \alpha_0(\Delta\omega/\delta\omega_0)^2. \quad (7.5.11)$$

The last case was studied to simulate the suppression of mutual influence of stored particles across the stack. The typical result of the cooling for these two cases is shown in Fig. 7.17. One can see as one specific feature of the cooling with the nonlinear friction force the gap in the particle energy distribution in the center of the beam [near $\alpha(\Delta\omega) = 0$]. There is no such effect [see Fig. 7.17(c)] for the beam cooled by the system with the linear friction force.

As it is typical for the problems with nonlinear wake fields, the values of decrements of coherent oscillations for the case with the cubic friction force [Eq. (7.5.11)] are determined by the beam energy spread. To this end, the stability condition in Eq. (7.5.6) reads

$$N^* \alpha_0 \langle \Delta\omega^2 \rangle / \delta\omega_0^2 < 1. \quad (7.5.12)$$

The value of the frequency spread in the beam $\{\langle \Delta\omega^2 \rangle\}^{1/2}$ is determined itself by heating factors (i.e., by the noise of the cooling circuit and by the noise from the newly injected beam).

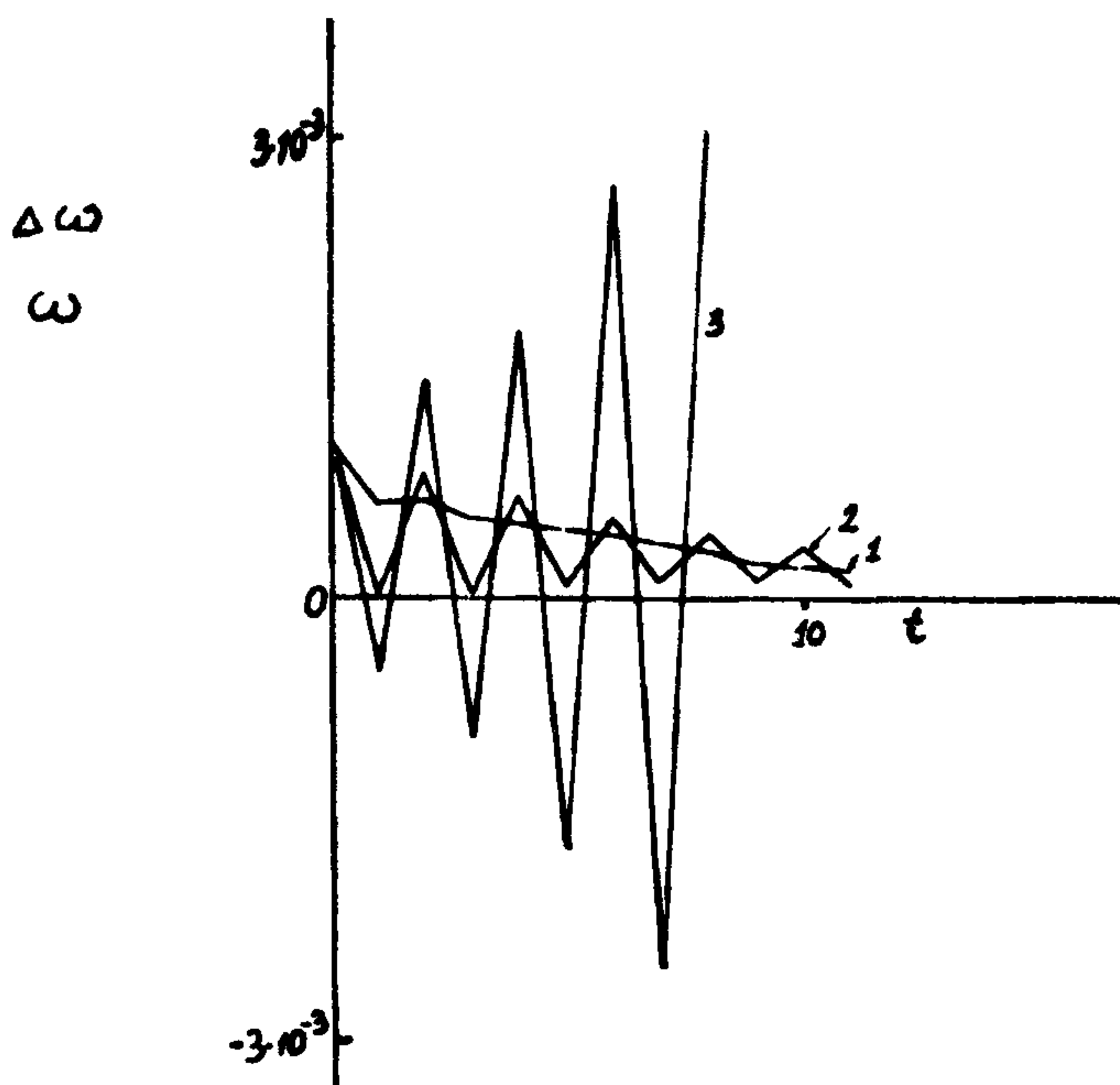


FIG. 7.16. The dependence of the beam dipole momentum on the number of turns; $\alpha = 0.15$, 1— $N = 10$, 2— $N = 100$, 3— $N = 150$.

The external diffusion (the noise of the feedback circuit) was simulated by the random kicks with rectangular distribution of amplitudes. For the case with the cubic friction force this resulted in the equilibrium frequency spread

$$\langle \Delta\omega^2 \rangle = 0.4\delta\omega_0 \sqrt{\langle \Delta\omega^2 \rangle / \alpha}, \quad (7.5.13)$$

where $\langle \Delta\omega_T^2 \rangle$ is the least square amplitude of kicks.

In these simulations the threshold number of particles N_{th} was determined by the appearance in the beam of unstable coherent oscillation (that, as a rule, was accompanied by the loss of a fraction of the beam). The results were fitted well by

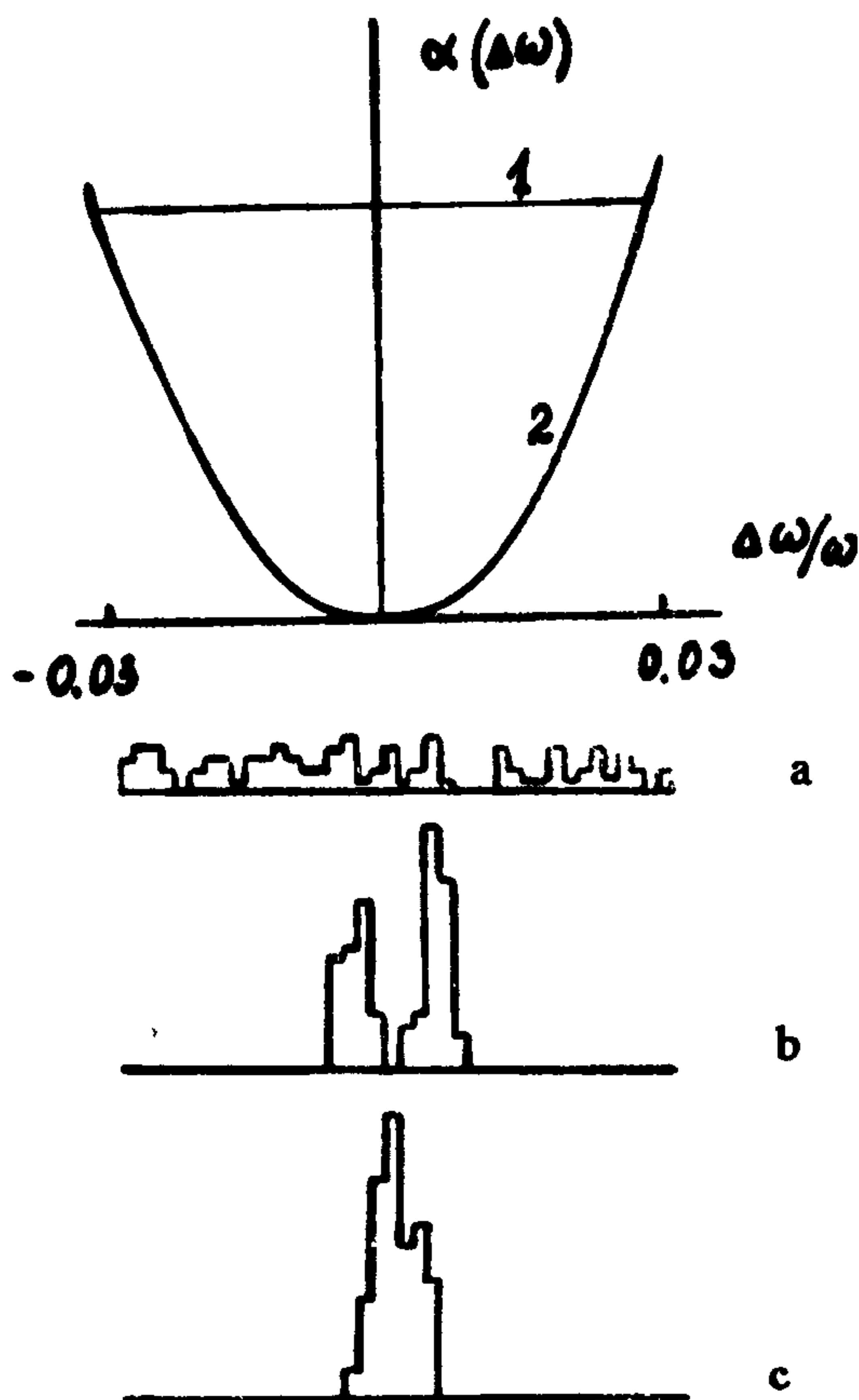


FIG. 7.17. The dependence of the amplification coefficient on the particle frequency deviation 1—for the system with linear energy gain, and 2—for the system with the cubic energy gain, as well as the distributions of particles over revolution frequencies; (a)—initial distribution, (b)—after 180 turns with nonlinear cooling, (c)—after 60 turns with linear cooling.

$$N = 0.7 \frac{W}{\alpha_0^{3/2} \sqrt{\langle \Delta \omega_T^2 \rangle / \delta \omega_0^2}}. \quad (7.5.14)$$

Another source of beam heating could come from the noise of the newly injected portion of particles. The blow-up of initially cooled beam due to the interaction with injected deposit (see Fig. 7.18) has caused the appearance of many particles in the region with a large amplification factor that might cause the instability of the beam coherent oscillations.

The typical dependence of the number of stored particles on the time for both linear and cubic friction forces is shown in Fig. 7.19. These calculations used $\alpha_0 = 0.15$, which was enough for particles to leave the injection region for the time between injections. The particle losses visible in Fig. 7.19 are caused by the excitation of coherent oscillation in a beam. These data were used for getting the dependence of the ultimate number of the accumulated particles on the number of the injected particles. (See Fig. 7.20.)

Taking into account the shifts of the injected particles from the injection region their influence on stack core qualitatively can be described by the diffusion coefficient:



FIG. 7.18. The distortion of the beam momentum distribution by the injected beam; $N=230$, $\alpha=0.15$; (a)—initial distribution, (b)—after 6 turns, (c)—after 12 turns.

$$\frac{d}{dt} \left(\frac{\Delta\omega}{\delta\omega} \right)^2 \approx \frac{\alpha_0^2 \Delta N}{T_s} \frac{T_s}{\alpha_0 \tau_1} \frac{\omega_s}{W \Delta\omega_{inj}}. \quad (7.5.15)$$

Here, $T_s/\alpha_0\tau_1$ is the fraction of the damping time $\tau_{||} = T_s/\alpha_0$ in the period between injections τ_1 ; $\Delta\omega_{inj}$ is the distance over the frequencies between the injection region and the center of the stack. A decrease in the number of stored particles with the growth of the number of injected particles ΔN is in good agreement with the formula obtained by combining of Eqs. (7.5.15) and (7.5.12):

$$N_{th} \approx \frac{W}{\alpha_0} \sqrt{\frac{W^2 \tau_1}{T_s \Delta N} \left(\frac{\Delta\omega_{inj}}{\omega_s} \right)} \quad (7.5.16)$$

or

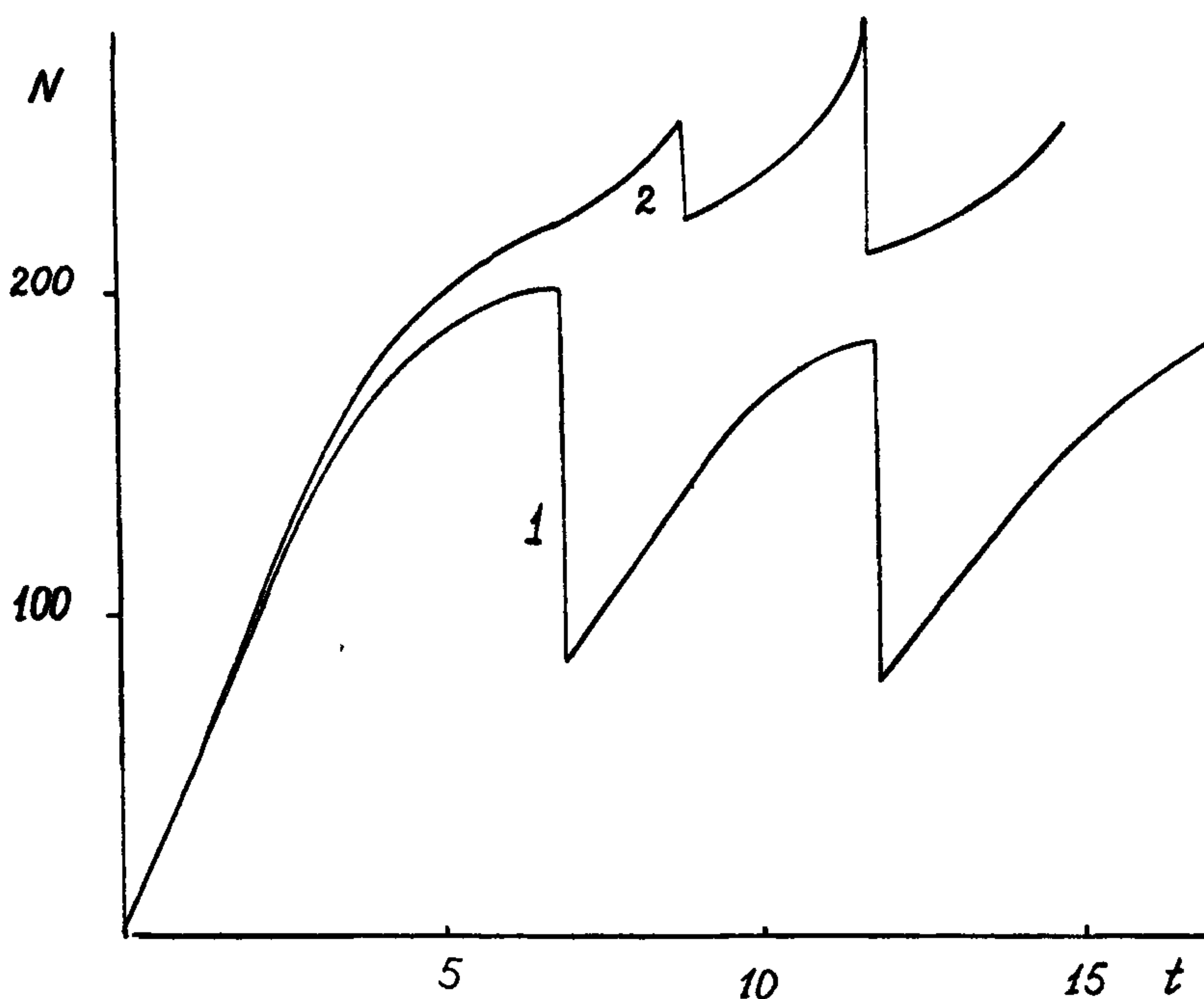


FIG. 7.19. The dependence of the number of accumulated particles on time; $\alpha=0.15$ injections after 10 turns, 1—the cooling by the linear system, 2—the cooling by the system with the cubic nonlinearity of the energy gain.

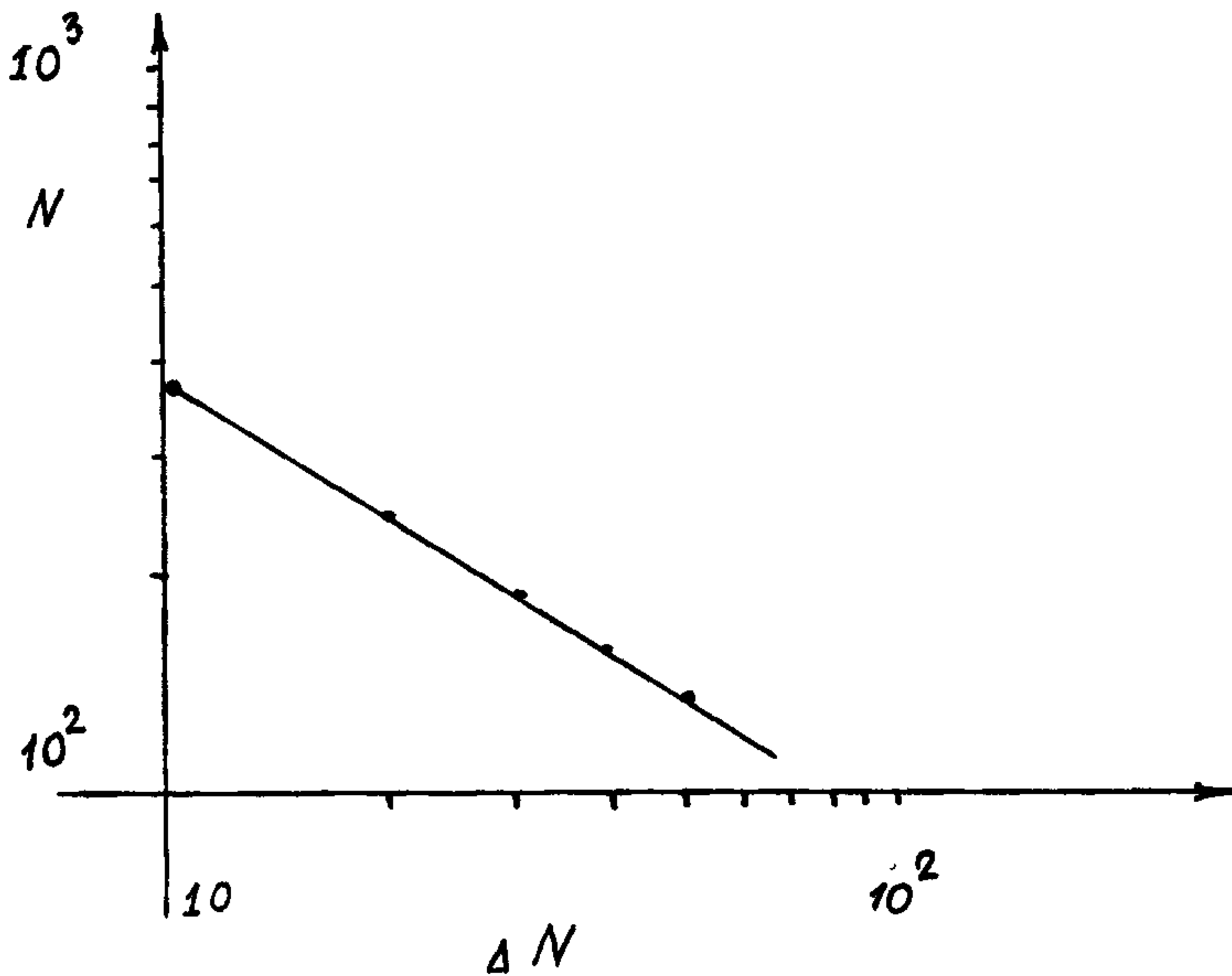


FIG. 7.20. The dependence of the threshold number of accumulated particles on the number of injected particles; $\alpha=0.15$ injections after 10 turns.

$$N_{th} = \frac{W\tau_{||}}{T_s} \left[\frac{\tau_1}{\tau(\Delta N)} \right]^{1/2},$$

$$\tau(\Delta N) = \frac{2\pi\Delta N}{W^2\Delta\omega_{inj}}. \quad (7.5.17)$$

These expressions also indicate that to store high currents using stochastic cooling, one has to decrease the value of the feedback amplification coefficient α_0 and increase the time between injections τ_1 . If, for instance, one takes $\tau_1 = \tau_{||}$, according to Eq. (7.5.17) the accumulation rate reads

$$\frac{dN}{dt} = \frac{\Delta N}{\tau} = \Delta N \left(\frac{W}{T_s N \sqrt{\tau(\Delta N)}} \right)^{2/3} \quad (7.5.18)$$

and the number of particles accumulated to time t is

$$N(t) = \Delta N^{2/5} \left(\frac{W\Delta\omega_{inj}}{\omega_s} \right)^{1/5} \left(\frac{5}{3} \frac{Wt}{T_s} \right)^{3/5}. \quad (7.5.19)$$

Comparison of this expression and Eq. (7.5.8) shows that use of a stochastic cooling system with a decrement decreasing to the center of the storage region does not lead to a substantial benefit in the number of particles stored during the given time interval t .

Electron Cooling and Collective Phenomena

In 1966 G. I. Budker¹¹² proposed a method for cooling beams of heavy charged particles by an accompanying electron beam. It was called the method of electron cooling. After initial study at the Institute of Nuclear Physics of the Siberian Division of the USSR Academy of Science in Novosibirsk¹¹³⁻¹³⁴ CERN,¹³⁵ and Fermi National Laboratory,¹³⁶ it was understood that except for an initial goal as the base for the construction of the proton (antiproton) storage rings this method provides an effective tool to get in the storage rings intense beams with extraordinary low temperatures ($T \sim 1\text{K}$).

In beams with low temperatures, collective effects might become very important at the unusually small number of particles in the rings. Alongside the previously discussed effects on storage rings with electron cooling, some specific related phenomena occur, first, to the collective interaction of the stored and cooling beams and, second, to the participation in the beam heating from the energy exchange between particles and long-living collective fluctuations of a beam. In particular, with a decrease in temperature, beam transformation into its condensed (liquid or crystalline) state becomes possible.^{126,127,128}

8.1. METHOD OF ELECTRON COOLING

8.1.1. The basic idea of the method is the use of heat exchange due to the Coulomb collisions of heavy particles (for definiteness, protons) with electrons from the cooling beam whose temperature T_e is maintained constant. Technically, such a heat exchange is realized mixing the cooling and cooled beams in one of the straight sections of the ring. As the cross section of Coulomb collisions strongly depends on the relative velocities of colliding particles, the beam average velocities for cooling section should coincide: $v_s = \langle v_e \rangle$.

The main requirements for parameters of the cooling beam can be specified using the following simple model.¹¹² For the time, when protons pass the cooling section,

$$\Delta t = T_s l / \Pi, \quad (8.1.1)$$

the electron and proton beams present the mixture of charged gases. If in the reference system traveling with the velocity v_s the components have temperatures T_e and T_p , respectively, then due to Coulomb collisions of particles, the mixture

will relax toward equalizing the temperatures of the components. In equilibrium the temperatures T_p and T_e become equal:

$$T_p = T_e. \quad (8.1.2)$$

This means that the velocity spread in the proton beam

$$v_{Tp} = \sqrt{\frac{T_p}{M}} = \sqrt{\frac{m}{M}} v_{Te} \quad (8.1.3)$$

will be $(m/M)^{1/2}$ times lower than that in the electron beam. (m is an electron and M is a proton mass.) As the relaxation time for this process is determined by the mean frequency of collisions, taking into account Eq. (8.1.1) it can be written as¹¹²

$$\tau^{-1} = \frac{l}{\Pi} \frac{4\pi n_e e^4 L_C}{Mm} \min\left(\frac{1}{v_{Te}^3}, \frac{1}{v_{Tp}^3}\right). \quad (8.1.4)$$

Here, n_e is the electron beam density, and L_C is the value of the Coulomb logarithm (by the order of magnitude $L \simeq 10-20$). Equation (8.1.4) shows the cooling time τ in terms of the beam characteristics in the accompanying reference system. Its recalculation to the laboratory reference system yields

$$\tau^{-1} = \frac{l}{\Pi} \frac{4\pi n_e e^4 L_C}{\gamma^2 M m v_T^3(\mathcal{E})}, \quad (8.1.5)$$

$$\frac{1}{v_T^3(\mathcal{E})} = \min\left(\frac{1}{v_{Te}^3(\mathcal{E})}, \frac{1}{v_{Tp}^3(\mathcal{E})}\right).$$

From this formula one can see that with an increase in beam energy, the cooling time τ increases at least like γ^2 . But, if with the growth of electron beam energy its angular divergence is maintained constant

$$v_T(\mathcal{E}) = \gamma \beta c \theta_T, \quad \gamma \ll 1/(\beta \theta_T), \quad (8.1.6)$$

the cooling time increases like γ^5 :

$$\tau^{-1} = \frac{l}{\Pi} \frac{4\pi n_e e^4 L_C}{\gamma^5 \beta^3 \theta_T^3}. \quad (8.1.7)$$

Equations (8.1.3) through (8.1.7) specify the requirements of the first approximation to the performance of the cooling electron beam: at sufficiently high density n_e it should have as small as possible angular divergences (and energy spread, respectively).

The choice of a certain scheme of the cooling device depends on the range of energy desirable to carry out the cooling. At the relatively low energies of particles (for example, proton energies $\mathcal{E}$ up to a few GeV) it is reasonable from

the technical point of view to use the scheme with a single pass of the cooling region by electrons. At larger energies it will be more convenient to use the electron beam from the special storage ring. In the last case, an electron temperature is maintained at the required level either by radiative cooling or by replacement of an electron beam upon its heating.

Let us consider in more detail the case of a single pass.¹¹³⁻¹³⁴ An electron beam of the required energy is then obtained by an electrostatic acceleration in a gun. At electron beam energies of tens and hundreds of keV one of the main technical problems becomes how to accept it on the collector and to use the power from the high-voltage source. In the given energy range the solution is in the use of the schemes with energy recuperation (i.e., with the deceleration of electron beam down to possibly low energy before it is dumped by the collector). Practically this is performed by the connection of the collector with the accelerating voltage source through the rectifier with a small positive voltage U_k . The product $U_k I_e$ (I_e is the current of the electron beam) yields the power consumed by the cooling device from the mains.

To suppress the electrostatic repulsion of electrons and related increase in angular divergences, an electron beam is immersed into the longitudinal magnetic field H accompanying it from the gun to the collector. For nonrelativistic electrons this field should satisfy the condition

$$\frac{H^2}{8\pi} > n_e mc^2. \quad (8.1.8)$$

8.1.2. Let us obtain the equations describing the kinetics of electron cooling in the simplest case, when neither collective effects nor the effects of IBS in the proton and electron beams are important. Below we shall see that because of the microscopic space scales characteristic for the elementary processes in the method of electron cooling, the effects of both classes become substantial with the noticeable increase in the density of proton beam in the phase space. Assuming the proton energies in the reference system accompanying the beams to be nonrelativistic and neglecting the IBS of protons, the collision integral in Eq. (2.5.25) is defined by the current

$$\mathbf{j}(\mathbf{I}, t) = \mathbf{j}^{(1)} + \mathbf{j}^{(4)}. \quad (8.1.9)$$

The terms $\mathbf{j}^{(1)}$ and $\mathbf{j}^{(4)}$ in this expression through Eqs. (2.5.26) and (2.5.31) are related to the wake of a proton in the electron beam $G(1, 2, \omega)$. The Green function $G(1, 2, \omega)$ can be calculated in the following way. After entering the cooling region, the proton forms in the electron beam the cloud of electrons compensating its charge. If

$$\mathbf{E}_{k, \omega} = -\frac{4\pi i e k}{k^2} \rho_p(\mathbf{k}, \omega)$$

is the Fourier amplitude of the proton electric field outside the electron beam and $\epsilon(\mathbf{k}, \omega)$ is the dielectric constant of the electron beam, then inside the electron beam $\mathbf{E}$ reads

$$\mathbf{E}_{\mathbf{k}, \omega} = -\frac{4\pi i e \mathbf{k}}{k^2 \epsilon(\mathbf{k}, \omega)} \rho_p(\mathbf{k}, \omega). \quad (8.1.10)$$

According to this equation we can write $G(1, 2, \omega)$ in the form

$$G(1, 2, \omega) = -4\pi \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)}}{k^2 \epsilon(\mathbf{k}, \omega)}. \quad (8.1.11)$$

Of course, such a form of $G(1, 2, \omega)$ assumes that the integration over $\mathbf{k}$ in Eq. (8.1.11) is limited by the region

$$ka \gg 1, \quad k > k_D = \sqrt{\frac{4\pi n_e e^2}{m v_{Te}^2}},$$

$$a = \min(a_p, a_e), \quad (8.1.12)$$

where a_p and a_e are the radii of the proton and electron beams, respectively. The second inequality in (8.1.12) enables one to omit from consideration the heat exchange between the proton and nonstationary long-lived fluctuations of the electric field in the electron beam. The possibility to use the scalar relations in Eqs. (8.1.10) and (8.1.11) is connected with the fact that in our cases of interest the friction force is determined by the longitudinal dielectric constant of the electron beam.

In the magnetic field $\mathbf{H}$ electrons are spiraling along the field force lines. The helix radius coincides with the electron Larmour radius r_L and the revolution frequency along the helix coincides with the Larmour frequency of an electron Ω_L :

$$r_L = \frac{v_\perp}{\Omega_L}, \quad \Omega_L = \frac{eH}{mc}. \quad (8.1.13)$$

To this end, the integration over $\mathbf{k}$ in Eq. (8.1.11) can be split into two regions. In the first one,

$$kr_L \gg 1, \quad kv_\perp \gg \Omega_L, \quad (8.1.14)$$

the value $\epsilon(\mathbf{k}, \omega)$ can be calculated neglecting the influence of the magnetic field. Since k is related to the impact parameters of the proton and electron collision by $b \propto 1/k$, the region (8.1.14) corresponds to collisions with $b \ll r_L$. This is the so-called region of close-range (or fast) collisions. The specific collision times in this region are of the order of $(kv_\perp)^{-1}$ and therefore are substantially lower than the revolution period of electrons along the Larmour rings.

The second range of parameters can be defined by the inequalities

$$ku \ll \Omega_L, \quad kr_L \ll 1, \quad (8.1.15)$$

where $\mathbf{u} = \mathbf{v}_p - \mathbf{v}_e$ is the relative velocity of the proton and electron. In this region the collisions of protons and electrons proceed adiabatically slowly compared to the Larmour rotation of the electron and, therefore, can be considered as the collision of the proton and the Larmour ring as a whole.²⁴ In the region (8.1.15) $\epsilon(\mathbf{k}, \omega)$ can be presented by the power series of $1/\Omega_L$. The first term of such an expansion $\epsilon^A(\mathbf{k}, \omega)$ can be calculated taking $H \rightarrow \infty$. As will be shown below, the rough partition of the integration region by conditions (8.1.14) and (8.1.15) is quite sufficient to calculate the kinetic coefficients in the logarithmic approximation.

Taking into account the partitions (8.1.14) and (8.1.15), let us represent $G(1,2,\omega)$ in Eq. (8.1.11) by the sum

$$G(1,2,\omega) = G^f(1,2,\omega) + G^A(1,2,\omega), \quad (8.1.16)$$

where the terms G^f and G^A describe the contributions of the fast and adiabatic collisions, respectively. The current $\mathbf{j}$ is also split into two parts:

$$\mathbf{j} = \mathbf{j}^f + \mathbf{j}^A. \quad (8.1.17)$$

Let us consider first the contribution of fast collisions. As already was mentioned, in the region (8.1.14), when calculating $\epsilon(\mathbf{k}, \omega)$ one can take $H=0$. Then $\epsilon(\mathbf{k}, \omega)$ reads

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{\omega_e^2}{k^2} \int d^3v_e \frac{\mathbf{k} \partial f_e / \partial \mathbf{v}_e}{\omega - \mathbf{k} \mathbf{v}_e},$$

$$\omega_e^2 = \frac{4\pi e^2 n_e}{m}, \quad (8.1.18)$$

$f_e(v)$ is the distribution function of electrons over velocities. Let us calculate first $\mathbf{j}^{(1)}$. (Index f will be omitted for short.) According to Eq. (2.5.26) we have

$$\mathbf{j}^{(1)} = -4\pi e^2 f_0 \int \frac{d^3k}{(2\pi)^3} \sum_{\mathbf{m}, n} \frac{\mathbf{m} |b_{\mathbf{m}n}(\mathbf{k})|^2 \epsilon''(\mathbf{k}, \omega_{\mathbf{m}, n})}{k^2 |\epsilon(\mathbf{k}, \omega_{\mathbf{m}, n})|^2},$$

$$\epsilon = \epsilon' + i\epsilon'', \quad b_{\mathbf{m}n} = \{\exp[i\mathbf{k}\mathbf{r}(\mathbf{I}, \psi, t)]\}_{\mathbf{m}, n}. \quad (8.1.19)$$

In the region $k > k_D$ one can neglect in the denominator of Eq. (8.1.19) the difference of ϵ from unity. Then $\mathbf{j}^{(1)}$ reads

$$\mathbf{j}^{(1)} = 4\pi^2 e^2 f_0 \int \frac{d^3k}{(2\pi)^3 k^2} \sum_{\mathbf{m}, n} \mathbf{m} |b_{\mathbf{m}n}(\mathbf{k})|^2$$

$$\times \frac{\omega_e^2}{k^2} \int d^3v_e \left(\mathbf{k} \frac{\partial f_e}{\partial \mathbf{v}_e} \right) \delta(\omega_{\mathbf{m}n} - \mathbf{k} \mathbf{v}_e). \quad (8.1.20)$$

Replacing the δ -function by the Fourier integral and summing over m and n we obtain

$$j_{\alpha}^{(1)} = e^2 \omega_e^2 f_0 \int_{k > k_D} \frac{d^3 k}{2\pi k^2} \int d^3 v_e \left(k \frac{\partial f_e}{\partial v_e} \right) \int_{-\infty}^{\infty} d\tau \left(k \frac{\partial \mathbf{r}}{\partial \psi_{\alpha}} \right) e^{i\mathbf{k}[\mathbf{r}(t) - \mathbf{r}(t+\tau) - \mathbf{v}_e \tau]},$$

or, taking into account that $\partial r_{\lambda} / \partial \psi_{\alpha} = \partial I_{\alpha} / \partial p_{\lambda}$ and the fact that in the cooling region the proton trajectory is a straight line,

$$j_{\alpha}^{(1)} = \frac{\partial I_{\alpha}}{\partial p_{\lambda}} e^2 \omega_e^2 f_0 \int d^3 v_e \frac{\partial f_e}{\partial v_{e\beta}} \int_{k > k_D} \frac{d^3 k}{2\pi} \frac{k_{\lambda} k_{\beta}}{k^4} \delta(\mathbf{k}\mathbf{u}). \quad (8.1.21)$$

The current $j_{\alpha}^{(4)}$ can be calculated similarly. However, for direct use of Eq. (2.5.31) before the substitution in Eqs. (8.1.18) and (8.1.11) the derivative $\partial f_e / \partial v_e$ should be replaced by the function $-m v_e f_e / T_e$, which corresponds to the equilibrium distribution of electrons. This results in

$$j_{\alpha}^{(4)} = -e^2 \omega_e^2 m \frac{\partial I_{\alpha}}{\partial p_{\lambda}} \frac{\partial I_{\beta}}{\partial p_{\sigma}} \frac{\partial f_0}{\partial I_{\beta}} \int dv_e f_e \int_{k > k_D} \frac{d^3 k}{2\pi} \frac{k_{\lambda} k_{\beta}}{k^4} \delta(\mathbf{k}\mathbf{u}). \quad (8.1.22)$$

Equations (8.1.21) and (8.1.22) comprise the integral

$$w_{\alpha\beta}(\mathbf{u}) = \int_{k > k_D} \frac{d^3 k}{2\pi} \frac{k_{\lambda} k_{\beta}}{k^4} \delta(\mathbf{k}\mathbf{u}) = \frac{\delta_{\alpha\beta} u^2 - u_{\alpha} u_{\beta}}{u^3} \int_{k_{\min}}^{k_{\max}} \frac{dk}{k}.$$

The formal coincidence of this expression with the collision tensor in Eq. (2.6.26) is not surprising. In this approximation the leading contribution gives such high k that kinetic coefficients can be calculated considering the scattering of individual protons and electrons. (See, for instance, Ref. 112 and 113.) As it is known, the logarithmic divergence of $w_{\alpha\beta}$ on the upper limit can be regularized by introducing such $k_{\max}$ that satisfy the perturbation theory

$$k_{\max} = \frac{mu^2}{e^2}.$$

The value of $k_{\min}$ can be chosen from

$$k_{\min} = \max\left(\frac{1}{r_L}, k_D\right).$$

Introducing the so-called Coulomb logarithm

$$L_C = \ln\left(\frac{mu^2}{e^2 k_{\min}}\right) \gg 1, \quad (8.1.23)$$

we can rewrite $w_{\alpha\beta}$ in the form

$$w_{\alpha\beta} = \frac{L_C}{2u^3} (\delta_{\alpha\beta} u^2 - u_\alpha u_\beta) \quad (8.1.24)$$

Combining Eqs. (8.1.21) and (8.1.22) into Eq. (2.5.15) we obtain the kinetic equation for f_0 ,

$$\frac{\partial f_0}{\partial t} = -\frac{\partial j}{\partial I} = -\frac{\partial}{\partial I_\alpha} \left\{ \langle \Delta I_\alpha \rangle f_0 - \frac{1}{2} \langle \Delta I_\alpha \Delta I_\beta \rangle \frac{\partial f_0}{\partial I_\beta} \right\}, \quad (8.1.25)$$

where

$$\langle \Delta I_\alpha \rangle = \overline{\frac{\partial I_\alpha}{\partial p_\lambda} \mathcal{F}_\lambda(u)}, \quad \langle \Delta I_\alpha \Delta I_\beta \rangle = \overline{\frac{\partial I_\alpha}{\partial p_\lambda} \frac{\partial I_\beta}{\partial p_\sigma} \langle \Delta p_\lambda \Delta p_\sigma \rangle},$$

while the friction force and diffusion tensor are

$$\mathcal{F} = -\frac{4\pi e^4 L_C}{m} \int d^3 v_e f_e \frac{\mathbf{u}}{u^3}, \quad (8.1.26)$$

$$d_{\alpha\beta} = \langle \Delta p_\alpha \Delta p_\beta \rangle = 4\pi e^4 L_C \int d^3 v_e f_e \frac{(\delta_{\alpha\beta} u^2 - u_\alpha u_\beta)}{u^3}. \quad (8.1.27)$$

As could be expected, Eq. (8.1.25) coincides with the result of the transformation of the Landau equation¹³⁷ to the action-phase variables. In this respect the calculation illustrates how to use Eqs. (2.5.25) through (2.5.31) for calculating of collision integrals. At the same time, it clearly shows what kind of kinetic effects have been neglected when getting Eq. (8.1.25).

The specific features of the beam cooling kinetics described by Eq. (8.1.25) were detailed in Ref. 113. Here we shall note only some of them. As can be seen from Eq. (8.1.25), the rate of the beam emittance is defined by

$$\langle \dot{I}_\alpha \rangle = \left\langle \frac{\partial \mathbf{r}}{\partial \psi_\alpha} \mathcal{F} \right\rangle + \frac{1}{2} \left\langle \frac{\partial}{\partial I_\alpha} \langle \Delta I_\alpha \Delta I_\beta \rangle \right\rangle, \quad (8.1.28)$$

where the brackets $\langle \rangle$ mean an averaging with $f(\mathbf{I}, t)$. The first term in Eq. (8.1.28) presents the partial power of friction force $Q_\alpha(\mathbf{I})$, while the second presents the diffusion power. It is clear that the stationary states of the beam can occur in the range of amplitude oscillations, where $Q_\alpha(\mathbf{I})$ decreases. If $v_{Te} = \sqrt{T_e/m}$ is the velocity spread in an electron beam, $\partial Q_\alpha / \partial I_\alpha < 0$ corresponds to the region $v_p < v_{Te}$. If $v_p > v_{Te}$, one has $\mathcal{F} \propto -v_p^{-2}$ and $Q_\alpha(\mathbf{I})$ increases (decreases in its absolute value). For small amplitudes of proton oscillations $v_p \ll v_{Te}$ Eq. (8.1.28) takes the most simple form. If $f_e(\mathbf{v})$ is the Maxwellian distribution,

$$f_e(\mathbf{v}_e) = n_e \left(\frac{m}{2\pi T_e} \right)^{3/2} \exp \left(-\frac{mv_e^2}{2T_e} \right),$$

and neglecting the coupling of particle oscillations one can write

$$\mathcal{F} \simeq -\frac{2}{3\sqrt{2\pi}} \frac{4\pi n_e e^4 L_C}{mv_{Te}^3} \nabla p$$

$$d_{\alpha\beta} = \delta_{\alpha\beta} d, \quad d \simeq \frac{4}{3\sqrt{2\pi}} \frac{4\pi n_e e^4 L_C}{v_{Te}}, \quad (8.1.29)$$

which instead of Eq. (8.1.28) yields

$$\langle \dot{I}_\alpha \rangle = \left\langle \frac{\partial r_\alpha}{\partial \psi_\alpha} \mathcal{F}_\alpha \right\rangle + \frac{1}{2} \left\langle \frac{\partial^2 I_\alpha}{\partial p_\alpha^2} d \right\rangle. \quad (8.1.30)$$

The substitution here of Eqs. (8.1.29) after the averaging yields ($I_\alpha = p_s \varepsilon_\alpha / 2$)

$$\dot{\varepsilon}_\alpha = -\frac{2}{3\sqrt{2\pi}} \frac{4\pi n_e e^4 L_C l}{M m v_{Te}^3} \frac{1}{\Pi} (\varepsilon_\alpha - 2\beta_\alpha \theta_{Te}^2), \quad (8.1.31)$$

$\theta_{Te} = v_{Te} / v_s$, and the value of β_α was taken as the constant along the cooling region ($\beta_\alpha(s) = \beta_\alpha$). From Eq. (8.1.31) one can see that in the cooling time

$$\tau^{-1} = \frac{2}{3\sqrt{2\pi}} \frac{4\pi n_e e^4 L_C l}{M m v_{Te}^3} \frac{1}{\Pi} \quad (8.1.32)$$

the emittance of the proton beam will equal

$$\varepsilon_\alpha = 2\beta_\alpha \theta_{Te}^2 \quad (8.1.33)$$

while its angular spread

$$\theta_{p\alpha} = \sqrt{\frac{\varepsilon_\alpha}{\beta_\alpha}} = \sqrt{\frac{2m}{M}} \theta_{Te}. \quad (8.1.34)$$

The cooling of the coasting beam is also accompanied by equalizing of the beam average velocities

$$\nabla_s \rightarrow \langle \nabla_e \rangle.$$

This signifies the advantage of the method of electron cooling from the radiative cooling. Since here there is no need to compensate for the particle energy loss, the storage can be performed without bunching of the stored beam. In addition, the equalizing of the electron and cooling particle velocities can be used for the precision measurement of beam energy without increase in its longitudinal temperature. The latter possibility is of special importance when carrying out the spectrometric measurements.¹²³

In spite of the fact that Eq. (8.1.32) was obtained for the nonrelativistic energies, it holds also for the relativistic energy range, provided the parameters entering there are related to the reference system accompanying beams. The

recalculation of cooling times for the laboratory system was already considered in §8.1.1.

Finally, we note that in region $v_p \gg v_{Te}$ one can neglect in Eq.(8.1.30) the contribution from diffusion, while

$$\mathcal{F} \simeq -\frac{4\pi e^4 L_C}{mv_p^3} \mathbf{v}_p$$

In this case, Eq. (8.1.5) for cooling decrements is still valid. They are growing with a decrease in v_p . That means that the cooling time for such a beam is determined by its initial temperature

$$\tau^{-1} \simeq \frac{4\pi n_e e^4 L_C l}{M m v_{p,in}^3 \Pi}. \quad (8.1.35)$$

8.1.3. In the storage ring, the oscillation of a particle relative to the closed orbit results in two additional effects: the so-called monochromatic instability¹¹³ and redistribution of decrements between the coupled degrees of freedom.

The monochromatic instability is caused by strong differences in the transverse average velocities of the cooling and cooled beams in the cooling region. Let

$$\Delta = \mathbf{v}_s - \langle \mathbf{v}_e \rangle, \quad |\Delta| \equiv \Delta \gg v_{Te} \quad (8.1.36)$$

Then, for protons with small velocities ($v_p \ll \Delta$) Eq. (8.1.26) yields

$$\mathcal{F}_\alpha = -\frac{4\pi e^4 n_e L_C}{m} \frac{v_{p\alpha}}{\Delta^3} \left(1 - 3 \frac{\Delta_\alpha^2}{\Delta^2} \right). \quad (8.1.37)$$

This expression shows that the friction force might change the sign turn to be directed along $v_{p\alpha}$, provided

$$|\Delta_\alpha| > \sqrt{2} \Delta_\perp. \quad (8.1.38)$$

Here $\Delta_\perp$ are the components of Δ , which are transverse to the direction α . The oscillations along the direction α will be excited by the friction force with an increment

$$\tau^{-1} \simeq \frac{4\pi n_e e^4 L_C l}{M m \Delta^3 \Pi} \left(3 \frac{\Delta_\alpha^2}{\Delta^2} - 1 \right), \quad (8.1.39)$$

while along others directions they will be damped. The sum of decrements over all three degrees of freedom, as Eq. (8.1.37) shows, equals zero:

$$\sum_\alpha \tau_\alpha^{-1} = \frac{\partial \mathcal{F}}{\partial \mathbf{p}} = 0. \quad (8.1.40)$$

The reason of the instability in the considered case is the fact that the work of the friction force along Δ turns out to be somewhat larger than that against Δ . As shown in Ref. 113 the oscillations in a beam grow up to the amplitudes corresponding to $v_p \simeq \Delta$.

When cooling the bunched beam the monochromatic instability can pose additional limitations on the accuracy of the matching of the electron average velocity and the frequency of the RF field of the storage ring. The reason is that due to electrostatic acceleration the energy spread of the electron beam remains equal to the gun cathode temperature $\Delta \mathcal{E} = T_k$. Since $\Delta \mathcal{E} = m v \Delta v_{\parallel}$, the spread of electron longitudinal velocities $\Delta v_{\parallel}$ decreases while the acceleration. After accelerating electrons up to the energy $W_e = m v_e^2 / 2$ it becomes equal to

$$\Delta v_{\parallel} = \frac{T_k}{\sqrt{2mW_e}}.$$

The corresponding longitudinal temperature of electrons

$$T_{\parallel} = m \Delta v_{\parallel}^2 = T_k^2 / 2W_e \quad (8.1.41)$$

turns out to be substantially lower than the cathode temperature. Therefore, even small deviations of $\langle v_e \rangle$ from the value, matching the RF-frequency, can exceed $\Delta v_{\parallel}$.

One more important feature of the kinetic of the beam electron cooling in the storage ring is the influence of the coupling of the particle oscillations on the values of cooling decrements. In the absence of the coupling of betatron oscillations in cyclic machines the radial and longitudinal particle motions are naturally coupled. If the longitudinal component of friction force depends on the particle radial position, this coupling will lead to the redistribution of decrements between coupled degrees of freedom. In particular, the required dependence can be provided by the dependence of the electron average velocity on the transverse coordinates,¹¹³ for example, due to the potential drop in the electron beam. In Ref. 138 it was shown that under these conditions due to flattening of electron velocity distribution the redistribution of decrements can substantially enhance the damping of radial oscillations. The corresponding addition to the cooling decrement of small radial oscillations ($v_p \ll v_{Te}$) reads¹³⁸

$$\delta \tau_x^{-1} = \tau_x^{-1} - (\tau_x^{-1})_0 = \frac{l}{\Pi} \frac{\eta \mathcal{F}_{\max}}{2p_s} \frac{dv_e/dx}{\left[(\Delta v_{\parallel})^2 + \epsilon_x \beta_x \left(\frac{dv_e}{dx} \right)^2 \right]^{1/2}}. \quad (8.1.42)$$

Here $\mathcal{F}_{\max}$ is the maximum value of the longitudinal component of the friction force; η, β_x are the values of the dispersion and β -function in the cooling region. The value $\delta \tau_x^{-1}$ will be positive if

$$\frac{dv_e}{dx} > 0. \quad (8.1.43)$$

Simultaneously this redistribution decreases longitudinal decrement exactly by $\delta\tau_x^{-1}$. To provide the damping of both degrees of freedom the following condition should be satisfied:

$$\eta \frac{dv_e}{dx} < v_s \quad (8.1.44)$$

For instance, in the electron beam with the round cross section of the radius a_e moving along the axis of the cylindrical vacuum chamber, the distribution of longitudinal velocities has a gradient

$$\frac{dv_e}{dx} = \frac{2\pi n_e e^2}{mv_e} x.$$

Then, according to Eqs. (8.1.43) and (8.1.44), the simultaneous cooling of the transverse and longitudinal motions occurs inside the region

$$0 \leq \Delta x \leq \frac{mv_e^2}{2\pi n_e e^2 \eta}. \quad (8.1.45)$$

With the growth of electron beam density n_e the width of this region decreases. Note also that when the position of the proton beam is displaced toward the center of the electron beam, the dependence of $\delta\tau_x^{-1}$ on n_e can change from linear to quadratic.

For applications it is obviously important to have as wide a region as possible with sufficiently fast damping of all degrees of freedom. The reasons given show that to this end, except for improvement of the electron-optical system it is necessary to redistribute in the proper way the electric fields across the cooling region. In particular, it might turn out to be reasonable to surround the electron beam with the system of electrodes with special distribution of their potentials to satisfy condition (8.1.44) inside the desirable region.

If the discussed redistribution is undesirable, to cancel $\delta\tau_x^{-1}$ one should undertake some special measures. The simplest way is to eliminate the dispersion function in the cooling region.

8.2. MOMENTUM COOLING OF A COLD BEAM BY MAGNETIZED ELECTRONS

8.2.1. All predictions of the "old theory" were basically confirmed by the initial experiments on electron cooling at the ring NAP-M. (See, for instance, Refs. 36, 114, 115, 116, and 118.) Further development of the theory¹²¹ and the improvement of parameters of experimental installation enabled¹²⁰ the finding that the efficiency of the method can surprisingly exceed that predicted by Eqs. (8.1.32) and (8.1.34).

Such a possibility is caused by the combined action of two factors: by the already mentioned decrease in the longitudinal temperature of electrons during their electrostatic acceleration; and by the influence of the accompanying magnetic field on the cooling process. The presence of the magnetic field in the cooling region divides the proton–electron collisions onto the fast collisions, when $\tau_c \Omega_L \ll 1$ (τ_c is the collision time), and adiabatic collisions, when $\tau_c \Omega_L \gg 1$. The relaxation due to fast collisions proceeds as if there is no magnetic field. It is described by Eqs. (8.1.26) and (8.1.27) with the obvious change of the Coulomb logarithm. For the adiabatic collisions the fast rotation of electrons along the Larmour circle gives no contribution into the proton–electron energy exchange—the proton collides effectively with the Larmour ring as a whole. Due to this fact the equilibrium is achieved by equalizing the temperature of protons and the gas temperature of Larmour rings, which is defined by Eq. (8.1.41) and can be substantially lower than the transverse temperature of electrons. The cooling decrements also increase in the region $v_p < v_{Te}$ and can reach very large values. For example, in the experiments on fast electron cooling on NAP-M^{91,120,128,129,130} the cooling times reached 40 ms and the longitudinal and transverse temperature of the cooled beam were $T_{\parallel} \simeq 1$ K and $T_{\perp} \simeq 150$ K (with $T_k^{eff} \simeq 2.7 \times 10^3$ K). In the same experiments the sharp dependence of cooling decrements on the perturbations of an accompanying magnetic field and much slower dependence on the velocities of Larmour rotation of electrons were also observed. Such behavior of decrements is also specific for the cooling by the magnetized electron beam.

A complete theory of the cooling by magnetized electrons can be found, for example, in Refs. 117 and 125. We shall not detail it here. Instead we briefly discuss some unusual features that are specific for its final stage.

For the sake of simplicity here we calculate only the longitudinal friction force $\mathcal{F}_{\parallel}$, which is responsible for the momentum cooling of the beam. Below, we follow Ref. 133. For the electron beam, we assume that $r_L \ll n_e^{-1/3}$, the charge of the cooled ions $Z = \pm 1$, and

$$T_{\parallel} = m \Delta^2 \gg e^2 n_e^{1/3} \quad (8.2.1)$$

as well as that the number of Langmuir oscillations during the time of flight l/v of the cooling section is high enough:

$$\frac{\omega_e l}{v} \gg 1, \quad \omega_e^2 = \frac{4\pi n_e e^2}{m}. \quad (8.2.2)$$

As can be seen from Eq. (8.1.10), the friction force acting on the ion and calculated in the lowest order of the perturbation theory reads

$$\mathcal{F} = -\frac{ie^2}{2\pi^2} \int \frac{d^3 k \mathbf{k}}{k^2 \epsilon(\mathbf{k}, \mathbf{k}u)}, \quad (8.2.3)$$

where $\mathbf{u}$ is the velocity of the ion in the rest frame of the electron beam. The cooling force is determined by the imaginary part of the dielectric constant $\epsilon(\mathbf{k}, \omega)$ and therefore describes the reaction of the radiation by the ion of plasma waves in the electron beam. For an extremely magnetized electron beam ($H \rightarrow \infty$) one can easily find for ϵ^{125} :

$$\epsilon(\mathbf{k}, \omega) = 1 + \frac{\omega_e^2 k_{\parallel}}{k^2} \int du_e \frac{\partial f_e / \partial u_e}{\omega - k_{\parallel} u_e}, \quad \text{Im} \omega > 0. \quad (8.2.4)$$

Here, $k_{\parallel} = h k$, $h = H/H$, and f_e is the distribution function of electrons over longitudinal velocities. Taking

$$f_e = \frac{1}{\sqrt{2\pi}\Delta} \exp\left(-\frac{u_e^2}{2\Delta^2}\right), \quad (8.2.5)$$

for ions moving along h we can rewrite Eq. (8.2.4) in the form

$$\begin{aligned} k^2 \epsilon(\mathbf{k}, k_{\parallel} u) = k^2 - \frac{\omega_e^2}{\sqrt{2\pi}\Delta^3} \mathcal{P} \int_{-\infty}^{\infty} du_e \frac{u_e \exp(-u_e^2/2\Delta^2)}{u - u_e} \\ + i\pi \text{sgn}(k_{\parallel}) \frac{u \omega_e^2}{\sqrt{2\pi}\Delta^3} \exp(-u^2/2\Delta^2). \end{aligned} \quad (8.2.6)$$

These equations enable one to get simple expressions for $\mathcal{F}_{\parallel}$ in two regions.

Region 1: $|u| \ll \Delta$. Here, in a linear approximation over u/Δ Eq. (8.2.6) yields

$$k^2 \text{Re} \epsilon(\mathbf{k}, k_{\parallel} u) \simeq k^2 + k_D^2, \quad k_D = \frac{\omega_e}{\Delta},$$

$$k^2 \text{Im} \epsilon(\mathbf{k}, k_{\parallel} u) \simeq \pi \text{sgn}(k_{\parallel}) \frac{u \omega_e^2}{\sqrt{2\pi}\Delta^3},$$

and hence, Eq. (8.2.3) reads

$$\mathcal{F}_{\parallel} \simeq -\frac{e^2 \omega_e^2}{\sqrt{2\pi}} \frac{u}{\Delta^3} \int_0^{\infty} dk_{\parallel} k_{\parallel} \int \frac{d(k_1^2)}{(k_1^2 + k_{\parallel}^2 + k_D^2)^2}. \quad (8.2.7)$$

The logarithmic divergency in this integral at the upper limit of integration over k_1 can be regularized if we cut the integrand at $k_{\max}$ corresponding to the validation of the perturbation theory

$$k_{\max} \simeq \frac{m\bar{u}}{e^2} \gg k_D,$$

where $\bar{u}$ is some typical value of the relative velocity of the ions and electrons. As the cutoff parameter $k_{\max}$ is specified, the integration in Eq. (8.2.7) over $\mathbf{k}$ yields

$$\mathcal{F}_{\parallel}(u) \simeq -\frac{e^2 \omega_e^2}{\sqrt{2\pi}} \frac{u}{2\Delta^3} \left(\ln \left[1 + \frac{k_{\max}^2}{k_D^2} \right] - \frac{k_{\max}^2}{k_{\max}^2 + k_D^2} \right). \quad (8.2.8)$$

Sometimes for rough estimations the special value $\bar{u} \propto \Delta$ is used. Then $k_{\max} \sim m\Delta^2/e^2$ and Eq. (8.2.8), with logarithmic accuracy, yields

$$\mathcal{F}_{\parallel}(u) \simeq -\frac{e^2 \omega_e^2}{\sqrt{2\pi}} \frac{u}{\Delta^3} \frac{3}{2} \ln \left(\frac{m\Delta^2}{e^2 n_e^{1/3}} \right), \quad |u| < \Delta. \quad (8.2.9)$$

Region 2. $|u| \gg \Delta$. In this region, ions radiate real plasma waves

$$\text{Re}\epsilon(\mathbf{k}, k_{\parallel} u) \simeq 1 - \omega_e^2/k^2 u^2$$

and $\text{Re}\epsilon \gg \text{Im}\epsilon$ for all $\mathbf{k}$ except those satisfying the dispersion equation $\text{Re}\epsilon(\mathbf{k}, k_{\parallel} u) = 0$. Using here

$$\frac{\text{Im}\epsilon}{|\epsilon|^2} \rightarrow \pi \delta(\text{Re}\epsilon),$$

we can get from Eq. (8.2.3)

$$\mathcal{F}_{\parallel}(u) = -\frac{e^2}{2\pi} \text{sgn}(u) \int \frac{d^3 k k_{\parallel}}{k^2} \delta[\text{Re}\epsilon(\mathbf{k}, k_{\parallel} u)],$$

or, after integration over $\mathbf{k}$,

$$\mathcal{F}_{\parallel}(u) = -\frac{e^2 \omega_e^2}{2u^2} \text{sgn}(u), \quad |u| \gg \Delta. \quad (8.2.10)$$

Combining Eqs. (8.2.9) and (8.2.10), we can write

$$\mathcal{F}_{\parallel}(u) \simeq -\frac{e^2 \omega_e^2}{2\Delta^2} \begin{cases} 3 \frac{u}{\Delta} \ln \left(\frac{m\Delta^2}{e^2 n_e^{1/3}} \right), & |u| \ll \Delta, \\ \text{sgn}(u) \frac{\Delta^2}{u^2}, & |u| \gg \Delta. \end{cases} \quad (8.2.11)$$

8.2.2. The cooling by magnetized beam, especially during its final stage, can have very complicated kinetics. For instance, when cooling the negatively charged ions, the contribution into $\mathcal{F}$ of collisions with impact parameters

$$b < b(\Delta u) = \frac{2e^2}{m\Delta u^2}, \quad \Delta u = u - u_e \quad (8.2.12)$$

should be taken into account. This is a nonperturbative region, where collisions of electrons and negatively charged ions look like the reflection of electrons from a potential barrier of the height e^2/b . Since such a reflection changes the electron

momentum by $\Delta p_{\parallel} = -2m\Delta u$, the averaged momentum rate of the ion, passing the cooling region, reads

$$\begin{aligned} \overline{\frac{d\Delta p_{\parallel}}{dt}} &= \int_{-\infty}^0 du_e (-2m\Delta u) \pi b^2(\Delta u) |u| \left(1 - \frac{vt}{l}\right) n_e f(u_e) \\ &+ \int_0^{\infty} du_e (-2m\Delta u) \pi b^2(\Delta u) |u| \frac{vt}{l} n_e f(u_e). \end{aligned} \quad (8.2.13)$$

This yields the averaged gain of the ion momentum after passing the cooling region,

$$\overline{\Delta p_{\parallel}} = \frac{-l}{v} \int_0^{\infty} d\Delta u 2m\Delta u^2 \pi b^2(\Delta u) \frac{n_e}{2} [f(u - \Delta u) - f(u + \Delta u)],$$

and, correspondingly, the contribution into the average value of the friction force,

$$\Delta \mathcal{F}_{\parallel}^{(-)} = \frac{\overline{\Delta p_{\parallel}}}{(l/v)} = -e^2 \omega_e^2 \int \frac{d\Delta u}{\Delta u^2} [f(u - \Delta u) - f(u + \Delta u)]. \quad (8.2.14)$$

Due to the fast decay of the integrand the upper limit in Eq. (8.2.14) can be left infinite. Near the lower limit this integral has a logarithmic divergency caused by the increase of the collision cross section for particles having small relative velocities:

$$\sigma(\Delta u) = \pi b^2(\Delta u) = \frac{4\pi e^4}{m^2 \Delta u^4}. \quad (8.2.15)$$

To cure this divergency we have to remember that Eq. (8.2.14) was obtained in the approximation of pair collisions and therefore impact parameters b would be limited at least by

$$\frac{\omega_e b}{|\Delta u|} \ll 1$$

or

$$b(\Delta u) \ll b_{\max} = |\Delta u| / \omega_e$$

This yields

$$|\Delta u| \gg \Delta u_{\min} = \left(\frac{2e^2 \omega_e}{m} \right)^{1/3} \quad (8.2.16)$$

and instead of Eq. (8.2.14) we find

$$\Delta \mathcal{F}_{\parallel}^{(-)} = -e^2 \omega_e^2 \int_{\Delta u_{\min}}^{\infty} \frac{d\Delta u}{\Delta u^2} [f(u - \Delta u) - f(u + \Delta u)]. \quad (8.2.17)$$

For some cases this expression can be simplified. In the linear region $|u| \ll \Delta$ one can use the Taylor expansion of the integrand in Eq. (8.2.17) in powers of (u/Δ) . Taking into account linear terms, we find

$$\Delta \mathcal{F}_{\parallel}^{(-)} = 2e^2 \omega_e^2 u \int_{\Delta u_{\min}}^{\infty} \frac{d\Delta u}{\Delta u^2} \frac{\partial f}{\partial \Delta u}, \quad |u| \ll \Delta. \quad (8.2.18)$$

For the Gaussian distribution Eq. (8.2.5) the integral in Eq. (8.2.18) is expressed in terms of the integral exponential function¹⁴

$$\Delta \mathcal{F}_{\parallel}^{(-)} = \frac{e^2 \omega_e^2 u}{\sqrt{2\pi} \Delta} \text{Ei} \left(-\frac{\Delta u_{\min}^2}{2\Delta^2} \right). \quad (8.2.19)$$

Using the asymptotes¹⁴

$$\text{Ei}(-x) = \begin{cases} \ln x - x + C, & x \ll 1 \\ -e^{-x}/x, & x \gg 1, \end{cases}$$

where $C \simeq 0.577$ is the Euler constant, we rewrite Eq. (8.2.19) in the form

$$\Delta \mathcal{F}_{\parallel}^{(-)} \simeq -\frac{e^2 \omega_e^2 u}{\sqrt{2\pi} \Delta^3} \begin{cases} \ln \left(\frac{2\Delta^2}{\Delta u_{\min}^2} \right) + \frac{\Delta u_{\min}^2}{\Delta^2} - C, & \Delta u_{\min} \ll \Delta, \\ \frac{\exp(-\Delta u_{\min}^2/2\Delta^2)}{(\Delta u_{\min}^2/2\Delta^2)}, & \Delta u_{\min} \gg \Delta. \end{cases} \quad (8.2.20)$$

In the inverse limiting case $|u| \gg \Delta$ the value of $\Delta \mathcal{F}_{\parallel}^{(-)}$ is determined by the behavior of the tails of the electron distribution function. Provided this distribution is a smooth function of u , Eq. (8.2.17) yields

$$\Delta \mathcal{F}_{\parallel}^{(-)} = e^2 \omega_e^2 \frac{\partial f}{\partial u} \ln \left(\frac{u^2}{\Delta u_{\min}^2} \right), \quad |u| \gg \Delta. \quad (8.2.21)$$

On the contrary, for a δ -function or another singular distribution f_e Eq. (8.2.17), as can be easily seen, gives the nonlogarithmic expression for $\Delta \mathcal{F}_{\parallel}^{(-)}$:¹³¹

$$\Delta \mathcal{F}_{\parallel}^{(-)} \propto 1/u^2, \quad |u| \gg \Delta.$$

Strong sensitivity to the shape of the distribution function of electrons is one of the features of calculations related to cooling by extremely magnetized electrons.

8.2.3. The total friction force acting on the negatively charged ion in the magnetized electron beam is given by the sum of the expressions in Eqs. (8.2.7) and (8.2.17). But because the contribution from the nonperturbative region is taken into account by Eq. (8.2.17), the integration over $k_{\perp}$ in Eq. (8.2.7) must be limited by

$$k_{\perp, \max} = \frac{m\Delta u_{\min}}{e^2} \gg k_D \quad (8.2.22)$$

or $k_{\perp, \max} > n_e^{1/3}$. Then, for instance, Eq. (8.2.8) will take the form

$$\mathcal{F}_{\parallel}^{(0)}(u) \simeq -\frac{e^2 \omega_e^2}{\sqrt{2\pi}} \frac{u}{2\Delta^3} \ln \left[1 + \left(\frac{k_{\max}}{k_D} \right)^2 \right], \quad |u| \ll \Delta. \quad (8.2.23)$$

For negative ions the total friction force $\mathcal{F}_{\parallel}$ gets the addition given by Eq. (8.2.17), which in a logarithmic approximation ($\Delta^2 \gg \Delta u_{\min}^2$, $m\Delta^2 \gg e^2 n_e^{1/3}$) yields

$$\begin{aligned} \mathcal{F}_{\parallel}^{(-)} &= \mathcal{F}_{\parallel}^{(0)} + \Delta \mathcal{F}_{\parallel}^{(-)} = \mathcal{F}_{\parallel}^{(0)} + 2\mathcal{F}_{\parallel}^{(0)} = 3\mathcal{F}_{\parallel}^{(0)}, \quad |u| \ll \Delta, \\ \mathcal{F}_{\parallel}^{(0)} &= -\frac{e^2 \omega_e^2}{\sqrt{2\pi}} \frac{u}{2\Delta^3} L, \quad L = \ln(\Delta^2 / \Delta u_{\min}^2) \gg 1. \end{aligned} \quad (8.2.24)$$

Because there are no reflections of electrons from positively charged ions, we may expect that the nonperturbative region of impact parameters (8.2.12) gives no significant contribution to $\mathcal{F}_{\parallel}^{(+)}$ and hence we write

$$\mathcal{F}_{\parallel}^{(+)} = \mathcal{F}_{\parallel}^{(0)}. \quad (8.2.25)$$

For high ion velocities $|u| \gg \Delta$, as can be seen from Eq. (8.2.21), $\Delta \mathcal{F}_{\parallel}^{(-)}$ is negligible. Therefore here the forces $\mathcal{F}_{\parallel}^{(\pm)}$ coincide:

$$\mathcal{F}_{\parallel}^{(\pm)} = -\frac{e^2 \omega_e^2}{2u^2} \text{sgn}(u), \quad |u| \gg \Delta.$$

The possible difference in electron cooling of the positively and negatively charged particles has been pointed out in Ref. 125 and 126 and was confirmed experimentally in Ref. 131.

All the formulas, obtained in this section were calculated in the logarithmic approximation ($m\Delta^2 \gg e^2 n_e^{1/3}$). In the nonlogarithmic region $m\Delta^2 \leq e^2 n_e^{1/3}$ the value $\mathcal{F}_{\parallel}$ must be calculated beyond the framework of the perturbation theory. In the intermediate region $m\Delta^2 \sim e^2 n_e^{1/3}$ for estimating $\mathcal{F}_{\parallel}$ one can extrapolate expressions to $m\Delta^2 \sim e^2 n_e^{1/3}$.¹²⁵ After such an extrapolation, as can be easily seen from Eqs. (8.2.19) and (8.2.22), the relative contribution of $\Delta \mathcal{F}_{\parallel}^{(-)}$ to $\mathcal{F}_{\parallel}^{(-)}$ decreases roughly by a factor of two. Nevertheless, the friction force for

negatively charged ions remains twice as large as that for positively charged ions.

The discussed enhancement of $\mathcal{F}_{\parallel}^{(-)}$ due to collisions from the nonperturbative region (8.2.12) mainly increases cooling decrements and the maximum value of $\mathcal{F}_{\parallel}^{(-)}$. By direct calculation of the diffusion coefficient due to collisions from region (8.2.12) one can easily see that in the logarithmic approximation near the equilibrium the rate of ion longitudinal temperature $T_{\parallel}$ reads

$$\frac{dT_{\parallel}}{dt} = -2\lambda^{(-)}(T_{\parallel} - m\Delta^2),$$

which obviously has the equilibrium solution $T_{\parallel} = m\Delta^2$.

8.3. LONGITUDINAL INSTABILITY OF THE COASTING PROTON BEAM DUE TO ELECTRON COOLING

A somewhat different set of effects is related to the collective reaction of the cooling electron beam to the coherent oscillations of the cooled proton beam. In these terms the cooling beam itself is the dissipative medium and, generally speaking, can cause the coherent instabilities in proton beam.

When calculating the longitudinal impedance of the cooled beam, one should have in mind that the frequencies of fields excited by coherent oscillations of protons are close to harmonics of revolution frequency. Therefore, up to harmonics $n_{max} = \Omega_L/\omega_s \gg 1$ in calculation one can neglect the excitation of the Larmour rotation of electrons by the collective fields. Moreover, under the typical condition of the experiment (for example, on NAP-M) the Larmour rotation radii r_L of electrons are small compared to the transverse sizes of the beams. Under these conditions, when calculating the impedance of the cooling region, one can even take $r_L = 0$, considering electrons as if they are threaded onto the force lines of the magnetic field.

Assuming that the electron beam radius is substantially larger than that of proton beam, we can calculate the field induced by the proton beam in the cooling region using Eqs. (8.1.10) and (8.2.4)

$$\mathbf{E}(\mathbf{r}, t) = \frac{-ie^2}{4\pi^3} \int_{ka>1} \frac{d^3k d\omega}{k^2 \epsilon(\mathbf{k}, \omega)} \int d\Gamma' dt' e^{\mathbf{k}(\mathbf{r}-\mathbf{r}') - i\omega(t-t')} f_p(\Gamma', t'). \quad (8.3.1)$$

$$\epsilon(k, \omega) = 1 + \frac{\omega_e^2 k_{\parallel}}{k^2} \int dv_{\parallel} \frac{\partial f_e / \partial v_{\parallel}}{\omega - k_{\parallel} v_{\parallel}}. \quad (8.3.2)$$

As was shown in §3.2, the spectra of longitudinal coherent oscillations are determined by the linearized Vlasov equation

$$-i(\omega - n\omega_0(\Delta p)) f_{n,\omega} + eE_{\parallel}(n, \omega) \frac{\partial f_0}{\partial \Delta p} = 0. \quad (8.3.3)$$

Substituting here $E_{\parallel}$ from Eq. (8.3.1) one should take into account that the interaction of protons and electrons proceeds only in the cooling region of the azimuthal length $2\vartheta_0 = l/R_0$. This results in

$$E_{\parallel}(n, \omega) = \frac{-ie^2 k_n}{2\pi^2} \int \frac{d^3 k b(k_{\parallel} - k_n)}{k^2 \epsilon(k, \omega)} F(k_{\perp}) \rho_{n, \omega} \quad (8.3.4)$$

Here $k_n = n/R_0$,

$$b(k_{\parallel} - k_n) = \frac{\sin[l(k_{\parallel} - k_n)/2]}{\pi R_0 (k_{\parallel} - k_n)}, \quad (8.3.5)$$

the certain form of the dimensionless form-factor $F(k_{\perp}) > 0$ will not be significant below. [It is only important that $F(0) \simeq 1$ and $F(k_{\perp}) \rightarrow 0$, $k_{\perp} a \gg 1$.]

The comparison of Eq. (8.3.4) with

$$E_{\parallel}(n, \omega) = -\frac{e\omega_s Z_n}{\Pi} \int d\Delta p f_{n, \omega}, \quad (8.3.6)$$

yields the impedance of the cooling region:

$$\frac{-iZ_n}{n} = \frac{\Pi}{v_s} \int \frac{d^3 k b^2(k_{\parallel} - k_n)}{2\pi^2 k^2 \epsilon(k, \omega)} F(k_{\perp}). \quad (8.3.7)$$

Before making particular calculations, we note that for a nonrelativistic coasting beam its space charge typically gives the leading contribution to the imaginary part of the longitudinal impedance. For this reason, below we focus on the calculation of the real part of $Z_n(\omega)$:

$$Z'_n = \frac{\Pi n}{v_s} \int \frac{d^3 k}{2\pi^2} \frac{\epsilon''(k, \omega_n)}{k^2 |\epsilon(k, \omega_n)|^2} b^2(k_{\parallel} - k_n) F(k_{\perp}). \quad (8.3.8)$$

As the electrons passing the cooling region are absorbed by collector, the process under consideration is a single turn process. Therefore, the exact value of ω in Eqs. (8.3.7) and (8.3.8) can be replaced by the unperturbed value $\omega_n = n\omega_s$.

The calculation of integrals in Eq. (8.3.8) can be simplified when the deviation of longitudinal velocities of beams [$u = v_p - v_e$] is substantially smaller, or substantially larger than the longitudinal velocity spread in electron beam [Δ]. Let us first consider the case, when $|u| \ll \Delta$. Assuming a Gaussian distribution over longitudinal velocities in electron beam, we can write in this region:

$$\epsilon'' = \frac{\pi}{\sqrt{2\pi}\Delta} \frac{k_D^2 v_s (k_n - k_{\parallel}) - k_n u^2}{k^2 |k_{\parallel}|} \exp\left(-\frac{(v_s [k_n - k_{\parallel}] + k_n u)^2}{2k_{\parallel}^2 \Delta^2}\right)$$

and, hence, $k^2 \epsilon' \simeq k^2 + k_D^2$, where $k_D^2 = \omega_e^2 / \Delta^2$. These expressions show that for small u the main contribution to Z' give the region $|k_n - k_{\parallel}| \sim |k_n u| / v_s \ll |k_n|$.

Therefore, for reasonable harmonic numbers [$|k_n|a \ll 1$] we can take in Eq. (8.38) $k^2 \simeq k_1^2$ which results in

$$Z'_n \simeq I(n) \frac{\omega_e^2}{v_s} \int_{1/a}^{\infty} \frac{d^2 k_1 F(k_1)}{\pi(k_1^2 + k_D^2)^2} \simeq \frac{I(n)}{v_s} \frac{\omega_e^2 a^2}{(1 + k_D^2 a^2)} \quad (8.3.9)$$

and

$$I(n) = -\frac{R_0 |n|}{\sqrt{2\pi\Delta}} \frac{d}{du} \left\{ \left(1 + \frac{u}{v_s} \right) \int_{-\infty}^{\infty} dq b^2(q) \times \exp \left(-\frac{(v_s q + k_n u)^2}{2k_n^2 \Delta^2} \right) \right\}.$$

The calculation of $I(n)$ can be simplified in two cases. In the region $|n| \vartheta_0 \Delta / v_s \ll 1$ we may neglect in the integrand the variation of $b^2(q)$ and factor this function out of integral at $q = k_n u / v_s$. This yields

$$I(n) \simeq -\frac{1}{\pi^2 v^2} \frac{d}{dz} \frac{(1+z) \sin^2[n \vartheta_0 z]}{z^2},$$

$$\simeq \frac{n^2}{v_s^2} \left(\frac{l}{\Pi} \right)^2, \quad z = \frac{u}{v_s}.$$

In the case of cooling ($a \ll l\Delta / v_s$), we may also consider the region $|n| \vartheta_0 \Delta / v_s \gg 1$, where

$$I(n) \simeq \frac{u |n|}{\sqrt{2\pi\Delta^3}} \frac{l}{\Pi}$$

Resulting expression for Z'_n reads

$$\frac{Z'_n}{|n|} \simeq \frac{1}{v_s} \frac{k_D^2 a^2}{1 + k_D^2 a^2} \begin{cases} |n| \frac{\Delta^2}{v_s^2} \frac{l^2}{\Pi^2}, & |n| \vartheta_0 \ll v_s / \Delta, \\ \frac{u}{\sqrt{2\pi\Delta}} \frac{l}{\Pi}, & |n| \vartheta_0 \gg v_s / \Delta. \end{cases} \quad (8.3.10)$$

$\gamma, \vartheta_e \ll \Delta$ ^{electr.} _{temp.}

In the case when $|u| \gg \Delta$, we may neglect the velocity spread in electron beam and can replace Eq. (8.3.2) by the following expression

$$\epsilon = 1 - \frac{k_{\parallel}^2 v_s^2 c_e^2}{(\omega - k_{\parallel} v_e)^2}, \quad c_e = \frac{\omega_e}{v_s k_1}, \quad \text{Im} \omega > 0.$$

This yields

$$Z'_n = \frac{\Pi n}{v_s} \int \frac{d^2 k_\perp}{\pi k_\perp^2} F(k_\perp) \int_{-\infty}^{\infty} \frac{dk_\parallel}{4\omega_n c_e} (\omega_n - k_\parallel v_e)^2 b^2(k_n - k_\parallel) \\ \times \left[\delta \left[\frac{\omega_n}{1 - c_e} - k_\parallel v_e \right] - \delta \left[\frac{\omega_n}{1 + c_e} - k_\parallel v_e \right] \right] \quad (8.3.11)$$

or, after simple transformations,

$$Z'_n = \frac{\omega_e}{\pi v_s^2} \int_{1/a}^{\infty} \frac{dk_\perp}{k_\perp^2} F(k_\perp) \\ \times \left[\frac{\sin^2 \left(n \vartheta_0 \frac{z - c_e}{1 + c_e} \right)}{(z - c_e)^2} - \frac{\sin^2 \left(n \vartheta_0 \frac{z + c_e}{1 - c_e} \right)}{(z + c_e)^2} \right]. \quad (8.3.12)$$

For small harmonic numbers $|n| \vartheta_0 \max(|z|, c_e) \ll 1$ we can use in Eq. (8.3.12)

$$\frac{\sin^2 \alpha x}{x^2} \simeq \alpha^2,$$

which results in

$$Z'_n \simeq -\frac{2\omega_e^2 a^2 n^2 \vartheta_0^2}{\pi v_s^3} \int_1^{\infty} \frac{dq}{\left(q - \frac{\omega_e^2 a^2}{v_s^2} \right)^2} \\ \simeq -\frac{2\pi n^2}{v_s} \left(\frac{l}{\Pi} \right)^2 \frac{\omega_e^2 a^2}{v_s^2 - \omega_e^2 a^2} \simeq -\frac{2\pi n^2}{v_s} \left(\frac{l}{\Pi} \right)^2 \frac{\omega_e^2 a^2}{v_s^2}. \quad (8.3.13)$$

π of number, rem B.

For high harmonic numbers $|nz| \vartheta_0 \gg 1$, the calculation of the integral in Eq. (8.3.12) can be simplified in two cases. If $\omega_e a \ll |u|$, we can expand the integrand in powers of $\omega_e a / (k_\perp v_s)$. The leading term of this expansion reads

$$Z'_n \simeq -\frac{2\omega_e^2 a^2}{\pi v_s^3} \frac{d}{dz} \left(\frac{\sin^2 n \vartheta_0 z}{z^2} \right) \int_1^{\infty} \frac{dq}{q^3} F(q) \\ \simeq -\frac{\omega_e^2 a^2}{\pi'' v_s^3} \left[\frac{n \vartheta_0 \sin 2n \vartheta_0 z}{z^2} - \frac{2 \sin^2 n \vartheta_0 z}{z^3} \right].$$

Asymptotically $|nz| \vartheta_0 \gg 1$ this value increases like

$$Z'_n \simeq -\frac{n \omega_e^2 a^2 l}{v_s u^2 \Pi} \sin(2n \vartheta_0). \quad (8.3.14)$$

In opposite limiting case $\omega_e a \gg |u|$, we can use in Eq. (8.3.12) the identity

$$\lim_{a \rightarrow \infty} \frac{\sin^2 \alpha x}{\alpha x^2} = \pi \delta(x).$$

Then, after integration over $k_{\perp}$ we obtain

$$Z'_n \simeq \text{sgn}(u) \frac{|n| \vartheta_0}{v_s} F(\omega_e/u), \quad \frac{\omega_e a}{|n|} \gg 1. \quad (8.3.15)$$

Obtained expressions shown that the strength of collective interaction of a beam with the cooling device increases with the increase in the mode harmonic number. The comparison of Eqs. (8.3.10), (8.3.13), (8.3.14) and (8.3.15) with Eq. (3.2.29) shows that even for high frequencies the real part of the impedance due to such an interaction is roughly l/Π times smaller compared to the impedance due to the space charge of the beam. Nevertheless, due to strong reduction of the frequency spread in very cold beams even these small distortions of the longitudinal impedance can limit the beam intensity. Relevant stability criteria were obtained in §3.7. For low-frequency modes $|n| \Delta \omega_0 \ll \lambda_{\parallel}$ the stability diagram is a simple parabola

$$\xi' + 1 > (\xi'')^2 \frac{n^2 \Delta \omega_0^2}{\lambda_{\parallel}^2}, \quad \xi = \xi' + i\xi'' = \frac{\Omega_n^2}{n^2 \Delta \omega_0^2}.$$

Using natural assumption that $\xi' \gg 1$, we can rewrite this criterion in the form

$$I < I_{th} = \frac{N_{th} e \omega_0}{2\pi} = p v_s \left(\frac{\lambda_{\parallel}}{\omega_0} \right)^2 \frac{Z''_n / N}{\alpha_p (Z'_n)^2}. \quad (8.3.16)$$

This criterion does not pose any serious limitation on the stability of initial modes. However, for modes with $|n| \simeq \lambda_{\parallel} / \Delta \omega_0$, the substitution in Eq. (8.3.16) of parameters specific for the experiments on the installations [120, 135, 136] by the order of magnitude estimates $N < 3 \times 10^9$. This value is higher than the beam currents used in the experiments though an absolute value of N_{th} is not large. In addition it is worthwhile to note that N_{th} in Eq. (8.3.16) depends quadratically on the cooling decrement $\lambda_{\parallel}$. Therefore, with an increase in $\lambda_{\parallel}$ the value of N_{th} quickly increases. Such a dependence of N_{th} on $\lambda_{\parallel}$ is in a qualitative agreement with the beam behavior observed at NAP-M.¹²⁰

For high-frequency modes [for instance, $|n| \vartheta_0 \gg v_s / \max(\Delta, |u|)$], the stability diagram is given by Eqs. (3.2.19) and (3.2.20). For the beam with a Gaussian momentum distribution due to

$$\mathcal{P}(\omega) \simeq \frac{1}{w^2}, \quad w = \frac{\omega - n\omega_0}{n\Delta\omega_0}$$

$$J(\omega) = \sqrt{\frac{\pi}{2}} w e^{-w^2/2}$$

the width of the stability region exponentially shrinks, when $\zeta' \gg 1$:

$$\zeta' \simeq w^2, \quad \zeta'' \simeq \sqrt{\frac{\pi}{2}} (\zeta')^{5/2} \exp(-\zeta'/2).$$

As was found, in this region of harmonics together with $Z''_n/n = \text{const}$ holds $Z'_n/|n| = \text{const}$. This means that during the beam cooling the “working point” of the beam (ζ) can cross the border of the stability diagram. For a given combination Z''_n and Z'_n that occurs, when

$$\zeta' = 3 \ln \zeta' + B, \quad B = 2 \ln \left| \frac{Z''_n}{Z'_n} \right|.$$

This equation can be solved using successive iterations:

$$(\zeta')^{(0)} = B,$$

$$(\zeta')^{(k)} = 3 \ln [(\zeta')^{(k-1)}] + B, \quad k = 1, 2, \dots \quad (8.3.17)$$

Note that, if $|Z''_n/Z'_n| \gg 1$, Eqs. (8.3.17) give the root ζ' with an accuracy about 1% after 3 iterations.

8.4. DAMPING OF COHERENT OSCILLATIONS OF A BUNCH DUE TO A COOLING DEVICE

Without reflections from the collector the density distortions of the cooling electron beam can propagate along it with a continuous spectrum $\omega \simeq k_{\parallel} v_e$. As an electron beam is constantly renewed, such a system can introduce damping into the collective motion of the bunched cooled beam.¹³² Let us first analyze the case close to the condition of the experiment carried out at NAP-M. Therefore we shall assume that electrons pass the cooling region once, that the particle energies are nonrelativistic, and that the intensity of the electron beam is not too large. Namely, if n_e is the electron beam density, a_e is its radius, and W_e is the kinetic energy of electrons, we shall assume that the dimensionless parameter

$$\frac{n_e e^2 a_e^2}{W_e} \simeq \frac{\Delta U}{W_e} \ll 1.$$

The value ΔU in the numerator of the expression is of the order of the potential drop in the radius of the electron beam. Therefore, satisfaction of this condition does not pose excessive limits to the value n_e .

Since for the longwave length coherent oscillations the vacuum chamber is a waveguide below the cutoff frequency, the Green function of the cooling region is defined by Eq. (8.1.11). Then, with the adopted assumptions the Lagrangian of the interaction for a proton inside the electron beam can be written in the form

$$\overline{L_\omega(1)} = -e^2 \int d(2) \overline{G_\omega(1,2)} f_\omega(2), \quad (8.4.1)$$

where

$$\overline{G_\omega(1,2)} = -4\pi\omega_e^2 \int \frac{d^2 k k_\parallel}{(2\pi)^3 k_\perp^4} e^{ik[\mathbf{r}(1)-\mathbf{r}(2)]} \int_{-\infty}^{\infty} \frac{dv_e f'_e}{\omega - k_\parallel v_e}. \quad (8.4.2)$$

In the cases discussed below, the region $k_\parallel \ll k_\perp$ will always dominate. For this reason, we replace k^2 by $k_\perp^2$ in the denominator of Eq. (8.4.2).

Using Eqs. (8.4.1) and (8.4.2) let us rewrite Eq. (3.1.11) in the form ($m_x=0$):

$$\begin{aligned} \Delta\omega_m f_m = & Ne^2\omega_e^2 \left(m_z \frac{\partial f_0}{\partial I_z} \right) \int_{-\infty}^{\infty} dn J_{m_c}(n_1\varphi_c) \\ & \times \Lambda_{m,n} \int d(2) U_m(\mathbf{I}_1, \mathbf{I}_2) J_{m_c}(n_1\varphi_2) f_m(2), \end{aligned} \quad (8.4.3)$$

$$\begin{aligned} U_m = & \int \frac{d^2 k_\perp}{\pi k_\perp^4} b_m(\mathbf{k}, \mathbf{I}_1) b_m(\mathbf{k}, \mathbf{I}_2), \\ b_m(\mathbf{k}, \mathbf{I}) = & J_0(k_x a_x) J_{m_z}(k_z a_z), \end{aligned} \quad (8.4.4)$$

and

$$\Lambda_{m,n} = \int_{-\infty}^{\infty} \frac{dk_\parallel}{2\pi} k_\parallel |b(k_\parallel - k_n)|^2 \int \frac{dv_e f'_e}{\omega_{m,n} - k_\parallel v_e}, \quad (8.4.5)$$

$\omega_{m,n} = m_z \omega_z + n \omega_s$. Because of the evident single-turn character of interaction the summation over n in Eq. (8.4.3) is replaced by integration.

If the stationary distribution f can be factorized

$$f_0 = F_0(\mathbf{I}_\perp) \rho(\varphi_c),$$

to solve Eq. (8.4.3) one can use the substitution

$$f_m = \chi_1(I_\perp) \chi_2(\varphi_c),$$

when Eq. (8.4.3) splits into two independent equations

$$\lambda_m \chi_1 = -\frac{\partial F_0}{\partial a_z^2} \int d^2(a'_\perp)^2 U_m(\mathbf{I}, \mathbf{I}') \chi_1(\mathbf{I}'), \quad (8.4.6)$$

$$\Delta\omega_m \chi_2 = -\lambda_m \frac{2Ne^2\omega_e^2 \rho(\varphi_c)}{M\omega_z} \int_{-\infty}^{\infty} dn \Lambda_{m,n} \int_0^\infty d\varphi' \varphi' J_{m_c}(n_1\varphi_c) J_{m_c}(n_1\varphi'_c) \chi_2. \quad (8.4.7)$$

Since in the problem the induced fields can substantially vary on the transverse size of the proton beam, Eq. (8.4.6) cannot be solved by expanding its kernel in the power series of a_z and retaining only the leading term. We shall estimate the eigenvalues of this equation for the distribution $F_0(\mathbf{I})$ of special kind

$$F_0 = \frac{\delta(a_x^2)}{a_0^2} \exp\left(-\frac{a_z^2}{a_0^2}\right).$$

With such a distribution Eq. (8.4.6) reads

$$\lambda_m \chi_1 = \frac{e^{-a_z^2/a_0^2}}{a_0^4} \int_0^\infty \frac{dk}{k^3} J_{m_z}(ka_z) \int_0^\infty da'_z a'_z J_{m_z}(ka'_z) \chi_1.$$

Using the substitutions:

$$\chi_1 = \frac{w(z)e^{-z^2/2}}{\sqrt{z}}, \quad z = \frac{a_z}{a_0}, \quad x = ka_0,$$

we transform this equation in the integral equation with a symmetrical and positive kernel

$$\begin{aligned} \lambda_m w(z) &= \int_0^\infty dz' K(z, z') w(z'), \\ K(z, z') &= \sqrt{zz'} \exp\left(-\frac{z^2 + z'^2}{2}\right) \int_0^\infty \frac{dx}{x^3} J_{m_z}(xz) J_{m_z}(xz'). \end{aligned} \quad (8.4.8)$$

Although a direct solution of Eq. (8.4.8) is very difficult, its eigennumbers λ_m can be evaluated as maxima of the following integrals⁴²

$$\begin{aligned} \Lambda[w] &= \int_0^\infty dz dz' w(z) K(z, z') w(z), \quad \int_0^\infty dz w^2(z) = 1, \\ \lambda_m &= \max\{\Lambda[w]\}. \end{aligned}$$

In order to evaluate λ_m for a ground state modes we assume

$$w(z) = C_m z^{m_z+1/2} e^{-\beta z^2/2}, \quad C_m^2 = \frac{2\beta^{m_z+1}}{m_z!}.$$

Then, simple calculations result in

$$\Lambda(\beta) = \frac{C_m^2}{(1+\beta)^{2m_z+2}} \int_0^\infty dx x^{2m_z-3} \exp\left(-\frac{x^2}{1+\beta}\right).$$

For $m_z \geq 2$, this integral can be calculated easily:

$$\Lambda(\beta) = \frac{C_m^2}{2(1+\beta)^{m_z+3}} \int_0^\infty dt t^{m_z-2} e^{-t}$$

$$= \frac{1}{m_z(m_z+1)} \frac{\beta^{m_z+1}}{(1+\beta)^{m_z+3}}.$$

The function $\Lambda(\beta)$ has its maximum at $\beta = (m_z+1)/2$, which yields

$$\lambda_m \geq \frac{4(m_z+1)^{m_z}}{m_z(m_z+3)^{m_z+3}}, \quad m_z \geq 2.$$

For dipole modes ($|m_z| = \pm 1$) we take into account that the integration over k_1 in Eq. (8.4.2) must be limited by $k_1 a_0 \geq 1$. This results in

$$\lambda_{\pm 1} \geq -\frac{\text{Ei}(-1)}{16} \simeq 1.37 \times 10^{-2}.$$

Combining these expressions for λ_m we obtain

$$\lambda_m \simeq \begin{cases} \frac{4(m_z+1)^{m_z}}{m_z(m_z+3)^{m_z+3}}, & m_z \geq 2, \\ -\frac{\text{Ei}(-1)}{16}, & m_z = 1. \end{cases} \quad (8.4.9)$$

The calculation of the spectrum of Eq. (8.4.7) (and the spectrum of collective oscillations of protons) met some mathematical difficulties. Therefore, to estimate the qualitative dependence of decrements on the main parameters of the problem let us calculate the decrements for the simplest case, when all protons have the same amplitude of synchrotron oscillations:

$$\rho(\varphi_c) = \delta(\varphi_c^2 - \varphi_0^2), \quad (8.4.10)$$

where $l_b = 2\varphi_0 R_0$ is the proton bunch length. As already mentioned, the expressions obtained in such a way give the upper limit for the decrements of collective motion of a beam with a smooth distribution over the amplitudes of synchrotron oscillation. More accurate estimates of maximal decrements can be found by the extrapolation of the spectrum from the short wave region.

Substituting Eq. (8.4.10) into Eq. (8.4.7) we get the expression for the complex coherent frequency shift

$$\Delta\omega_m = -\lambda_m \frac{2Ne^2\omega_e^2 m_z}{M\omega_z} \int_{-\infty}^{\infty} dn \Lambda_{mn} J_{m_c}^2(n_1\varphi_0). \quad (8.4.11)$$

For the decrements of oscillations $\delta = -\text{Im}\omega$ this yields

$$\delta_m = \lambda_m \frac{2Ne^2\omega_e^2 m_z}{M\omega_z} \int_{-\infty}^{\infty} dn J_{m_c}^2(n_1\varphi_0) \text{Im}\Lambda_{mn}. \quad (8.4.12)$$

Let us first evaluate the decrements for the effect of fast damping of coherent oscillations ($\varphi_0 \rightarrow 0$). According to Eq. (8.4.12) the decrements of synchro-beta-tron oscillations equal zero and the decrements of betatron oscillations are defined by the formula

$$\delta_{m_z} = \lambda_m \frac{2Ne^2\omega_e^2 m_z}{M\omega_z} \int_{-\infty}^{\infty} dn \text{Im}(\Lambda_{mn}). \quad (8.4.13)$$

Taking into account Eq. (8.4.5) the last expression can be rewritten as

$$\delta_{m_z} = \lambda_m \frac{2Ne^2\omega_e^2 m_z}{M\omega_z} \int_{-\infty}^{\infty} \frac{dv_e}{v_e^2} f'_e \int_{-\infty}^{\infty} dn \omega_{mn} \left| b\left(\frac{\omega_{mn}}{v} - k_n\right) \right|^2. \quad (8.4.14)$$

With a decrease in the electron velocity spread ($\Delta_{\parallel}$) the integral in Eq. (8.4.14) diverges like $\Delta_{\parallel}^{-3}$. This result is caused by neglecting in Eqs. (8.4.1) and (8.4.2) the fact that the interaction of charges is indeed spread over the region with an azimuthal size of the order of the transverse size of proton beam $\Delta s \simeq k_{\perp}^{-1} \simeq a$. This is reasonable if for the time necessary to pass the cooling region the electron density fluctuation is spread more than the shielding radius

$$\Delta_{\parallel} \gg a \frac{v_e}{l}.$$

For qualitative analysis of the effect, such a limitation is not significant. The sign of decrement δ_{m_z} depends on the accuracy of the tuning of average velocities of the electron and proton beams. With the coincidence better than the width of electron distribution $|u_{\parallel}| = |v_s - v_e| < \Delta_{\parallel}$, the decrements of oscillations are positive and for a Gaussian distribution of electrons over the velocities read

$$\delta_m = \frac{Nr_p c^2 \omega_e^2}{\Delta_{\parallel}^3} \frac{l}{\Pi} m_z^2 \lambda_m \ln\left(\frac{\Delta_{\parallel} l}{v_s a}\right). \quad (8.4.15)$$

With a large detuning of the proton and electron velocities $|u_{\parallel}| \gg \Delta_{\parallel}$ coherent oscillations are unstable with the increment

$$\delta_m = -\frac{Nr_p c^2 \omega_e^2}{|u_{\parallel}|^3} \frac{l}{\Pi} m_z^2 \lambda_m. \quad (8.4.16)$$

This instability can be easily understood if the proton motion is considered in the rest frame reference system of electrons. In this system the protons are traveling in the longitudinal direction with a velocity $u_{\parallel}$. In this case, the instantaneous density of electrons is retuned in such a way that the proton interacts with the

positive trace, traveling in the same direction with the same velocity $u_{\parallel}$. On the contrary, if $u_{\parallel} \rightarrow 0$ electrons are attracted symmetrically and an energy of proton oscillations is dissipated for the generation of the negatively charged trace in the electron beam.

It is easy to see that the calculated dependencies of decrements of coherent oscillations on the relative velocities of the proton and electron beams coincide with the similar dependencies of decrements for cooling by the magnetized electron beam.^{117,121} Such a coincidence is because, on the one hand, the theorem on the sum of decrements of coherent modes is satisfied (see §3.4), and on the other hand, the circumstance that the main contribution to Eqs. (8.4.15) and (8.4.16) comes from high harmonic numbers.

Let us calculate now the decrements of coherent oscillations of the long beam. These are defined by Eq. (8.4.12), which can be rewritten here in the form

$$\begin{aligned} \delta_m = & -\lambda_m \frac{Nr_p c^2 \omega_e^2 m_z}{\pi^2 v_z} \int_{-\infty}^{\infty} \frac{du_{\parallel}}{v_e^2} \frac{df'_e}{du_{\parallel}} \int_0^1 d\xi (1-\xi) \\ & \times \int_{-\infty}^{\infty} dx J_{m_c}^2(x\varphi_0) (x - m_z \xi) \\ & \times \cos \left(2\vartheta_0 \xi \left[m_z v_z + \frac{u_{\parallel}}{v_s} (x - m_z \xi) \right] \right), \\ & \xi = \frac{dv_z}{d \ln \omega_s}. \end{aligned} \quad (8.4.17)$$

If $v_s = v_e$ and $f_e(u_{\parallel})$ is a Gaussian distribution, the calculation of decrements can be simplified in two cases.

For short bunches, $l_b \ll l\Delta_{\parallel}/v_s$, the integration in Eq. (8.4.17) yields

$$\delta_m = \frac{Nr_p c^2 \omega_e^2}{\Delta_{\parallel}^3} \frac{l}{\Pi} m_z^2 \lambda_m \begin{cases} \ln \left(\frac{\Delta_{\parallel} l}{v_s l_b} \right), & m_c = 0, \\ \frac{l_b v_s}{|m_c| l \Delta_{\parallel}}, & m_c \neq 0. \end{cases} \quad (8.4.18)$$

Otherwise ($l_b \gg l\Delta_{\parallel}/v_s$)

$$\delta_m = \frac{Nr_p c^2 \omega_e^2}{2\Delta_{\parallel}^2 v_s} \frac{l}{l_b} \frac{l}{\Pi} m_z^2 \lambda_m \ln \left(\frac{v_s l_b}{\Delta_{\parallel} l} \right). \quad (8.4.19)$$

The last equation is valid for both betatron and synchrobetatron modes. Since, usually, the ratio $v_s/\Delta_{\parallel}$ is large, the decrements of the bunched beam due to the interaction with the cooling beam can be high.

With a large difference of longitudinal velocities, $|u_{\parallel}| \gg \Delta_{\parallel}$, the interaction of beams leads to the instability of proton oscillations with the increment

$$\delta_m = -\frac{Nr_p c^2 \omega_e^2}{2|u_{\parallel}|^2 v_s} \frac{l}{l_b} \frac{l}{\Pi} m_z^2 \lambda_m. \quad (8.4.20)$$

Additional factors $(l\Delta_{\parallel}/l_b v_s)$ in Eqs. (8.4.19) and (8.4.20) as compared to Eqs. (8.4.15) and (8.4.18) appear for the following reasons. Effectively via an electron beam the protons interact with longitudinal distance smaller than $l\Delta_{\parallel}/v_s$, which corresponds to the interval of synchrotron phases $\Delta\psi_c \sim (l\Delta_{\parallel}/l_b v_s) \ll |m_c|^{-1}$. In this case, up to the multipolarities of the order of $|m_c| \sim (l_b v_s/l\Delta_{\parallel})$ all the synchrobetatron modes are damped similarly with no dependence of decrements on the number of multipolarity of synchrotron oscillations.

By the same reason, the contribution in Eqs. (8.4.18) through (8.4.20) from the head-tail effect is suppressed. Since the velocities of electrons and protons are very close, the perturbation in the electron beam moves relative protons with the velocity $u_{\parallel}$, which decreases corresponding contributions in δ_m at least by a factor $\Delta_{\parallel}/v_s$.

The phenomenon of fast damping of coherent oscillation of protons with the coincidence of average velocities $|u_{\parallel}| \ll \Delta_{\parallel}$ can be used for damping the coherent oscillation of bunched beams in the rings with electron cooling. Let us estimate the values of damping decrements for the antiproton storage ring NAP described in Ref. 139 ($R_0=750$ cm, $\nu=2-1.3$, $l=500$ cm, $v/c=0.85$, $\gamma=2$, $\Delta_{\parallel}/v_s=10^{-3}$, $n_e=10^8$ cm $^{-3}$). For the antiproton current corresponding to $N \sim 10^{10}$ and the bunch length $l_b=\Pi/3$ Eq. (8.4.19) gives the damping time $\tau_k=10^{-3}$ s. With the worse accuracy of tuning of the average velocities of beams the damping time increases according Eq. (8.4.15).

Let us remind the reader that the possibility to describe coherent oscillations by betatron and synchrobetatron modes assumes the sufficiently slow development of coherent oscillations $\delta_m \ll \omega_c$. This yields additional limitations on the decrements given by Eqs. (8.4.12) through (8.4.20). The inverse case $\delta_m \gg \omega_c$ corresponding to the so-called fast oscillations, for the single-turn interaction requires an additional analysis for the stability of the beam coherent motion. Using results of §5.3, in this case, we may expect the strong influence of the space charge of the cooled beam on the stability of its coherent oscillations.

The discussed scheme with a single pass of electrons through the cooling region will not fit cooling beams with relativistic or ultrarelativistic energies.¹²⁴ In this case, from the energetic point of view it is more convenient to use the electron beams from the special storage ring. As it seems unrealistic to provide the damping of coherent oscillations in an electron beam for one turn, the interaction between the cooling and cooled beams becomes resonant. The character of development of coherent oscillations in such a system was covered in Chapter 4. An evident difference in this case consists in the reduction of the interaction by a factor of γ^{-2} due to coincidences of the average velocities of

beams. Therefore, one can hope that coherent interaction of beams in this range of energies will not give the noticeable limitations on the beam currents.

8.5. A KINETIC EQUATION FOR COULOMB RELAXATION IN A COOLED BEAM

Collective effects in cold beams can strongly modify their kinetic in the storage ring with electron cooling. Up to now we have discussed problems when the time to reach the stationary level of coherent fluctuations in a beam was small compared to the time of particles relaxation (or cooling time). Due to this circumstance the collision integrals in the corresponding kinetic equations could be expressed using only the single-particle distribution function, while the processes of the relaxation of fluctuations and particles were independent. Below we will consider problems when the kinetic effects are to a large extent affected by the interaction of particles and the long living coherent fluctuations of the beam. Under the conditions where the lifetimes of such fluctuations become comparable with the beam relaxation time, the fluctuations can participate in the energy exchange with particles. In this case, the equilibrium in the beam is achieved in the process of the common evolution of the distribution of particles and coherent fluctuations. Therefore, to describe such a process one needs two kinetic equations: one for f_0 and another for relaxation of spectral densities of fluctuations, as if the relaxation of a two-component system is described.^{16,127}

If the interaction of particles in the beam does not lead to the coherent instabilities, the number of oscillation modes N_{ex} participating in such a process can be large. This brings strong correlations in the motion of beam particles and, thus, the substantial ordering in their distribution along the orbit. As the ultimate possibility, after sufficient cooling and the enhancement of such an ordering the beam can transit into its crystalline state.

For the sake of simplicity, let us confine ourselves to the study of the relaxation of the energy distribution in the coasting beam. In §7.2 and §8.1 it was shown that the use of electron cooling, at least in principle, enables one to obtain in storage rings the beams of heavy particles with extraordinarily low longitudinal and transverse temperatures. Its practical possibility was demonstrated by experimental results reported in Ref. 91, 120, and 129. On the other hand, as seen from the results discussed in §2.5, cooling the beam increases the contribution to the kinetics from IBS. Therefore, if, for example, the beam transverse temperature is larger than the longitudinal one (as in the experimental conditions at NAP-M), one should observe its increase with the growth of the beam current.

The experimental study of the dependence of the longitudinal temperature $T_{\parallel}$ on the beam current was carried out in Ref. 99. (See also §7.2.) It turned out that this dependence is of the threshold behavior. At beam currents lower than $I_{th} = 10 \mu\text{A}$ the longitudinal temperature $T_{\parallel}$ did not depend on N and it was approximately equal to 1 K. At beam currents larger than I_{th} an increase in

$T_{\parallel}$ with the growth of the beam current (N) was observed. Such a behavior of $T_{\parallel}(N)$ contradicts predictions using the IBS theory.

Note that the calculations in §2.6 were based on assumptions of the dominant role of fast collisions. Therefore, the given contradiction with experimental data is not very surprising. With the decrease in temperature, such an assumption can be invalidated for least two reasons.

First, since the particles collide in the focusing fields, the number of collisions grows with the collision time τ_c exceeding the periods of betatron oscillation of particles $2\pi/\omega_{\perp}$ (i.e., the adiabatic collisions). The contribution of such collisions ($\tau_c \omega_{\perp} \gg 1$) into the transverse longitudinal relaxation is suppressed.

Second, due to the long-range Coulomb interaction of particles, with a decrease in beam temperature the contributions of collective effects in the beam kinetics increase. Since the Coulomb tune shift under these conditions is large ($\Delta\nu_{\perp} \sim N/a^2$, a is the beam radius) the transverse coherent oscillations decay quickly. The ordering in the beam in this case can be caused by the strong instability of the beam caused by the Coulomb interaction. On the contrary, the damping time of longitudinal oscillations is inversely proportional to the revolution frequency spread and therefore it grows with the decrease in $T_{\parallel}$. In a cooled beam this time might be sufficient to provide the heat exchange between particles and longitudinal coherent fluctuations and associated ordering of the particles along the orbit into the chain without transversal ordering. It is obvious that the first case corresponds to extremely high beam currents, while the second corresponds to rather low beam currents. Respectively, one may expect the difference in the phase transition into the crystalline states in currents of high and low intensity.

Both these circumstances were ignored when obtaining equations in §2.5 and §2.6, but they should be taken into account in the kinetic equation appropriate for the description of the Coulomb relaxation in the cooled beam. Having in mind the comparison with the experimental results,⁹⁹ we shall calculate the coasting beam assuming the particle energy to be nonrelativistic (and lower than the transition energy of the ring).

The kinetic equation for the momentum distribution function

$$f_0(\Delta p, t) = \int d\Gamma_{\perp} f_0(\mathbf{I}_{\perp}, \Delta p, t),$$

can be obtained by the integration of the Eq. (2.4.27) over the transverse phase space variables ($\mathbf{I}_{\perp}, \psi_{\perp}$):

$$\frac{\partial f_0}{\partial t} = \text{St} = -\frac{\partial}{\partial p} \left\{ \sum_n \sum_{\mathbf{m}_{\perp}} \frac{1}{N} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K_{mn}(\omega) \right\}, \quad (8.5.1)$$

where

$$K_{mn}(\omega) = \lim_{\Delta \rightarrow 0} 2\Delta \int d\Gamma_1 < e\delta E_{m_1, n}^*(\omega + i\Delta) \delta F_{m_1, n}(\omega + i\Delta) >. \quad (8.5.2)$$

For further simplification let us split the right-hand side of Eq. (8.5.1) into two terms:

$$St = St_f + St_A, \quad St_f = St(m_1 \neq 0), \quad St_A = St(m_1 = 0).$$

This enables us to separate of two physically different processes. The first term

$$St_f = -\frac{1}{N} \sum_n \sum_{m_1 \neq 0} (...)$$

describes the variation of f_0 due to particle collisions with the energy transfer from the transverse oscillations to the longitudinal motion (and vice versa). The collision integral can noticeably contribute to the relaxation of f_0 provided the main parts of collisions are the fast ones. If the impact parameter of a collision is b and the relative velocity of particles is u , then St_f is contributed from the region

$$\tau_c \omega_1 = \frac{b\omega_1}{u} \ll 1. \quad (8.5.3)$$

The satisfaction of the condition for a larger part of collisions in the beam, enables the calculation of St_f in the logarithmic approximation resulting in the Landau collision integral.

In terms of beam fluctuations this equivalent to the contribution of coherent modes from the spectrum of δF satisfies the condition $\delta_{mn}\tau_{rel} \gg 1$, when the beam noise sharply achieves its stationary value corresponding to the instantaneous distribution $f_0(\Delta p, t)$. The substitution into Eq. (8.5.1) of the corresponding spectral densities gives directly the Landau collision integral (or more general integral of Lenard-Balescu). For the cooled beam with small transverse dimensions the condition $\delta_{mn}\tau_{rel} \gg 1$ can actually be satisfied starting from $m_1^2 = 1$ due to large frequency spread of transverse oscillations caused by the the Coulomb tune shift of betatron oscillations.

The second term

$$St_A = -\frac{1}{N} \frac{\partial}{\partial p} \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K_{m_1=0, n}(\omega) \quad (8.5.4)$$

describes the collisions of the particles, which are spread in the plane of betatron oscillations over the area of the order of the beam radius. For such collisions the condition (8.5.3) should be strongly violated $b\omega_1/u \gg 1$ (i.e., the collisions should be adiabatic with respect to the betatron oscillations of particles). By the analogy with the problem of the interaction of magnetized electrons, here one can discuss the interaction between the charged discs. Such a way is the general

method for describing kinetics of adiabatic collisions.^{24,121} It is evident that the number of such particle-discs is the same as the number of particles in the beam N and their temperature coincides with the beam longitudinal temperature $T_{\parallel}$. Due to the lattice focusing, these particle-discs have no transverse mobility. Therefore, in such a beam the coherent oscillations can only be longitudinal and the dynamics of fluctuations are described by Eq. (7.2.1).

The contribution of the noise stationary part into St_A was already discussed in §7.5. It could be caused by two factors: the nonconservative interaction and the presence in the wake fields of harmonics with $|n| > \omega_s/\Delta\omega_0$. Both these factors are not characteristic for the conditions under consideration.

The contribution of the nonstationary part of the beam noise can be caused by the presence in spectrum of δF of modes satisfying

$$\delta_n \tau_{\text{rel}} \leq 1. \quad (8.5.5)$$

With satisfaction of condition (8.5.5) the coherent oscillations can act on the beam particles for a long time being absorbed by those particles due to Landau damping. The thermal motion of particles in the beam excites new oscillations. The competition of these processes causes the evolution of f_0 . The mean number of modes N_{ex} participating in this heat exchange is determined by criterion (8.5.5) and, in general, it grows with a decrease in the beam longitudinal temperature.

Thus, in the adiabatic limit the problem of the relaxation of the cold beam toward the equilibrium is reduced to the common solution of equations

$$\frac{\partial f_0}{\partial t} = -\frac{1}{N} \frac{\partial}{\partial \Delta p} \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \lim_{\Delta \rightarrow 0} 2\Delta \langle e\delta E_n^*(\omega + i\Delta) \delta F_n(\omega + i\Delta) \rangle, \quad (8.5.6)$$

$$\left(\frac{\partial}{\partial t} + i\omega_n \right) \delta F_n = -Ne\delta E_n \frac{\partial f_0}{\partial \Delta p}, \quad (8.5.7)$$

with the additional condition (8.5.5). The right-hand side of Eq. (8.5.6) contains f_0 and the spectral density of the field fluctuations $(\delta E_n)_{\omega}^2$, which both vary during the relaxation. Therefore, an additional kinetic equation is required to describe the evolution of $(\delta E_n)_{\omega}^2$.

8.6. KINETICS OF BEAM FLUCTUATIONS

Let us now obtain the kinetic equation for

$$(\delta E_n)_{\omega}^2 = \lim_{\Delta \rightarrow 0} 2\Delta \langle \delta E_n^*(\omega + i\Delta) \delta E_n(\omega + i\Delta) \rangle.$$

As before, we adopt that the Fourier amplitudes $\delta E_{n,\omega}$ and $\delta F_{n,\omega}$ are related through the coupling impedance

$$\delta E_{n,\omega} = -\frac{e\omega_s Z_n}{\Pi} \int d\Delta p \delta F_{n,\omega}.$$

For the Coulomb interaction of nonrelativistic particles Z_n is defined by

$$\left(\frac{-iZ_n}{n}\right) \approx \frac{4}{v_s} \begin{cases} \frac{1}{2} \ln(2n_0/n), & |n| < n_0 = R_0/a, \\ (n_0/n)^2, & |n| > n_0. \end{cases} \quad (8.6.1)$$

The Fourier transformation over time in Eq. (8.5.7) yields the formal solution of this equation in the form

$$\delta F_{n\omega}(t) = \frac{i\delta F_n(\Delta p, t)}{\omega - \omega_n} - \int_0^\infty d\tau \frac{iNe\delta E_n(t+\tau)e^{i\omega\tau}}{\omega - \omega_n} \frac{\partial f_0(t+\tau)}{\partial \Delta p}. \quad (8.6.2)$$

As already mentioned, the initial time t in this equation is to a large extent arbitrary. Therefore, $\delta F_n(\omega)$ generally is the function of the initial moment t , which is displayed in Eq. (8.6.2). The initial level of the beam noise is determined by an instantaneous distribution f_0 and so varies during the relaxation.

Equation (8.6.2) follows the equation for $\delta E_n(t)$:

$$\int_0^\infty d\tau e^{i\omega\tau} \delta E_n(t+\tau) \epsilon_n(\omega, t+\tau) = \frac{-ie\omega_s Z_n}{\Pi} \int_{-\infty}^\infty d\Delta p \frac{\delta F_n(\Delta p, t)}{\omega - \omega_n + i\Delta}, \quad (8.6.3)$$

where

$$\epsilon_n(\omega, t) = 1 + \frac{\Omega_n^2}{n\omega'_0} \int_{-\infty}^\infty d\Delta p \frac{\partial f_0(\Delta p, t)/\partial \Delta p}{\omega - \omega_n + i\Delta}$$

is an instantaneous value of dielectric constant of a beam [see Eq. (2.6.40)].

The solution of Eq. (8.6.3) for the stationary condition $\partial\epsilon/\partial t = 0$ was already discussed in §6.5:

$$\delta E_{n\omega}(t) = \frac{-ie\omega_s Z_n}{\Pi\epsilon(n, \omega)} \int_{-\infty}^\infty d\Delta p \frac{\delta F_n(\Delta p, t)}{\omega - \omega_n + i\Delta}. \quad (8.6.4)$$

In this case, the spectrum of the beam coherent modes is defined by the dispersion equation $\epsilon(n, \omega) = 0$. If the interaction of particles does not lead to the collective instabilities, then the slowly damping oscillations correspond to the region where $(n\Delta\omega_0)^2 \ll \Omega_n^2$. For these harmonics the roots of the dispersion equation read

$$\omega - \omega_n = \Delta\omega_n + \frac{i\epsilon''_n}{(\partial\epsilon'_n/\partial\omega)} \Big|_{\Delta\omega_n},$$

where $\Delta\omega_n \simeq \pm \Omega_n$ are the roots of the equation

$$\epsilon'(n, \omega) = 0. \quad (8.6.5)$$

With the evolution of f_0 the dependence of δE_n on time is the combination of the fast oscillations with frequencies $\omega_n = n\omega_s + \Delta\omega_n$ and of the slow (on times of the order of τ_{rel}) variations of amplitudes of these oscillations. The presence in the problem of the strongly different time scales enables the simplification of Eq. (8.6.3) using the averaging method.

For the slowly decaying modes one may indeed expect that $(\delta E_n)_\omega^2$ noticeably differs from zero only in the vicinity of roots of Eq. (8.6.5). According to this we can write

$$\delta E_n(t + \tau) = A_n(t + \tau) e^{-i\Omega_n \tau}. \quad (8.6.6)$$

Then the integration in the left-hand side Eq. (8.6.3) by parts yields

$$\frac{i\epsilon_n(\omega, t)A_n(t)}{\Delta\omega_n - \Omega_n} + i \int_0^\infty \frac{d\tau e^{i(\Delta\omega_n - \Omega_n)\tau}}{\Delta\omega_n - \Omega_n} \frac{d}{d\tau} \{\epsilon_n(\omega, t + \tau)A_n(t + \tau)\}. \quad (8.6.7)$$

Neglecting the noise nonstationarity the first term in Eq. (8.6.7) can be presented in the form

$$\epsilon_n(\omega, t)\delta E_n(\omega, t).$$

As to the second term, it contributes substantially to Eq. (8.6.7) only in the vicinity of $\Delta\omega_n = \Omega_n$. Expanding $\epsilon(n, \omega)$ in this region in the power series of $\Delta\omega_n - \Omega_n$ and taking the leading term of the expansion

$$\epsilon(n, \omega) = \frac{\partial \epsilon'_n}{\partial \omega} (\Delta\omega_n - \Omega_n),$$

instead of Eq. (8.6.7) we obtain

$$\epsilon_n(\omega, t)\delta E_n(\omega, t) + i \frac{\partial}{\partial t} \left\{ \frac{\partial \epsilon'_n}{\partial \omega} \delta E_n(\omega) \right\}.$$

Substituting this expression into Eq. (8.6.3) we get

$$\frac{\partial}{\partial t} \left\{ \frac{\partial \epsilon'_n}{\partial \omega} \delta E_n(\omega) \right\} - i\epsilon_n \delta E_{n, \omega} = \frac{-ie\omega_s Z_n}{\Pi} \int_{-\infty}^{\infty} \frac{d\Delta p \delta F_n(\Delta p, t)}{\omega - \omega_n + i\Delta}. \quad (8.6.8)$$

The equation for the spectral density $(\delta E_n)_\omega^2$ can be obtained by multiplying Eq. (8.6.8) by $2\Delta \delta E_{n, \omega + i\Delta}^*$ with further averaging over fluctuations. Since Eq. (8.6.8) itself is valid in the first approximation over the nonstationary effects, we can use as $\delta E_{n, \omega}^*$ an expression defined by Eq. (8.6.4). This results in

$$\frac{\partial}{\partial t}(\delta E_n)_\omega^2 + 2\delta_n(\omega)(\delta E_n)_\omega^2 = \left(\frac{\partial \epsilon'_n}{\partial \omega}\right)^{-1} \operatorname{Re} \left[\frac{-2i\Delta}{\epsilon_n^*(\omega + i\Delta)} \right] \left\langle \left| \frac{e\omega_s Z_n}{\Pi} \right|^2 \right\rangle \times \int_{-\infty}^{\infty} d\Delta p \frac{\delta F_n(\Delta p, t)}{|\omega - \omega_n + i\Delta|} \quad (8.6.9)$$

Here

$$\delta_n(\omega) = -\frac{\epsilon_n''(\omega)}{\partial \epsilon'_n / \partial \omega} + \frac{\partial}{\partial t} \ln \left(\frac{\partial \epsilon'_n}{\partial \omega} \right) \quad (8.6.10)$$

is the decrement of the oscillation with $\omega \simeq n\omega_s$, which also takes into account the adiabatic variation of the amplitudes $\delta E_{n,\omega}$ with the slow variation of $\epsilon(n, \omega)$. When calculating the adiabatic correction, one, of course, has to have in mind that Eq. (8.6.9) was obtained on the lowest level of the perturbation theory. In this approximation the adiabatic damping can be due to changes in Ω_n^2 or $\omega'_0(p)$, but not in $f_0(\Delta p, t)$.

For slowly decaying modes the right-hand side in Eq. (8.6.9) is noticeably different from zero only in the vicinity of $\epsilon'_n = 0$. Then

$$\operatorname{Re} \left[\frac{-2i}{\epsilon_n^*} \right] = \frac{2\Delta \partial \epsilon'_n / \partial \omega}{(\epsilon'_n)^2 + \Delta^2 (\partial \epsilon'_n / \partial \omega)^2} = 2\pi \operatorname{sgn} \left(\frac{\partial \epsilon'_n}{\partial \omega} \right) \delta[\epsilon'_n(\omega)].$$

An averaging over fluctuations with the use of Eq. (2.5.24) yields

$$\frac{\partial}{\partial t}(\delta E_n)_\omega^2 + 2\delta_n(\omega)(\delta E_n)_\omega^2 = \frac{\delta[\epsilon'_n(\omega)]}{|\partial \epsilon'_n / \partial \omega|} \left| \frac{e\omega_s Z_n}{R_0} \right|^2 \times \int_{-\infty}^{\infty} d\Delta p \delta(\omega - \omega_n) f_0(\Delta p, t). \quad (8.6.11)$$

The factor $\delta[\epsilon'_n(\omega)]$ in the right-hand side of Eq. (8.6.11) describes the radiation of collective oscillations by particles. A decrease in energy of these oscillations, as can be seen from Eq. (8.6.11), is caused by their absorption in the beam due to Landau damping. The competition of these processes causes the evolution of $(\delta E_n)_\omega^2$ to the stationary level

$$(\delta E_n)_{\omega, st}^2 = \left| \frac{e\omega_s Z_n}{\Pi} \right|^2 (\delta \rho_n)_{\omega, st}^2, \quad (8.6.12)$$

where

$$(\delta \rho_n)_{\omega, st}^2 = \frac{2\pi^2 \delta[\epsilon'_n(\omega)]}{|\partial \epsilon'_n / \partial \omega| \delta_n} \int_{-\infty}^{\infty} d\Delta p \delta(\omega - \omega_n) f_0(\Delta p, t). \quad (8.6.13)$$

Since under the stationary conditions $(\partial \epsilon_n / \partial t) = 0$, this result coincides with Eq. (7.2) with a Gaussian distribution f_0 .

8.7. MUTUAL RELAXATION OF PARTICLES AND WAVES

The kinetic equation (8.6.11) holds whether or not the energy exchange between the particles and waves is taken into account. But, if condition (8.5.5) is satisfied, the right-hand side of Eq. (8.6.11) generally varies on the same times as $(\delta E_n)_\omega^2$. Therefore, the total description of the processes requires the common solution of Eqs. (8.5.6) and (8.6.11).

To calculate the right-hand side of Eq. (8.5.6), we must solve Eq. (8.6.2). Assuming that the amplitudes $\delta F_{n,\omega}(t)$ are the slowly varying functions of time, and repeating in Eq. (8.6.2) the calculations leading from Eq. (8.6.3) to Eq. (8.6.8) one can write

$$\delta F_{n\omega}(t) = \frac{i\delta F_n}{\omega - \omega_n} - \frac{ine\delta E_{n\omega}}{\omega - \omega_n} \frac{\partial f_0}{\partial \Delta p} + \frac{\partial}{\partial t} \left\{ Ne\delta E_{n\omega} \frac{\partial}{\partial \omega} \frac{1}{\omega - \omega_n} \frac{\partial f_0}{\partial \Delta p} \right\}. \quad (8.7.1)$$

The first two terms in this expression coincide with that in Eq. (7.2.3). An additional third term gives the correction of the first approximation due to effects of the nonstationary beam noise. Substituting Eq. (8.7.1) into Eq. (8.5.6) enables one to calculate the collision integral directly. The obtained expression is quite cumbersome and inconvenient to analyze. Moreover, the main features of the relaxation of the beam—especially near the equilibrium—can be studied with the simpler Langevin equation. This possibility is related to the fact that the equilibrium distribution of Eq. (8.5.6) is the Gaussian one.

Multiplying Eq. (8.5.6) by Δp^2 and integrating this equation over the momenta, we obtain

$$\frac{d\sigma_{\parallel}^2}{dt} = \frac{2}{N} \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \lim_{\Delta \rightarrow 0} 2\Delta \left\langle e\delta E_{n,\omega+i\Delta}^* \int_{-\infty}^{\infty} d\Delta p \Delta p \delta F_{n,\omega+i\Delta} \right\rangle. \quad (8.7.2)$$

Substituting $\delta F_{n,\omega}$ from Eq. (8.7.1) and using the definition of ϵ_n , one can rewrite Eq. (8.7.2) in the form

$$\begin{aligned} \frac{d\sigma_{\parallel}^2}{dt} = & \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \left\{ \frac{1}{N} \int d\Delta p \Delta p \operatorname{Re} \left\langle \lim_{\Delta \rightarrow 0} 2i\Delta \frac{e\delta E_{n,\omega+i\Delta}^*}{\omega - \omega_n + i\Delta} \delta F_n(\Delta p, t) \right\rangle \right. \\ & + \frac{\Delta \omega_n \epsilon'_n(\omega)}{\Omega_n^2} e^2 (\delta E_n)_\omega^2 + \frac{1}{2} \frac{\partial}{\partial t} [(e\delta E_n)_\omega^2] \frac{\partial}{\partial \omega} \left(\frac{\Delta \omega_n}{\Omega_n^2} [\epsilon'_n - 1] \right) \\ & \left. + (e\delta E_n)_\omega^2 \frac{\partial}{\partial \omega} \left(\frac{\Delta \omega_n}{\Omega_n^2} \frac{\partial \epsilon_n}{\partial t} \right) \right\}, \quad (8.7.3) \end{aligned}$$

where $\Delta\omega_n = \omega - \omega_n$. The first term in the curly brackets of Eq. (8.7.3) except for the factor

$$-e^2 \frac{\Delta\omega_n}{2\Omega_n^2} \frac{\partial \epsilon'_n}{\partial \omega}$$

coincides with the right-hand side of Eq. (8.6.9). Therefore, it can be rewritten in the following form:

$$-e^2 \frac{\Delta\omega_n}{\Omega_n^2} \left\{ \frac{1}{2} \frac{\partial \epsilon'_n}{\partial \omega} \frac{\partial}{\partial t} (\delta E_n)_\omega^2 + (\delta E_n)_\omega^2 \left(\epsilon''_n + \frac{\partial^2 \epsilon'_n}{\partial t \partial \omega} \right) \right\}.$$

Substituting this expression into Eq. (8.7.3) reduces the terms that are proportional to $\Delta\omega_n$. This results in

$$\frac{d\sigma_{\parallel}^2}{dt} = e^2 \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \left\{ \frac{\epsilon'_n - 1}{2\Omega_n^2} \frac{\partial}{\partial t} (\delta E_n)_\omega^2 + \frac{\partial \epsilon'_n}{\partial t} \frac{(\delta E_n)_\omega^2}{\Omega_n^2} \right\}. \quad (8.7.4)$$

In the vicinity of the equilibrium one has $(\delta E_n)_\omega^2 \propto \delta[\epsilon'_n(\omega)]$, therefore

$$\frac{d\sigma_{\parallel}^2}{dt} = - \sum_n \frac{e^2}{\Omega_n^2} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\delta E_n)_\omega^2. \quad (8.7.5)$$

Thus, in the adiabatic approximation ($|\Omega_n^2| \gg \delta_n$, $\tau_c \omega_1 \gg 1$, and $\delta_n \tau_{rel} \sim 1$) discussed here, the variation of the beam longitudinal temperature $T_{\parallel} = R_0 \omega'_0 \sigma_{\parallel}^2$ is completely caused by the nonstationary collective fluctuations of the induced field.

The kinetics of the relaxation of the beam to the equilibrium are described by the system of Eqs. (8.6.11) and (8.7.5). To further simplify considerations, let us introduce the temperature of fluctuations (or collective modes):

$$T_n = \frac{R_0^2}{n^2 \omega_s} \operatorname{Re} \left[\frac{n}{-iZ_n} \right] \int_{-\infty}^{\infty} d\omega (\delta E_n)_\omega^2. \quad (8.7.6)$$

Then, for states close the equilibrium one ($f'_0 \simeq -\Delta p f_0 / \sigma_{\parallel}$), Eqs. (8.6.11) and (8.7.5) yield the system of equations for $T_{\parallel}$ and for temperatures of fluctuations T_n :

$$\begin{aligned} \frac{dT_{\parallel}}{dt} &= \sum_n \frac{2\delta_n}{N} (T_n - T_{\parallel}), \\ \frac{dT_n}{dt} &= -\delta_n (T_n - T_{\parallel}). \end{aligned} \quad (8.7.7)$$

As it expected, Eqs. (8.7.7) have the form of the system of relaxation equations for the multicomponent system beam + fluctuation of field. An average number

of modes participating in such a relaxation is determined by condition (8.5.5) and, generally, depends on temperature $T_{\parallel}$. These equations indicate that the spread of the energy along the spectrum of coherent oscillations is caused by the reradiation of oscillations by particles.

From Eq. (8.7.7) one can see that during relaxation the total energy of the system is conserved

$$\mathcal{E} = \frac{NT_{\parallel}}{2} + \sum_n \frac{T_n}{2}. \quad (8.7.8)$$

This means that system of equations (8.7.7) describes the equalizing temperatures of particles and coherent fluctuations. In the equilibrium ($\dot{T}_{\parallel} = 0$, $\dot{T}_n = 0$) the temperatures being equal,

$$T_{\parallel} = T_n. \quad (8.7.9)$$

The value of the equilibrium temperature T can be calculated using initial conditions of the problem and the conservation law Eq. (8.7.8):

$$\mathcal{E} = \frac{N + N_{ex}}{2} T. \quad (8.7.10)$$

Equation (8.7.8) can be written in the more familiar form if by the definition of T_n we rewrite it as

$$\mathcal{E} = \frac{NT_{\parallel}}{2} + \frac{1}{2} \sum_n \frac{R_0^2}{n^2 \omega_s} \frac{n}{Z_n''} \int_{-\infty}^{\infty} d\omega (\delta E_n)_{\omega}^2. \quad (8.7.11)$$

Using here the relation $\delta E_{n\omega}$ and $\delta \rho_{n\omega}$ we can rewrite the second term in Eq. (8.7.11) in the form

$$\frac{1}{2} \sum_n \frac{e^2 \omega_s}{2\pi} \frac{Z_n''}{n} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\delta \rho_n)_{\omega}^2 = \frac{1}{2} \sum_{a,b=1}^N \langle \mathcal{U}_{a,b} \rangle, \quad (8.7.12)$$

where

$$\mathcal{U}_{a,b} = \sum_n \frac{e^2 \omega_s}{2\pi} \frac{Z_n''}{n} e^{in(\vartheta_a - \vartheta_b)} \quad (8.7.13)$$

is the energy of interactions of particle-discs a and b in adiabatic approximation. Thus we can write

$$\mathcal{E} = \frac{NT_{\parallel}}{2} + \frac{1}{2} \sum_{a,b=1}^N \langle \mathcal{U}_{a,b} \rangle,$$

which explains the physical sense of the second term in Eq. (8.7.8)—the average potential energy of particle interaction through the surrounding system.

Returning to system (8.7.7) one can mention that the "cooling" decrements of particles $\lambda = -\dot{T}_{\parallel}/T_{\parallel}$ are related to the decrements of coherent fluctuations by

$$N\lambda = \sum_n \delta(\Omega_n), \quad (8.7.14)$$

showing the content of the theorem on the sum of decrements of coherent modes. (See §3.4.) In §3.4 this theorem was proved for the beam with a discrete spectrum of eigenfrequencies. Eq. (8.7.14) states the similar relation taking into account modes with Landau damping, which satisfy condition (8.5.5).

Let us estimate the relaxation time to the equilibrium in the cool beam. The decrements of coherent oscillations are defined by Eq. (8.6.10). In addition, for the long-living modes $[\Omega_n^2 \gg (n\Delta\omega_0)^2]$ we can take $\partial\epsilon'_n/\partial\omega \simeq 2/\Omega_n$. For a Gaussian momentum distribution this yields

$$\lambda = \frac{(\pi/2)^{1/2}}{N} \sum_{n=1}^{n_{ex}} \frac{\Omega_n^4}{(n\Delta\omega_0)^3} \exp\left[-\frac{\Omega_n^2}{2n^2\Delta\omega_0^2}\right], \quad (8.7.15)$$

where $n_{ex} = N_{ex}/2$ is given by condition (8.5.5). Let us say that the beam longitudinal temperature is so low that $n_{ex} \gg n_0$. Then λ can be calculated by substituting the summation over n in Eq. (8.7.15) by the integration and due to the fast convergency of the integral one can take $n_{ex} \rightarrow \infty$. Taking into account Eqs. (7.1.14) and (8.6.1) this integration yields

$$\lambda = \frac{\Delta\omega_0}{N} \left(\frac{\pi}{2}\right)^{1/2} \left\{ \frac{n_0^2}{2} \left(\frac{N}{N_c}\right)^2 \exp\left(-\frac{N}{2N_c}\right) + n_D^2 \left[1 - \exp\left(-\frac{n_D^2}{2n_0^2}\right) \right] \right\}, \quad (8.7.16)$$

with parameter n_D defined by Eq. (7.2)

$$n_D^2 = \frac{\Omega_n^2}{\Delta\omega_0^2} \Big|_{n \gg n_0} = \frac{2Ne^2R_0}{\pi a^2 T_{\parallel}} \simeq n_0^2 \frac{N}{N_c} \quad (8.7.17)$$

and related to the radius of the Debye shielding $r_D = R_0/n_D$ in the beam of the disc-particles. For rather low longitudinal temperatures one has $N \gg N_c$ and $n_D \gg n_0$ respectively. Under these conditions Eq. (8.7.16) reads

$$\lambda \simeq \frac{n_D^2 \Delta\omega_0}{N} \simeq \frac{e^2}{a^2 \sigma_{\parallel}}. \quad (8.7.18)$$

The first part of this formula is interesting by the formal coincidence with Eq. (7.1.12) for the asymptotic value of the cooling decrement in the method of stochastic cooling. Such a coincidence is not purely formal, of course. In both cases the result is due to the interaction of particles with the beam nonstationary noise.

Using Eqs. (7.1.12) and (8.7.18) one can compare directly the cooling efficiency of the intense beam in the methods of electron and stochastic cooling. In stochastic cooling the particle interaction radius is related to the value $W \sim n_0$, which only depends on the feedback circuit characteristics and does not vary during cooling. On the contrary, at the Coulomb interaction of particles a similar role is played by parameter n_D , which grows $\propto (a\Delta\omega_0)^{-1}$ with the beam cooling. Such a difference is due to the substantial difference in the space scale characteristics for both methods. With the microscopically small dimension of the interaction region, as it is realized in the electron cooling method, the region $n_D \gg n_0$ becomes accessible, where the decrements of electron cooling can significantly exceed the stochastic cooling decrements.

Note finally that Eq. (8.7.18) corresponds to the friction force

$$|\mathcal{F}_{\parallel}| \simeq \frac{e^2}{a^2}, \quad (8.7.19)$$

which is not dependent on the particle velocity. As it is known, such a behavior of $\mathcal{F}$ is specific for the so-called "dry" friction.

In cases of practical interest the values of decrements are not small in general. So, for the parameters⁹⁹ of NAP-M $a=0.01$ cm (with beam current $I_p < 10 \mu A$), Eq. (8.7.18) yields $\lambda = 10^5 \text{ s}^{-1}$.

To conclude this section let us estimate the number of oscillations participating in relaxation. Using Eqs. (8.7.15) and (8.7.18) we can write

$$e^{x^2/2} \gg x^3 \frac{N}{n_D}, \quad x = \frac{n_D}{n}, \quad n > n_0. \quad (8.7.20)$$

If $N \gg n_D$, the value of N_{ex} is close to $2n_D$:

$$N_{ex} \simeq 2n_D / \sqrt{2 \ln N / n_D}. \quad (8.7.21)$$

In the inverse ultimate case $N \ll n_D$ the inequality (8.7.20) is valid for all x . In this case, the use of Eq. (8.7.20) to find N_{ex} is not appropriate. Using the thermodynamic considerations one can find that in this region N_{ex} does not exceed N .

8.8. BEAM HEATING BY OFF-EQUILIBRIUM NOISE

From Eq. (8.7.7) one can conclude that if the initial level of the beam coherent noise exceeds the equilibrium, the relaxation will be accompanied by the growth of the beam longitudinal temperature. These conditions can be realized, for example, during the beam debunching. Let us discuss this process for the beam, which initially had the Gaussian distribution over the azimuth

$$\rho(\varphi) = \frac{N}{(2\pi)^{1/2} \vartheta_b} \exp\left(-\frac{\varphi^2}{2\vartheta_b^2}\right). \quad (8.8.1)$$

The initial value of $\mathcal{E}$ is defined by the expression

$$\mathcal{E} = \frac{NT_i}{2} + \sum_n \frac{e^2 \omega_s}{2\pi} \frac{Z_n''}{n} |\rho_n|^2 = \frac{N}{2} \left\{ T_i + \sum_n \frac{e^2 \omega_s}{2\pi} \frac{Z_n''}{n} e^{-2n^2 \vartheta_b^2} \right\}. \quad (8.8.2)$$

Here T_i is the initial value of the beam longitudinal temperature. The final value of the longitudinal temperature T_f is calculated using Eq. (8.7.10). Taking into account that in this equation $N_{ex} < N$, neglecting the factor with the value between 1 and 2 for T_f we can write

$$T_f \simeq T_i + \sum_n \frac{e^2 \omega_s}{2\pi} \frac{Z_n'}{n} e^{-2n^2 \vartheta_b^2}. \quad (8.8.3)$$

Since typically $\vartheta_b \gg a/R_0$, one can take in the sum over n $Z_n''/n = \text{const}$. Then, after summation Eq. (8.8.3) reads

$$T_f \simeq T_i + \frac{Ne^2 \omega_s}{(2\pi)^{1/2} 2\vartheta_b} \left(\frac{Z_n''}{n} \right). \quad (8.8.4)$$

Equation (8.8.4) shows that the heating effect is especially substantial in the sufficiently short and cool beams. Using Eq. (3.23) we can write the criterion of the strong heating in the form

$$\frac{N}{N_c} \gg \vartheta_b \quad (8.8.5)$$

Taking into account the relation of $T_{\parallel}$ with the frequency spread $T_{\parallel} = R_0 \Delta\omega_0^2 / \omega_0'$, Eq. (8.8.4) can be presented in the form

$$\Delta\omega_f^2 = \Delta\omega_i^2 + \frac{1}{\vartheta_b} \left(\frac{\Omega_n^2}{n^2} \right)_{|n| < n_0}. \quad (8.8.6)$$

If prior to debunching the particles were under the action of the RF-field of the storage ring, the values $\Delta\omega_i$ and ϑ_b are coupled by $\Delta\omega_i = \omega_c \vartheta_b$. This transforms Eq. (8.8.6) into

$$\left(\frac{\Delta\omega_f}{\Delta\omega_i} \right)^2 = 1 + \frac{\omega_c}{\Delta\omega_i^3} \frac{\Omega_n^2}{n^2}, \quad (8.8.7)$$

or, provided that

$$\frac{\omega_c a}{R_0} \ll \Delta\omega_i \ll [\omega_c \Omega_n^2 / n^2]^{1/3}, \quad (8.8.8)$$

$$\Delta\omega_f^2\Delta\omega_i\simeq\omega_c\frac{\Omega_n^2}{n^2}. \quad (8.8.9)$$

The last equation recalls (though not completely) the so-called Dori formula¹⁴² and thus predicts the overshoot phenomenon while debunching the beam from the separatrix.

By now, we assumed that $\vartheta_b < \pi/2$. In longer beams $\pi \leq 2\vartheta_b \leq 2\pi$ the contribution of coherent motion in the beam energy decreases by the factor ϑ_g/ϑ_b (where $\vartheta_g = \pi - \vartheta_b$). Consequently, the effect of beam heating by the off-equilibrium noise becomes weaker:

$$T_f \simeq T_i + \frac{\vartheta_g}{\vartheta_b} \frac{Ne^2\omega_s}{(2\pi)^{1/2}2\vartheta_b} \left(\frac{Z_n''}{n} \right) \quad (8.8.10)$$

while the criterion of the strong heating of the beam during debunching, Eq. (8.8.5) takes the form

$$\frac{N}{N_c} \gg \frac{\vartheta_b^2}{\vartheta_g} > 1. \quad (8.8.11)$$

8.9. ON THE CRYSTALLINE BEAMS

8.9.1. According to Eqs. (8.7.21) and (8.7.17), with a decrease in the longitudinal temperature, the number of modes in equilibrium N_{ex} increases and, in general, it can reach very high values. The presence in equilibrium of a large number of coherent oscillations corresponds to the appearance in the beam of the substantial ordering in the space positioning of particles. Indeed, according to Eqs. (2.4.7) and (7.1.13) the probability for two particles of the beam to be at an azimuthal distance $\Delta\vartheta = \vartheta_1 - \vartheta_2$ can be estimated by

$$w(\Delta\vartheta) = 1 - \frac{\sqrt{2}\pi n_D}{N} \exp\left(-\frac{n_D|\Delta\vartheta|}{\sqrt{2}}\right). \quad (8.9.1)$$

Such correlations in the mutual positioning of particles are caused by the action of collective fields. The strengths of these correlations increase with an increase in n_D . As the orbit length is finite, one can expect that with a decrease in the beam longitudinal temperature the particles will tend to equidistant distribution along the orbit with period Π/N specific for the crystal bodies.

The interest to the ordering phenomena in extremely cold Coulomb systems periodically appears in different fields of physics. Using Monte-Carlo and molecular dynamic (MD) simulations Hansen and others^{140,141} showed that the infinite classical one-component ion plasma (OCP)* can undergo the phase transi-

*By definition, OCP is the system of the same charged particles whose space charge repulsion is compensated by external focusing or by uniform motionless background of the opposite charge. In

tion into solid state and form the bcc lattice (so-called Wigner crystal) if its plasma coupling parameter

$$\Gamma = \frac{(Ze)^2}{aT}$$

exceeds the critical value $\Gamma_c = 170$. Here Ze is the charge of the ion,

$$a = \left(\frac{3}{4\pi\rho_i} \right)^{1/3}$$

is the so-called Wigner-Seitz radius, ρ_i is the density, and T is the temperature of ions. The possibility to obtain in electron cooling rings ion beams with extraordinary small temperatures enables one to study these states in ion storage rings with electron cooling. Even in cases, when the temperature of electron beams is estimated by $e^2 n_e^{1/3}$,¹³⁴ for beams of multicharged ions, which have the density of the same order of magnitude as the electron beam, in equilibrium one can write

$$\Gamma_i \simeq Z^2 \Gamma_e$$

Therefore, for ion beams with $Z \geq 13$ one may expect that after the cooling Γ_i will exceed the critical value and the stored beam will undergo the phase transition by forming the ordered structures along the orbit.

The symmetry of the crystalline state in the storage ring is strongly affected by the focusing of particles and generally may differ from that found for infinite OCPs. For the nonrelativistic ion beams considered here it can strongly depend on the beam intensity. At rather low intensities due to large Coulomb betatron tune spreads collective interaction cannot lead to any noticeable ordering of the transverse degrees of freedom. On the contrary, the revolution frequency spread can be small. Under these conditions with a decrease of the longitudinal temperature the beam can form the "one-dimensional" crystal, or the string, composed of particle discs. Below we mainly follow Ref. 127.

The conditions when the beam goes into a crystalline state can be studied by calculation of the statistical sum. For the string ordering we can do the calculations in the adiabatic approximation, when the statistical sum of the beam reads

$$\exp(-F/T) = \int \left(\frac{R_0 d\vartheta d\Delta p}{h} \right)^N \frac{f_N}{N!}. \quad (8.9.2)$$

Here h is the Planck constant; F is the beam free energy;

$$f_N = \exp \left\{ - \sum_{a=1}^N \frac{\Delta p_a^2}{2\mu T} - \sum_{a,b} \frac{\mathcal{U}_{a,b}}{2T} \right\}$$

this sense, the beam in the storage ring presents an example of OCP.

is the N -particle distribution function; $\mu^{-1} = R_0 \omega'_0$ is the synchrotron mass of particles;

$$\mathcal{U}_{a,b} = 2T \sum_n \xi_n \exp(in[\vartheta_a - \vartheta_b])$$

is the energy of the interaction of particles a and b , and ξ_n gives the intensity of the interaction:

$$\xi_n = \frac{e^2 \omega_s}{4\pi T} \frac{Z''_n}{n} = \frac{1}{2N} \begin{cases} N/N_c, & |n| < n_0, \\ (n_D/n)^2, & |n| > n_0. \end{cases} \quad (8.9.3)$$

Let us rewrite Eq. (8.9.2) in the form

$$\exp(-F/T) = \exp\left\{N \sum_n \xi_n - F^{(0)}/T\right\} Q_N. \quad (8.9.4)$$

Here

$$F^{(0)} = -NT \left\{ \ln \left(\frac{\Pi}{h} (\pi \mu T)^{1/2} \right) - \ln N \right\} \quad (8.9.5)$$

is the free energy of an ideal gas of disc particles;

$$Q_N = \int_0^{2\pi} \frac{d\vartheta_1 \cdots d\vartheta_N}{(2\pi)^N} \exp\left(-\sum_n \xi_n |\rho_n|^2\right) \quad (8.9.6)$$

is the so-called configurational integral; and ρ_n is the Fourier harmonics of the beam azimuthal density:

$$\rho_n = \sum_{a=1}^N \exp(-in\vartheta_a). \quad (8.9.7)$$

We shall calculate the thermodynamic variables of the beam, which are defined by the beam free energy and its derivatives over the temperature. As is well known, F is related to Q_N through Eq. (8.9.4):

$$F = -NT \sum_n \xi_n + F^{(0)} - T \ln Q_N = N\varepsilon_0 + F^{(0)} + \Delta F, \quad (8.9.8)$$

where $\varepsilon_0 = \mathcal{U}_{a,a}/2$.

Simple expressions for the configurational integral Q_N can be obtained in two regions defined by the ratio of N to n_D . In the region $N \gg n_D$ the value of Q_N can be calculated using the identity

$$\langle |\rho_n|^2 \rangle = \frac{1}{Q_N} \int_0^{2\pi} \frac{d\vartheta_1 \cdots d\vartheta_N}{(2\pi)^N} |\rho_n|^2 \exp\left(-\sum_n \xi_n |\rho_n|^2\right) = -\frac{\partial}{\partial \xi_n} \ln Q_N. \quad (8.9.9)$$

For the region $N \gg n_D$ the left-hand side of this equation was already calculated in Eq. (7.2.13). Thus we can write

$$\langle |\rho_n|^2 \rangle = \frac{N}{1 + N\xi_n}.$$

Presenting Q_N as a product $Q_N = \prod_n Q_N(\xi_n)$, after integrating Eq. (8.9.9) with initial condition $Q_N(\xi_n=0) = 1$, we obtain

$$Q_N = \prod_n \frac{1}{1 + N\xi_n}. \quad (8.9.10)$$

The corresponding correction to the free energy is

$$\Delta F = T_{\parallel} \sum_n \ln(1 + N\xi_n). \quad (8.9.11)$$

Due to the fast convergence of series the summation over n in Eq. (8.9.11) can be extended to the region $|n| < \infty$.

Another way of calculating ΔF (or Q_N , as it is well known), consists of summing the contribution of various Maier's diagrams.¹⁴³ From this point of view Eq. (8.9.11) corresponds to the selection of the contributions of the so-called loop diagrams (i.e., to the calculation of Q_N in the loop approximation). This approximation is reliable, provided that $N \gg n_D$.

Taking into account Eq. (8.9.3) and an additional condition $N, n_d \gg n_0$, after summation over n in Eq. (8.9.11) we find

$$\Delta F \simeq 2\pi n_D T_{\parallel}. \quad (8.9.12)$$

Whence for the energy, specific heat, and entropy of the beam we obtain

$$\begin{aligned} \mathcal{E} &= -T_{\parallel} \frac{\partial}{\partial T_{\parallel}} \left(\frac{F}{T_{\parallel}} \right) = \frac{NT_{\parallel}}{2} \left(1 + \frac{2\pi n_D}{N} \right), \\ C &= \frac{\partial \mathcal{E}}{\partial T_{\parallel}} = \frac{N}{2} \left(1 + \frac{2\pi n_D}{N} \right), \end{aligned} \quad (8.9.13)$$

and

$$S = -\frac{\partial F}{\partial T_{\parallel}} = N \ln \left[\frac{e\Pi}{Nh} (2\pi\mu T_{\parallel})^{1/2} \right] + \frac{N}{2} \left(1 - \frac{2\pi n_D}{N} \right), \quad \ln e = 1.$$

In these formulas the influence of particle interaction is described by the ratio of $2\pi n_D/N$, i.e., by the ratio of the average distance between particles in the beam $2\pi/N$ to the shielding radius $\sim 1/n_D$. One can see that provided that n_D approaches $N/2\pi$, the beam entropy can substantially decrease while its thermal energy $\mathcal{E}$ and specific heat C tend to be doubled:

$$\mathcal{E} \rightarrow NT_{\parallel}, \quad C \rightarrow N. \quad (8.9.14)$$

Such a variation of the thermodynamic variables of the beam describes the tendency to the substantial ordering in its particle motion due to interaction. Comparison of Eqs. (8.7.10) and (8.9.13) yields one more estimation of the average number of modes in the equilibrium $N_{ex} = 2\pi n_D$. With a decrease in temperature, N_{ex} increases approaching N (when $n_D \rightarrow N/2\pi$).

In the region $n_D \gg N$ Eqs. (8.9.11) through (8.9.13) cannot be used any more. However, in this region, the value Q_N can be calculated assuming that ϑ_a are close to the node points of the lattice $2\pi a/N$:

$$\rho_n \simeq -in \sum_{a=1}^n \xi_a \exp\left(-\frac{2\pi i n a}{N}\right),$$

$$\vartheta_a = \frac{2\pi a}{N} + \xi_a \quad |\xi_a| \ll 2\pi/N. \quad (8.9.15)$$

Substituting Eq. (8.9.15) into f_N , we can write

$$\exp\left(-\sum_n \xi_n |\rho_n|^2\right) = \exp\left(-\sum_n \xi_n n^2 \sum_{a,b} \xi_a \xi_b \exp\left[\frac{2\pi i n}{N}(a-b)\right]\right).$$

Neglecting here the values of the order of $n_0/N \ll 1$ and using ($N\xi_n \rightarrow n_D^2/n^2$), we may rewrite this expression as

$$\exp\left(-\frac{n_D^2}{2} \sum_{a=1}^N \xi_a^2\right).$$

This means that in the region $n_D \gg N$ the beam N -particle distribution function takes the form

$$f_N = \exp\left\{-\sum_{a=1}^N \frac{\Delta p_a^2}{2\mu T_{\parallel}} - \frac{n_D^2}{2} \sum_{a=1}^N \left(\vartheta_a - \frac{2\pi a}{N}\right)^2\right\}, \quad (8.9.16)$$

which is in correspondence with the localization of particles near the node points $2\pi a/N$ with the dispersion

$$\langle \xi_a^2 \rangle = \frac{1}{n_D^2} \ll \left(\frac{2\pi}{N}\right). \quad (8.9.17)$$

One can see in Eq. (8.9.15) that the number of normal oscillations in this state is exactly equal to N . The calculation of F with the distribution function from Eq. (8.9.16) is easy. As a result, for the thermodynamic variables of the beam we obtain well-known expressions¹⁹

$$\begin{aligned} F &= -N \ln(T_{\parallel} / \hbar \omega_{\Lambda}), \quad \mathcal{E} = NT_{\parallel}, \\ S &= N[\ln(T_{\parallel} / \hbar \omega_{\Lambda}) + 1], \quad C = N, \end{aligned} \quad (8.9.18)$$

which coincide with the thermodynamic variables of the crystal body at high temperatures. ($T_{\parallel} \gg \hbar \omega_{\Lambda}$, ω_{Λ} is the beam Langmuir frequency.)

The condition

$$\Gamma^{(1)} \equiv \frac{2\pi n_D}{N} \gg 1 \quad (8.9.19)$$

specifies the range of longitudinal temperatures when the beam transits into the chain crystalline state. The fact that such a process has no transition temperature is not so surprising. It is connected with the far-acting character of the Coulomb interaction, when at any reasonable temperatures the beam is a mixture of the ordered and amorphous phases. To this end the transition into a purely amorphous state would correspond to high temperature $T_{\parallel}$, when collective oscillations in a beam could damp at distance of order of the distance between neighbor particles $2\pi/N$.

The "melting" temperature of such a string-ordered beam can be evaluated using the Lindemann melting criterion:¹⁴¹

$$\gamma_L^2 = \frac{1}{N} \left\langle \sum_{a=1}^N \left(\frac{N\xi_a}{2\pi} - a \right)^2 \right\rangle, \quad (\gamma_L)_c \simeq 0.17.$$

Since for a chain one can write $\Gamma^{(1)} = 1/\gamma_L^2$, this yields $\Gamma_c^{(1)} \simeq 35$. For typical parameters of the cooled beam at NAP-M this value is reached when $N_c = 1.5 \times 10^5$ and $\Pi/N = 3 \times 10^{-2} \text{ cm} \ll \sigma_{\perp}$. These values are in general agreement with the criterion for the string ordering

$$\lambda = Na/\Pi \leq 0.7,$$

which was obtained in Ref. 144 using MD simulations, but are out of the range of measurements at NAP-M. (See Fig. 7.6 in §7.2.) This discrepancy may indicate that for very cold beams we cannot recalculate the longitudinal beam temperature using simple formulas, obtained in the Debye-Hückel approximation ($2\pi n_D \ll N$). To exclude this we can estimate from data in Fig. 7.6 and using the identity¹⁹ (here P is the pressure)

$$\frac{1}{T} \left(\frac{\partial P}{\partial \rho} \right)_T = \lim_{n \rightarrow 0} \frac{N}{\langle |\delta \rho_n|^2 \rangle},$$

the isothermal compressibility $(\partial P / \partial \rho)_T$ of the beam. The result (see Fig. 8.1) clearly indicates that in these measurements $(\partial P / \partial \rho)_T$ strongly exceeded the value specific for the ideal gas $(\partial P / \partial \rho)_T = T$.

The physical sense of condition (8.9.19) is simple. It requires that a particle with an average kinetic energy $T_{\parallel} / 2$ could not overcome the potential barrier of the height

$$\Delta U = \frac{2e^2}{a^2} \frac{\Pi}{N},$$

equal to the interaction energy of two discs of charge e at distance Π/N . It is interesting to note in the region (8.9.19) the physical meaning of the parameter n_D is changed. As can be seen from Eq. (8.9.17), this parameter now characterizes not a shielding length but an average amplitude of particle oscillations with respect to the equilibrium positions $2\pi a/N$.

For the parameters of NAP-M⁹⁹ condition (8.9.19) is satisfied at

$$\frac{\sigma_{\parallel}}{p_s} \ll \frac{10^{-3}}{N^{1/2}}.$$

Therefore the crystalline state could be observed at this installation at low currents of the beam ($N < 10^6$).

8.9.2. Even if the beam temperature is so low that $\Gamma^{(1)} \gg 1$, its intensity N can be limited by Coulomb repulsion of ions, which pushes them off the unperturbed closed orbit. As a results, after some threshold intensity [$N \geq N_{th}$] the ion oscillations become unstable. The effect of this instability on phase transitions of a beam moving in a straight channel with constant focusing and equal vertical

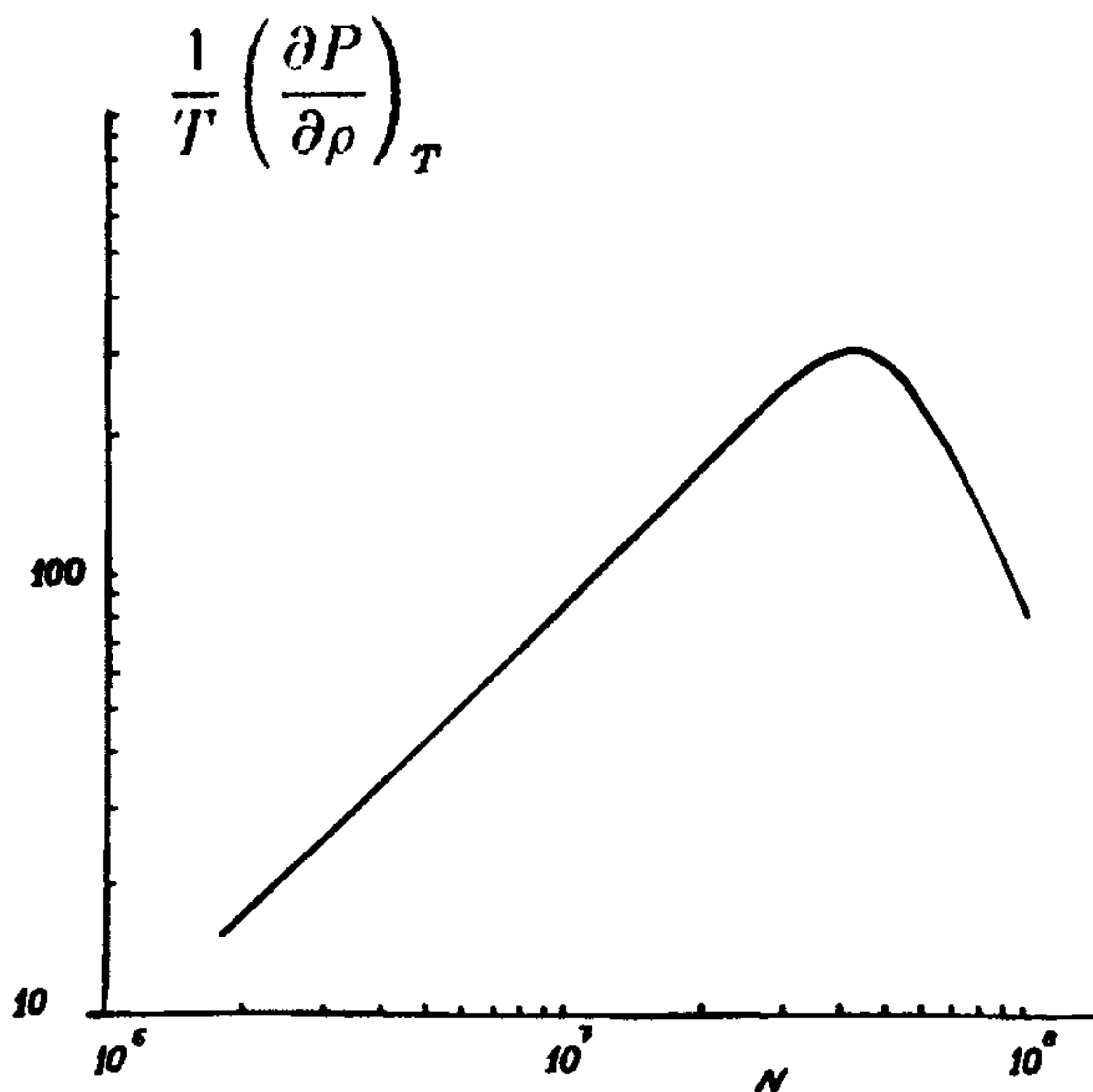


FIG. 8.1. The isothermal compressibility of the beam vs N recalculated using data from Fig. 7.6.

and horizontal tunes was studied in Ref. 144. The results showed that an increase in the beam intensity causes a sequence of the phase transitions from the string to a zig-zag ordering, then from the zig-zag to helix and to others more complicated structures.

Let us briefly discuss the effect of this space charge instability on the behavior of crystalline beams in storage rings. The oscillations of ions are described by the following equations:

$$\begin{aligned}
 z_m'' + g_z(\vartheta)z_m &= \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{z_m - z_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \\
 x_m + g_x(\vartheta)x_m &= \frac{R_0^2}{R} \frac{\Delta p_m}{p} + \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{x_m - x_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \\
 \frac{\Delta p_m'}{p} &= \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{\vartheta_m - \vartheta_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \quad s_m'/R_0 = -\frac{x_m}{R} + \frac{\Delta p_m}{\gamma^2 p}, \\
 g_z &= \frac{R_0^2}{R^2} n_B(\vartheta), \quad g_x = \frac{R_0^2}{R^2} [1 - n_B(\vartheta)].
 \end{aligned} \tag{8.9.20}$$

For the sake of simplicity, we consider the simplest structures: a string and a zig-zag. We assume first the smoothed focusing approximation. Then, if $d = \Pi/N$ is the string period and $x_m \ll d$, $z_m \ll d$ and $s_m \ll d$ are the offsets of the m -th ion, the linearization of Eqs. (8.9.20) near the closed orbit yields:

$$\begin{aligned}
 z_m'' + v_z^2 z_m &= \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{m \neq n} \frac{z_m - z_n}{|m - n|^3}, \\
 x_m'' + v_x^2 x_m &= R \frac{\Delta p_m}{p} + \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{m \neq n} \frac{x_m - x_n}{|m - n|^3}, \\
 \frac{\Delta p_m'}{p} &= -\frac{2}{R} \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{m \neq n} \frac{s_m - s_n}{|m - n|^3}, \quad s_m' = -x_m + R \frac{\Delta p_m}{\gamma^2 p}.
 \end{aligned} \tag{8.9.21}$$

Here, integers m and n mark the beam ions. Since for any particle $\mathbf{r}_m = \mathbf{r}_{m+N}$, we expect solutions of Eq. (8.9.21) in the form

$$\begin{aligned}
 z_m &= Z \exp(ik_l m - iv\vartheta), \quad x_m = X \exp(ik_l m - iv\vartheta), \\
 s_m &= S \exp(ik_l m - iv\vartheta), \quad (\Delta p/p)_m = \Delta \exp(ik_l m - iv\vartheta).
 \end{aligned} \tag{8.9.22}$$

Here, $k_l = 2\pi l/N$, and $l = 1, 2, \dots, N-1$. Substituting these expressions in Eqs. (8.9.21) we obtain the system of equations for amplitudes Z , X and S .

$$\begin{aligned}
(\nu^2 - \nu_z^2 + \Sigma_l) Z_l &= 0, \\
(\nu^2 - \nu_x^2 + \gamma^2 + \Sigma_l) X_l &= i\nu\gamma^2 S_l, \\
(\nu^2 - 2\Sigma_l/\gamma^2) S_l &= -i\nu X_l, \\
\Sigma_l &= 2 \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{n=1}^{\infty} \frac{1 - \cos(k_n n)}{n^3}.
\end{aligned} \tag{8.9.23}$$

These equations are uncoupled in l , which means that different numbers l specify different eigenmodes of the beam. Equations (8.9.23) show also that the mode with $l=0$ is not affected by the Coulomb interaction.

According to the first equation in Eqs. (8.9.23) the vertical oscillations of the string become unstable, when

$$\Sigma_l \geq \nu_z^2.$$

The function Σ_l reaches its maximum value for the mode with $l=N/2$, which corresponds to zig-zag displacements of neighbor particles relative the unperturbed closed orbit. This value:

$$\Sigma_{l=N/2} = \frac{(Ze)^2}{M\omega_0^2 d^3} \frac{7\zeta(3)}{2}, \quad \zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m}$$

gives the lowest threshold for the vertical oscillations

$$d \geq d_{th} = \left(\frac{(Ze)^2}{M\omega_0^2} \frac{7\zeta(3)}{2\nu_x^2} \right)^{1/3}. \tag{8.9.25}$$

The 2nd and the 3rd equations in Eqs. (8.9.23) describe the coupling of the horizontal and longitudinal oscillations due to the motion along the curved orbits. These equations yield the following dispersion equation

$$\begin{aligned}
w^2 - w(\nu_x^2 - \Sigma_l[1 - 2/\gamma^2]) + 2\Sigma_l(\nu_x^2 - \gamma^2 - \Sigma_l)/\gamma^2 &= 0, \\
w &= \nu^2.
\end{aligned} \tag{8.9.26}$$

The oscillations are stable, if both roots of this equations (w_+ and w_-) are positive numbers. For nonrelativistic ions ($\gamma \simeq 1$) from

$$w_+ + w_- = \nu_x^2 + \Sigma_l[1 - 2/\gamma^2] \simeq \nu_x^2 + \Sigma_l$$

and

$$w_+ w_- = 2\Sigma_l(\nu_x^2 - \gamma^2 - \Sigma_l)/\gamma^2,$$

we find the following stability criterion

$$v_x^2 \gg \gamma^2 + \Sigma_l.$$

Since $\Sigma_l > 0$, this condition holds, if $v_x^2 > \gamma^2$. This means that to maintain the string be stable in the horizontal plane the ion energies must be below transition energy of the ring. On its turn, this requirement demands an alternating gradient focusing lattice. The lowest threshold again has the zig-zag mode:

$$d \leq d_{th} = \left(\frac{(Ze)^2}{M\omega_0^2} \frac{7\xi(3)}{2(v_x^2 - \gamma^2)} \right)^{1/3}. \quad (8.9.28)$$

From Eq. (8.9.26) we find the frequencies of oscillations

$$w_{\pm} = \frac{1}{2} \left\{ v_x^2 - \Sigma_l \left(1 - \frac{2}{\gamma^2} \right) \pm \sqrt{\left[v_x^2 - \Sigma_l \left(1 + \frac{2}{\gamma^2} \right) \right]^2 + 8\Sigma_l} \right\}, \quad (8.9.29)$$

$$v_{\pm} = \pm \sqrt{w_{\pm}}.$$

If the space-charge of the beam is weak, so that $v_x^2 \gg \Sigma_l$, Eq. (8.9.29) results in the following expressions:

$$w_+ \simeq v_x^2 - \Sigma_l \left(1 - \frac{2}{\gamma^2} \right), \quad w_- \simeq 2\Sigma_l \left(\frac{1}{\gamma^2} - \frac{1}{v_x^2} \right).$$

The fast mode describes the Laslett tune shift, modified by the radial-longitudinal coupling. The slow mode describes the negative mass instability behavior of the beam

Near the threshold of instability the roots of Eq. (8.9.29) have the form:

$$v_+ \simeq \sqrt{\gamma^2 + 2\Sigma_l/\gamma^2}, \quad v_- \simeq \pm \sqrt{2\Sigma_l \frac{v_x^2 - \gamma^2 - \Sigma_l}{\gamma^4 + 2\Sigma_l}}.$$

8.9.3. Apart from the time-dependent, Eqs. (8.9.21) may have the stationary solutions $[x'_m = 0, z'_m = 0, s'_m = 0 \text{ and } \Delta p'_m = 0]$. Being determined by the equilibrium between the focusing forces and the forces of the ion Coulomb repulsion, these solutions determine new closed orbits, calculated in the smoothed focusing approximation. If the beam cooling is described by the dimensionless decrements $[\Lambda_z, \Lambda_x \text{ and } \Lambda_{\parallel}]$, the equilibrium equations read

$$\begin{aligned} v_z^2 z_m &= \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{z_m - z_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \\ v_x^2 x_m &= R \frac{\Delta p_m}{p} + \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{x_m - x_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \\ R \Lambda_{\parallel} \frac{\Delta p_m}{p} &= \frac{(Ze)^2}{M\omega_0^2} \sum_{n \neq m} \frac{\vartheta_m - \vartheta_n}{|\mathbf{r}_m - \mathbf{r}_n|^3}, \quad x_m = R \frac{\Delta p_m}{\gamma^2 p}. \end{aligned} \quad (8.9.30)$$

We consider only simplest solutions of these equations. A trivial set:

$$z_m = x_m = s_m = 0, \quad \Delta p_m = 0, \quad \vartheta_m = 2\pi m/N$$

corresponds to the string ordering. It is stable below the thresholds defined in Eqs. (8.9.25) and (8.9.28). Above this threshold we have to find nontrivial solutions of Eqs. (8.9.30). For the sake of simplicity we consider the case, when the thresholds for vertical and horizontal oscillations differ, say

$$v_x^2 - \gamma^2 < \Sigma_l < v_z^2$$

Then, the next simple structure could be a zig-zag in the horizontal plane where $z_m = 0$, while x_m , s_m and Δp_m obey the following equations:

$$x_{m+1} = -x_m = -ud, \quad s_{m+1} = -s_m = -vd, \quad \Delta p_{m+1} = -\Delta p_m,$$

$$v_x^2 - \gamma^2 = \frac{(Ze)^2}{M\omega_0^2 d^3} 2f_1(v, u), \quad \Lambda_{\parallel} u = \frac{(Ze)^2}{M\omega_0^2 d^3} \frac{f_s(v, u)}{\gamma^2} \quad (8.9.31)$$

$$f_1(v, u) = \sum_{l=-\infty}^{\infty} \frac{1}{[(2l+1+2v)^2 + 4u^2]^{3/2}}$$

$$f_s(v, u) = \sum_{l=-\infty}^{\infty} \frac{2l+1+2v}{[(2l+1+2v)^2 + 4u^2]^{3/2}}.$$

We note that the functions f_1 and f_s in Eq. (8.9.31) are periodic functions of the longitudinal variable v :

$$f_1(v+1, u) = f_1(v, u), \quad f_s(v+1, u) = f_s(v, u), \quad (8.9.32)$$

and that f_s has simple roots when v becomes an integer or half integer. The slope of f_s is negative, when v is close to an integer, and is positive, when v is close to a half integer. For this reason, without cooling ($\Lambda_{\parallel} = 0$) Eqs. (8.9.31) yield $v = 0$, while u (and Δp_m) can be found from the equation:

$$v_x^2 - \gamma^2 = \frac{(Ze)^2}{M\omega_0^2 d^3} 2f_1(0, u),$$

$$= 2 \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{l=-\infty}^{\infty} \frac{1}{[(2l+1)^2 + 4u^2]^{3/2}}. \quad (8.9.33)$$

Since f_1 is a positive function and since it monotonously decreases when $u \rightarrow \infty$, the roots of Eqs. (8.9.33) appear when

$$v_x^2 - \gamma^2 < \frac{(Ze)^2}{M\omega_0^2 d^3} 2f_1(0, 0).$$

This condition exactly coincides with the threshold condition in Eq. (8.9.28). According to Eqs. (8.9.31), due to the beam cooling particles get the longitudinal displacements relative to equilibrium points $[2\pi m/N]$, so that both u and v must be calculated as common roots of Eqs. (8.9.31).

8.9.4. As we saw in §1.5.4, the space charge instability is very sensitive to modulations of the lattice functions. These modulations result in various resonances, which generally can embarrass the phase transition. Here, we briefly discuss the effect of the alternating gradient focusing on the stability of oscillations near the string.

Let us first consider the equation for small vertical oscillation near the string. It reads:

$$z_m'' + g_z(\vartheta)z_m = \frac{(Ze)^2}{M\omega_0^2 d^3} \sum_{mn} \frac{z_m - z_n}{|m - n|^3}. \quad (8.9.34)$$

For solutions in the form of Eqs. (8.9.22) ($z_m = Z_l \exp(ik_l m)$) we find that amplitudes Z_l obey the following equation

$$Z_l'' + [g_z(\vartheta) - \Sigma_l]Z_l = 0.$$

This is an ordinary linear equation with periodic coefficients and it can be solved using standard procedures. The only difference compared to the case of a normal beam is that now the rigidity coefficient in Eq. (8.9.35) depends on the mode-number. For this reason, Eq. (8.9.35) determines $N/2$ β -functions, phase advances per cell and tunes. This feature is very general for the coupled betatron oscillations. As seen from Eq. (8.9.35), the requirement of the envelope stability of a string demands the increase in the strength of the lattice focusing.

As a specific for crystal beam we note that now β -functions cover some band between the unperturbed $\beta(\vartheta)[l=0]$ and β -function of the zig-zag mode $[l=N/2]$. The width of this band increases, when modes approach the border of the stability diagram. Since the highest Σ_l corresponds to the zig-zag mode this mode becomes unstable earlier than others.

Relevant equations for the oscillations in the horizontal plane read:

$$X_l'' + [g_x(\vartheta) - \Sigma_l]X_l = \frac{R_0^2}{R} \Delta_l \quad (8.9.36)$$

$$\Delta_l' = -2\Sigma_l S_l / R_0, \quad S_l' / R_0 = -\frac{X_l}{R} + \frac{\Delta_l}{\gamma^2}. \quad (8.9.36)$$

The calculation of the transport matrixes for all elements of a lattice except for bending magnets is more or less standard. For a bending magnet with $n_B = \text{const}$, simple calculations yield [again $s = R_0\vartheta$ and $X' = dX/ds$]

$$\hat{X}(s) = \hat{M}(s) \hat{X}_0 \quad \hat{X} = \begin{pmatrix} X_I \\ X'_I \\ S_I \\ \Delta_I \end{pmatrix}.$$

If we define $\kappa^2 = \Sigma_I / R_0^2$, then for $n_B < 0$ and $|n_B| > R^2 \kappa^2$ we obtain

$$\hat{M}(s) = \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ G_5 & G_1 & -2\kappa^2 G_4 & G_6 \\ -G_6 & -G_4 & G_7 & G_8 \\ 2\kappa^2 G_4 & -G_3 & G_9 & G_7 \end{pmatrix}. \quad (8.9.37)$$

Here,

$$G_1 = \frac{K_+ c_+ - K_- c_-}{F}, \quad G_2 = \frac{K_+ \frac{s_+}{q_+} - K_- \frac{s_-}{q_-}}{F},$$

$$G_3 = 2\kappa^2 \frac{\frac{s_+}{q_+} - \frac{s_-}{q_-}}{RF}, \quad G_4 = -\frac{c_+ - c_-}{RF},$$

$$G_5 = \frac{-q_+ K_+ s_+ + q_- K_- s_-}{F}, \quad G_6 = \frac{q_+ s_+ - q_- s_-}{RF},$$

$$G_7 = \frac{-K_- c_+ + K_+ c_-}{F}, \quad G_8 = \frac{\frac{q_+ s_+}{K_+} - K_- \frac{q_- s_-}{K_-}}{R^2 F},$$

$$G_9 = 2\kappa^2 \frac{\frac{K_- s_+}{q_+} - \frac{K_+ s_-}{q_-}}{F},$$

and

$$c_{\pm} = \cos(q_{\pm} s), \quad s_{\pm} = \sin(q_{\pm} s), \quad K_{\pm} = w_{\pm} - 2\kappa^2,$$

$$q_{\pm} = \sqrt{w_{\pm}}, \quad w_{\pm} = \frac{1}{2} \left\{ \frac{1 - n_B}{R^2} + \kappa^2 \pm F \right\},$$

$$F = \sqrt{\left[\frac{1 - n_B}{R^2} - 3\kappa^2 \right]^2 + \frac{8\kappa^2}{R^2}}.$$

If $n_B \geq 0$, we have $w_- < 0$ and therefore, the bending magnet becomes a defocusing element of the lattice for all string modes $l \neq 0$. For this case, the transport matrix can be obtained from Eq. (8.9.37) using the substitutions

$$q_- \rightarrow i\sqrt{|w_-|}, \quad c_- \rightarrow \cosh(q_-s), \quad s_- \rightarrow i \sinh(q_-s).$$

By direct calculation it is easy to verify that $\hat{M}$ is a symplectic matrix, so that

$$\hat{M}^T \hat{U} \hat{M} = \hat{U}, \quad \hat{U} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Generally, the modulations due to alternating gradient focusing result in smaller values of tunes of slow modes $[\nu_-]$ compared to that, predicted using the smoothed focusing approximation [see in Fig. 8.2 (a and b)]. These figures show also that the width of the stability region depends on the position of the working point of the unperturbed lattice. If we adopt that above the threshold of the envelope instability occurs the phase transition from the string to a zig-zag structure, the position of the transition point can be varied changing the strengths of the lattice quadrupoles.

8.10. ON THE SUPPRESSION OF THE COULOMB RELAXATION IN A COLD BEAM

Let us turn again to the problem of the influence of the Coulomb collisions on the beam longitudinal temperature in the storage ring with electron cooling. In this case, Eqs. (8.7.7) should be modified in the following way:

$$\begin{aligned} \frac{dT_n}{dt} &= -2\delta_n(\Omega_n)(T_n - T_{\parallel}), \\ \frac{dT_{\parallel}}{dt} &= \sum_n \frac{2\delta_n}{N}(T_n - T_{\parallel}) - \lambda_0 T_{\parallel} + d + (dT_{\parallel}/dt)_f. \end{aligned} \quad (8.10.1)$$

Here λ_0 and d are the cooling decrement and diffusion coefficient respectively; $(dT_{\parallel}/dt)_f$ describes the longitudinal temperature change due to fast collisions. It is assumed that the interaction with electron beam does not lead to the coherent instabilities.

Prior to studying the stationary solutions of system (8.10.1), let us consider in more detail the term $(dT_{\parallel}/dt)_f$. In the general case, such a term is proportional to the integral over the impact parameters:

$$(\dot{T}_{\parallel})_f \sim \left\langle \int db K_1\left(\frac{\omega_{\perp} b}{u}\right) \right\rangle, \quad (8.10.2)$$

where ω_1 are the betatron oscillation frequencies (which, for the sake of simplicity, are taken to be equal), u is the relative velocity of interacting particles, and $K_\nu(x)$ is the Bessel function of the 3rd kind. The integration over b in

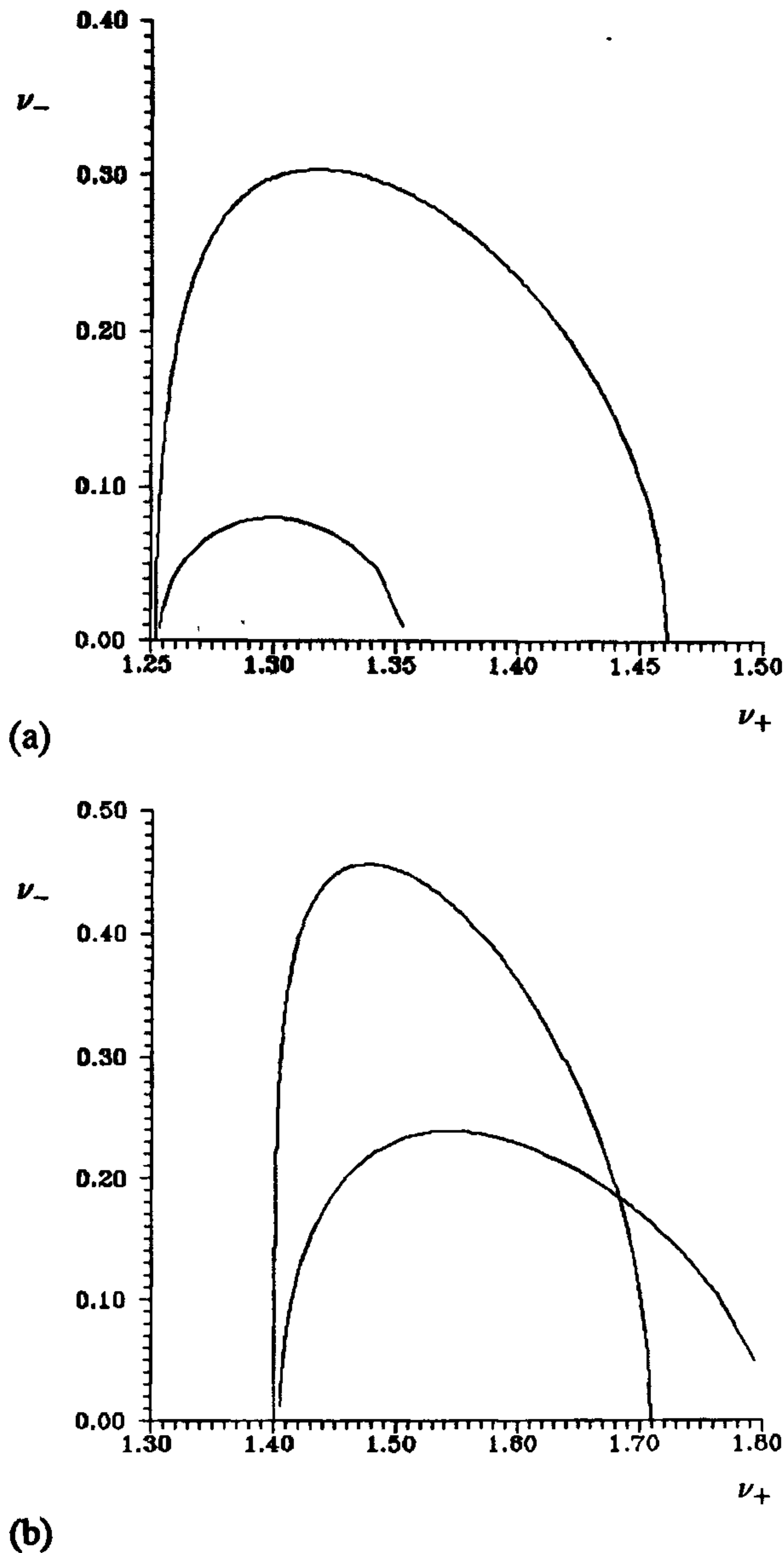


FIG. 8.2. The working-path for a string due to a variation of the beam current; upper curve: smoothed focusing approximation, lower curve: alternating gradient focusing; the calculations are done for the NAP-M lattice⁴⁵ (a): $\nu_x=1.253$, $\nu_z=1.33$; (b): $\nu_x=1.404$, $\nu_z=1.18$.

Eq. (8.10.2) is cut on the upper limit by the value $b_{\max} \sim u/\omega_{\perp}$ or by R_0/n_D . On the lower limit such an integral diverges logarithmically. As already mentioned, this divergence can be regularized by the integration in Eq. (8.10.2) from $b_{\min}$ providing the validation of the perturbation theory. Without the collective correlations the beam particles can take arbitrary positions along the orbit. Therefore, one can choose $b_{\min} = e^2/Mu^2$ and after that the integration in Eq. (8.10.2) gives the expression corresponding to the calculation of $(dT_{\parallel}/dt)_f$ with the use of the Landau collision integral. (See, for example, Ref. 117.) In the cool beam it is not true any more. In §8.5 we saw that with a decrease in temperature the collective correlations led to a substantial ordering of the space positioning of particles in a beam along the orbit. In this case, the integrand Eq. (8.10.2) should be multiplied by the probability $w(b)$ for two particles to have the distance between each other shorter than b . A particular form of this function is not so essential. Here, it is important only that in the region where n_D tends to N , this function approaches a steplike function

$$w(b) = \begin{cases} 1, & b > \Pi/N, \\ 0, & b < \Pi/N. \end{cases} \quad (8.10.3)$$

Such behavior of $w(b)$ actually follows from Eqs. (8.9.1) and (8.9.16). Taking this into account, the integral in Eq. (8.10.2) can be estimated as

$$(\dot{T}_{\parallel})_f \sim \left\langle K_0 \left(\frac{\omega_{\perp}}{u} \frac{\Pi}{n} \right) \right\rangle \sim K_0 \left(\frac{\omega_{\perp}}{R_0 \omega'_0 \sigma_{\parallel}} \right),$$

or, taking into account the asymptotes of $K_0(x)$,

$$(\dot{T}_{\parallel})_f \sim \begin{cases} \ln(N/N_{tr}), & N \gg N_{tr} \\ \exp(-N_{tr}/N), & N_{tr} \gg N. \end{cases} \quad (8.10.4)$$

The ratio

$$\frac{N_{tr}}{N} \equiv \frac{\omega_{\perp}}{R_0 \omega'_0 \sigma_{\parallel}} \frac{\Pi}{N}$$

estimates the number of periods of the betatron oscillations during the time interval taken for a couple of particles to pass the average distance Π/N . In this sense, the behavior of $(dT_{\parallel}/dt)_f$ predicted by Eq. (8.10.4) seems quite understandable. The particles with small relative longitudinal velocities perform a large number of betatron oscillations during a collision. These adiabatic collisions of particle disks do not contribute in the transverse-longitudinal heat exchange. In the reverse ultimate case, the collisions are actually fast ones (up to $b \sim \Pi/N$) and there is no suppression of the transverse-longitudinal energy exchange.

Let us emphasize that the given features in the behavior of $(dT_{\parallel}/dt)$ are totally due to the amplification of collective correlations in a beam at low temperatures and by the approach to its crystalline state. Far from this range of parameters ($N \gg n_D$) the particles move freely along the beam and the role of collective effects is reduced to the effective shielding of the interaction at large distances.

System (8.10.1) has the equilibrium solution. In an adiabatic limit $N \ll N_{tr}$ the equilibrium values of the temperatures

$$T_n = T_{\parallel} = d/\lambda_0 \quad (8.10.5)$$

do not depend on the beam current $I_p = Ne/T_s$. During beam cooling an energy of particle interaction is spent on the generation of new (collective) degrees of freedom.

If the contribution of the fast collisions cannot be neglected, then, in general, $T_{\parallel}$ grows with an increase in the beam current. From Eq. (8.10.4) it is seen that the suppression condition for such a growth of $T_{\parallel}$ has the form

$$\frac{\omega_{\perp}}{R_0 \omega'_0 \sigma_{\parallel}} \frac{\Pi}{N} > 1$$

or

$$N < N_{tr} = \frac{2\pi\nu_{\perp}}{\alpha_p(\sigma_{\parallel}/p_s)}. \quad (8.10.6)$$

If the number of particles in a beam becomes larger than N_{tr} , the adiabatic condition for collisions is violated and $T_{\parallel}$ can increase with the growth of N due to fast collisions ($\tau_{col}\omega_{\perp} < 1$).

The suppression of the IBS in the cooled proton beam at small currents was experimentally observed at the cooling ring NAP-M.⁹⁹ If we accept the interpretation of the results of those measurements given in Chapter 7, and take for the estimation ($\nu_{\perp} \simeq 1.2$, $\sigma_{\parallel}/p_s \simeq 1.4 \times 10^{-6}$, $\alpha_p \simeq 0.1$), Eq. (8.10.6) yields $N_{tr} \simeq 3 \times 10^7$, reasonably agreeing with the results of the measurements. (See Fig. 7.6, in §7.2.)

Let us emphasize that except for the beam temperature the current threshold value increases with an approach of particle energy to the transition energy of the storage ring. It is physically related to a decrease in the average longitudinal mobility of particles near transition. The latter circumstance can be used for expanding the region of beam currents, where the mutual heating of transverse and longitudinal degrees of freedom would be suppressed.

CHAPTER 9

Kinetics Near Thresholds of Collective Instabilities

Apart from the case of very cold beams, an energy exchange between the particles and the long-living coherent fluctuations can occur near the thresholds of instabilities of coherent oscillations. In this region, due to the blow-up of the stationary level of the noise of the beam the interaction of particles with even a relatively small amount of modes can amplify their diffusion, which results in abnormal increase in the beam momentum spreads, bunch lengths, etc. With moving away from the thresholds the role of the beam heating by the coherent noise will, of course, be weakened. But, at the attempt to cool the beam down to temperatures lower than the threshold level, the mentioned amplification of diffusion factors can substantially affect the equilibrium characteristics of the cooled beam.¹⁴⁵

9.1. RELAXATION NEAR THE THRESHOLD OF THE LONGITUDINAL COHERENT INSTABILITY OF THE COASTING BEAM

To simplify the subsequent calculations let us continue the study of the simplest case—the relaxation of an energy distribution of the coasting beam. Therefore, below when speaking on the instabilities we shall have in mind the instabilities of longitudinal coherent oscillations of such a beam. If abstracted from the possibility of resonance interaction, one can consider that the instabilities of interest to us have two causes: an effective attraction of particles (negative mass effect) the finite Q -value of electrodes surrounding the beam.

The method used in §8.5 and §8.6 for construction of the relaxation equations is based on the perturbation theory. Above the threshold of the instability ($N > N_{th}$) the coherent oscillations of the beam grow. This means that even small initial perturbations may strongly change the particle motion by the collective fields. The stabilization of oscillations can be caused either by the beam heating until its temperature became above the threshold value, or by the nonlinear saturation of the oscillations and associated turbulent effects. These possibilities will be discussed in the next section.

Below the threshold the coherent oscillations of the beam decay. Therefore, if the coherent noise level in a beam is not very high, its kinetics can be described using the formalism developed in §8.7.

Thus, the description of the beam kinetics in the close vicinity of the threshold of instability can be divided into two problems. The first is relaxation above the threshold, and the second is beam relaxation in the subthreshold region. The solution of the second problem is of interest in itself, for example, in connection with the study of features of beam cooling to the threshold as well as the description of the final stage of kinetic processes started above the threshold of instability. By this reason, let us first consider the second problem—relaxation of the beam with $N < N_{th}$.

When constructing the required kinetic equations we make two simplifying assumptions: (1) an interaction with the induced field distorts slightly the particle motion for the time of order of the unperturbed motion periods, and (2) the induced fields do not excite the particle transverse oscillations. Under these assumptions the heat exchange of the beam particles and coherent fluctuations are described by Eqs. (8.5.6) and (8.5.7):

$$\frac{\partial f_0}{\partial t} = -\frac{1}{N} \frac{\partial}{\partial \Delta p} \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle (e\delta E_{-n} \delta F_n)_{\omega} \rangle, \quad (9.1.1)$$

$$\left(\frac{\partial}{\partial t} + i\omega_n \right) \delta F_n(\Delta p, t) = -Ne\delta E_n(t) \frac{\partial f_0}{\partial \Delta p}. \quad (9.1.2)$$

As earlier, we assume the impedance coupling between $\delta E_{n\omega}$ and $\delta F_{n\omega}$:

$$\delta E_{n\omega} = -\frac{e\omega_s Z_n}{\Pi} \int_{-\infty}^{\infty} d\Delta p \delta F_{n\omega}(\Delta p),$$

where Z_n is generally the complex value $Z_n = Z' + iZ''$. We shall assume also that Z_n has weak dependence on ω and at least does not vanish at any $\Delta\omega_n$. The case when $Z_n \propto \Delta\omega_n$ is specific for the interaction with the damping systems and was already considered in Chapter 7.

Provided condition (8.5.5) is satisfied, Eq. (9.1.2) should be solved taking into account the effects of the nonstationary beam noise. Then the repetition of calculations of §8.5 leads to Eqs. (8.6.8) and (8.7.1) for $\delta E_{n\omega}(t)$ and $\delta F_{n\omega}(t)$, where the instantaneous value of the dielectric constant of the beam $\epsilon_n(\omega, t)$ is defined by the general expression

$$\epsilon_n(\omega) = 1 + \frac{\Omega_n^2}{n\omega_0'} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\omega - \omega_n}. \quad (9.1.3)$$

The equation for the spectral density of the field fluctuations $(\delta E_n)_{\omega}^2$ in general differs from Eq. (8.6.11). As it was pointed out in §3.2, near the threshold of instability ($\Delta N = N_{th} - N \ll N_{th}$) the beam dielectric constant $\epsilon_n(\omega)$ can be presented by the expansion $\epsilon_n(\omega) = (\Delta N / N_{th}) + i\epsilon''$. In this case, as the “dispersion” equation takes the form

$$\epsilon''(\omega) = 0, \quad (9.1.4)$$

the equation $\epsilon'(\omega)=0$ determines the threshold current of the instability. Taking this into account, instead of Eq. (8.6.11) for $(\delta E_n)_\omega^2$ we can write

$$\begin{aligned} \frac{\partial}{\partial t}(\delta E_n)_\omega^2 + 2\delta_n(\omega)(\delta E_n)_\omega^2 = N \left| \frac{e\omega_s Z_n}{\Pi(\partial\epsilon_n/\partial\omega)} \right|^2 \left| \frac{\delta\epsilon_n''}{\partial\omega} \right| \delta[\epsilon''(\omega)] \\ \times \int_{-\infty}^{\infty} d\Delta p \delta(\omega - \omega_n) f_0. \end{aligned} \quad (9.1.5)$$

Near the equilibrium the solutions of Eq. (9.1.5) $(\delta E_n)_\omega^2$ are proportional to $\delta[\epsilon_n'']$, while the relaxation times δ_n^{-1} are determined by the decrement of coherent oscillations from Eq. (3.2.23):

$$\delta_n = \left(1 - \frac{N}{N_{th}} \right) \frac{(\partial\epsilon''/\partial\omega)_{th}}{|\partial\epsilon_n/\partial\omega|_{th}^2}.$$

In spite of the formal difference Eq. (9.1.5) describes the same physical phenomena as Eq. (8.6.11): the radiation and absorption of coherent oscillations by the beam particles. Above and just on the instability threshold $N \geq N_{th}$ Eq. (9.1.5) has no stationary solutions

$$\frac{\partial}{\partial t}(\delta E_n)_\omega^2 > 0.$$

As was already mentioned, in this case, the development of coherent oscillation in a beam can be limited either by the action of the nonlinear terms omitted here or by the heating and subsequent pass of the beam below the instability threshold.

Below the threshold of instability $N < N_{th}$ the stationary solution of Eq. (9.1.5) is

$$(\delta E_n)_{\omega, st}^2 = \frac{N}{2} \left| \frac{e\omega_s Z_n}{\Pi} \right|^2 \frac{\delta[\epsilon_n''(\omega)]}{1 - N/N_{th}} \int d\Delta p \delta(\omega - \omega_n) f_0. \quad (9.1.6)$$

Whence, using the relation

$$(\delta E_n)_\omega^2 = \left| \frac{e\omega_s Z_n}{\Pi} \right|^2 (\delta\rho_n)_\omega^2,$$

one can easily obtain Eq. (7.1.16) for $(\delta\rho_n)_{\omega, st}^2$ and the relevant expression for the noise power

$$\langle |\delta\rho_n|^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\delta\rho_n)_\omega^2 = \frac{N}{1 - N/N_{th}} \sum_{\omega_\alpha} \int d\Delta p \frac{f_0(\Delta p) \delta(\omega_\alpha - \omega_n)}{|\partial\epsilon''/\partial\omega|_{\omega=\omega_\alpha}}, \quad (9.1.7)$$

where the summation is carried out over all the roots of Eq. (9.1.4).

In some special cases, Eq. (9.1.7) can be simplified. For instance, assuming a Gaussian momentum distribution function and $U_n < 0$ from Eq. (9.1.7) we can write [see Eq. (7.1.17)]

$$\langle |\delta\rho_n|^2 \rangle = \frac{N}{1 - N/N_{th}}, \quad N_{th} = N_c \quad (9.1.8)$$

As U_n and V_n should not go outside the stability diagram, this expression relatively weakly depends on what part of coupling impedance (active or reactive) determines the beam threshold current.

Another simple case can be met in the region where $U_n > 0$, $U_n \gg (n\Delta\omega_0)^2$, and within the boundary of the instability region also $U_n \gg |V_n|$. For this case, from Eq. (9.1.7) we have

$$\langle |\delta\rho_n|^2 \rangle = \frac{N}{1 - N/N_{th}} \frac{T_{\parallel}}{T_c}, \quad (9.1.9)$$

where

$$T_c = \frac{Ne^2\omega_s}{2\pi} \left(\frac{Z_n''}{n} \right).$$

The reason of an increase in the stationary level of the beam noise near the threshold of instability is evident. The generation frequency of oscillations by particles is determined by the frequency spread in a beam and does not depend of the distance to the threshold, whereas the decrements of oscillations δ_n decrease as $1 - N/N_{th}$. Therefore, the fraction of an energy transferred into the collective degrees of freedom increases. The noise stationary level, as seen from Eq. (9.1.19), is noticeably higher than the thermal level and for the negative mass instability ($U_n < 0$) is noticeably higher than the level of the Schottky noise.

The common evolution of particles and coherent oscillations in the beam is described by Eqs. (9.1.1) and (9.1.5). As below we are not going to detail the variation of the distribution function, let us replace Eq. (9.1.1) by the corresponding Langevin equation

$$\frac{d\sigma_{\parallel}^2}{dt} = \frac{2}{N} \sum_n \int \frac{d\omega}{2\pi} \lim_{\Delta \rightarrow 0} 2\Delta \int d\Delta p \Delta p \langle e\delta E_{n,\omega+i\Delta}^* \delta F_{n,\omega+i\Delta} \rangle. \quad (9.1.10)$$

Repeating here the calculations in §8.3 and taking into account that near the threshold of instability $\epsilon_n(\omega) = \Delta N/N_{th} + i\epsilon_n''$ as well as that $(\delta E_n)_{\omega}^2 \propto \delta[\epsilon_n''(\omega)]$, after simple transformations one gets

$$\frac{d\sigma_{\parallel}^2}{dt} + \sum_n \int_{-\infty}^{\infty} \text{Re} \left[\frac{e^2}{\Omega_n^2} \right] \frac{\partial}{\partial t} (\delta E_n)_{\omega}^2 = -2 \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Im} \left[\frac{e^2}{\Omega_n^2} \right] \Delta\omega_n (\delta E_n)_{\omega}^2. \quad (9.1.11)$$

The right-hand side of Eq. (9.1.11) coincides with the power of the energy losses (per single particle) of the fluctuations in the outer system

$$W = \langle e \delta E \delta j \rangle = -NR_0 \omega'_0 \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \text{Im} \left[\frac{e^2}{\Omega_n^2} \right] \Delta \omega_n (\delta E_n)_{\omega}^2.$$

Thus, defining similarly to Eq. (8.6) the temperatures of collective modes

$$T_n = \frac{R_0}{n^2 \omega_s} \text{Re} \left(\frac{n}{-iZ_n} \right) \int_{-\infty}^{\infty} d\omega (\delta E_n)_{\omega}^2 \quad (9.1.12)$$

and the energy of the thermal motion in the beam

$$\mathcal{E} = \frac{NT_{\parallel}}{2} + \text{sgn}(\omega'_0) \sum_n \frac{T_n}{2}, \quad (9.1.13)$$

one can replace Eq. (9.1.11) by the energy balance equation

$$\frac{d\mathcal{E}}{dt} = W. \quad (9.1.14)$$

As it was found in §8.3, the second term in Eq. (9.1.13) yields the statistically averaged energy of the particle interaction through the system with an impedance Z . At $\omega'_0 Z''_n < 0$ it is negative, which is equivalent to the effective attraction of particles.

Equation (9.1.14) relates the variations of the beam temperature and of the temperatures of coherent fluctuations. In cases when Eq. (9.1.4) has several roots, $\omega = \omega_{\alpha}$, every mode α generally has its own temperature $T_{n,\alpha}$, which therefore will contribute to Eqs. (9.1.12) and (9.1.14). Further we shall assume that the distribution $f_0(\Delta p)$ has a single peak. Then on the instability threshold Eq. (9.1.4) will have only one root with the single (for given n) temperature T_n . The equation describing the variation of T_n can be obtained by integrating Eq. (9.1.5) over ω . Together with Eq. (9.1.14) it gives the system of relaxation equations for the study of the beam kinetics near the threshold of instability. In the final form the equation system is

$$\begin{aligned} \frac{dT_n}{dt} &= -2\delta_n [T_n - \Phi(T_{\parallel})], \\ \frac{dT_{\parallel}}{dt} &= \text{sgn}(\omega'_0) \sum_n \frac{2\delta_n}{N} [T_n - \Phi(T_{\parallel})] + \frac{W}{N}, \end{aligned} \quad (9.1.15)$$

where

$$\Phi(T_{\parallel}) = \frac{e^2 \omega_s}{\Pi} \left(\frac{Z''_n}{n} \right) \langle |\delta \rho_n|^2 \rangle_{st} \quad (9.1.16)$$

is the stationary value of temperature T_n for fluctuations with the noise power $\langle |\delta\rho_n|^2 \rangle_{st}$ [See Eq. (9.1.7).]

Let us discuss some results following from Eqs. (9.1.15). Without the dissipation ($W=0$) Eqs. (9.1.15) indeed coincide with those given in §8.7. The system has the equilibrium solution reached by equalizing the temperatures of fluctuations

$$T_n \rightarrow T_n^{st} = \frac{T_c}{1 - T_c/T_{\parallel}}.$$

The value of the beam longitudinal temperature in equilibrium can be found from the conservation law

$$\mathcal{E} = \frac{NT_{\parallel}}{2} - \frac{N_{ex}}{2} \frac{T_c}{1 - T_c/T_{\parallel}}, \quad (9.1.17)$$

with N_{ex} as the number of unstable modes ($N_{ex} \ll N$). As seen from Eq. (9.1.17), with a decrease in $\mathcal{E}$, the equilibrium value $T_{\parallel}$ tends to T_c .

For more realistic cases, when the beam interacts with dissipative electrodes, Eqs. (9.1.15) have no equilibrium solutions. For the distributions f_0 close to the Gaussian one, on the instability threshold one has

$$\frac{(n\Delta\omega_0)^2}{\Delta\omega_n} \text{Im} \left[\frac{e^2}{\Omega_n^2} \right] = -\pi \int_{-\infty}^{\infty} d\Delta p \delta(\omega - \omega_n) f_0 < 0.$$

This means that the value W is positive and the beam relaxation is accompanied by an increase in $\mathcal{E}$. The particle interaction with the coherent fluctuations leads to the beam heating and to its exit out of the threshold region.

One way to prevent such heating could be the use of beam cooling. In this case, the second equation in Eqs. (9.1.15) should be replaced by

$$\dot{T}_{\parallel} = -\text{sgn}(\omega'_0) \sum_n \frac{\dot{T}_n}{N} + \frac{W}{N} - 2\lambda_0(T_{\parallel} - T_0), \quad (9.1.18)$$

where λ_0 is the cooling decrement, and T_0 is the equilibrium longitudinal temperature of a beam at zeroth current ($N \rightarrow 0$). Let us show now that at sufficiently strong friction and $T_0 < T_{th}$ the equilibrium solutions of the system will be below the threshold of the instability and the equilibrium temperature $T_{\parallel}$ will get closer to the threshold as λ_0 gets larger.

For the single-peak distribution function assumed here one can write

$$\frac{W}{N} = \sum_n \frac{\Delta\omega_n}{Np} T_n \left[\frac{nZ'_n}{Z''_n} \right]. \quad (9.1.19)$$

Substituting Eq. (9.1.19) into Eq. (9.1.18) and taking into account that in equilibrium $T_n = \Phi_n(T_{\parallel})$ we obtain the equation for the equilibrium longitudinal temperature $T_{\parallel} = 0$:

$$T_{\parallel} - T_0 = \frac{1}{2\lambda_0 N} \sum_n \frac{\Delta\omega_n}{n} \Phi_n(T_{\parallel}) [nZ'_n/Z''_n]. \quad (9.1.20)$$

The behavior of roots of Eq. (9.1.20) with a beam current and its longitudinal temperature is determined by the singularity of the factor $\Phi_n(T_{\parallel})$ in the vicinity of T_{th} . For the sake of simplicity let us confine ourselves by calculating $T_{\parallel}$ for two specific cases ($\Omega_n^2 = U_n + iV_n$).

Case 1: $U_n > 0$, $U_n \gg |V_n|$, $(n\Delta\omega_n)^2$. In this region one has

$$\Phi_n \approx \frac{1}{2} \frac{T_{th}}{1 - T_{th}/T_{\parallel}}, \quad T_{\parallel} - T_{th} \ll T_{th}, \quad T_{th} \approx T_c / \ln(Z''_n/Z'_n).$$

Substituting Φ_n and $\Delta\omega_n \approx U_n^{1/2}$ into Eq. (9.1.20) we obtain

$$T_{\parallel} - T_0 = \frac{1}{4\lambda_0 N} \sum_n \left(\frac{U_n^{1/2} Z'_n}{Z''_n} \right) \frac{T_{th}}{1 - T_{th}/T_{\parallel}}. \quad (9.1.21)$$

Taking as previously

$$\frac{Z''_n}{n} \propto \begin{cases} \text{const}, & |n| < n_0, \\ n^{-2}, & |n| > n_0, \end{cases}$$

and using that T_{th} only logarithmically slow depends on n , one can find that only the most unstable modes $|n| < n_0$ give the leading contribution in the r.h.s. of Eq. (9.1.21). To this end, it can be rewritten in the form

$$T_{\parallel} - T_0 = q \frac{T_{th}}{1 - T_{th}/T_{\parallel}} \quad (9.1.22)$$

where

$$q = \frac{n_0}{2\lambda_0 N} \frac{U_n^{1/2}}{n} \left(\frac{nZ'_n}{Z''_n} \right). \quad (9.1.23)$$

Since the ratios $n_0/N \ll 1$ and nZ'_n/Z''_n are generally small, the value of q is not large

$$q \ll 1, \quad \Delta\omega_n \ll N\lambda_0 \left(\frac{Z''_n}{nZ'_n} \right).$$

Under these conditions the character of solutions for Eq. (9.1.22) is evident:

$$T_{\parallel} \simeq \begin{cases} T_0, & T_0 \gg T_{th}, \\ (1+q)T_{th} \simeq T_{th}, & T_0 \ll T_{th}. \end{cases} \quad (9.1.24)$$

Case 2: $U_n < 0$, $|U_n| \gg |V_n|$. In this region one has

$$\Phi_n = \frac{T_c}{1 - T_c/T_{\parallel}}, \quad \Delta\omega_n \simeq n\Delta\omega_0 \left(\frac{Z'_n}{Z''_n} \right)^{1/3}, \quad T_{th} = T_c$$

which yields

$$T_{\parallel} - T_0 = \frac{\Delta\omega_0}{2N\lambda_0} \sum_n \frac{T_c}{1 - T_c/T_{\parallel}} \left(\frac{nZ'_n}{Z''_n} \right)^{4/3} \frac{1}{n^{1/3}}. \quad (9.1.25)$$

The value q in this case can be estimated by

$$q = \frac{\Delta\omega_0}{2N\lambda_0} \sum_n \left(\frac{nZ'_n}{Z''_n} \right)^{4/3} \frac{1}{n^{1/3}} \simeq \frac{n_0^{2/3} \Delta\omega_0}{2N\lambda_0} \left(\frac{nZ'_n}{Z''_n} \right)^{4/3}. \quad (9.1.26)$$

If it is small,

$$q \ll 1, \quad \frac{\lambda_0}{\Delta\omega_0} \gg \frac{n_0^{2/3}}{2N\lambda_0} \left(\frac{nZ'_n}{Z''_n} \right)^{4/3}, \quad (9.1.27)$$

then, like in the previous case, the value of the beam equilibrium longitudinal temperature will be defined by Eq. (9.1.24).

Whence one can conclude that with the stabilization of coherent instabilities by the Landau damping the beam cannot be cooled down to temperatures lower than T_{th} , if only the cooling itself does not vary the beam instability thresholds (which is satisfied at $\lambda_0 < |n\Delta\omega_0|$). The level of coherent fluctuations in the beam in the equilibrium

$$\langle |\delta\rho_n|^2 \rangle \simeq N/q$$

can significantly exceed that of the Schottky noise provided q is small. On the other hand, for the validity of the quasilinear approximation used here the beam noise power $\langle |\delta\rho_n|^2 \rangle$ should not approach N^2 . Therefore, q should be limited from below by the condition $q \gg 1/N$, which is better realized the larger the number of unstable modes n_0 .

If the thresholds of instabilities for various modes are strongly different, the equilibrium state is realized in the vicinity of the threshold with the higher temperature T_{th} . In this case, of course, the corresponding value q should not be too small. Otherwise the calculation of the equilibrium longitudinal temperature can require us to take into account the correlation functions of higher order (or nonlinear terms omitted here).

To conclude this section, let us note that the beam heating considered is due to an increase in the coherent noise level when approaching the instability

threshold. Such an increase is related to the decrease in the damping decrements of modes near the thresholds and it is the common feature for all the coherent instabilities. Therefore, one should expect that due to Landau damping similar heating of the beam near the threshold will also take place for other coherent instabilities. As a way against this heating one can indicate the use of the passive (or active) damping systems. Raising (or canceling) the instability thresholds these systems enable one to cool the beam down to lower temperatures. Figuratively speaking, the cooling of collective degrees of freedom enables one to improve the beam cooling. In these terms the use of the wide bandwidth damping systems allowing the damping of the wide class of instabilities seems to be most preferable.

9.2. WEAK TURBULENCE AND THE BEAM LONGITUDINAL TEMPERATURE

9.2.1. Let us now discuss some simple predictions concerning behavior of the beam above the threshold of coherent instability. Except for the general interest, this scope of phenomena as well as that discussed in the previous section can be related to the so-called bunch lengthening effect observed in many storage rings. In many cases these effects are associated with the unstable longitudinal coherent oscillations of the bunch and related turbulent phenomena.¹⁴⁶ Theoretical and experimental study of this subject shows that depending on the specific conditions, the instability stops when the bunch momentum spread either reaches the value corresponding to the threshold of the instability or overshoots it.^{76,147} The nonlinear saturation of coherent oscillations and associated blow-up of the beam momentum spread have been analyzed in Refs. 148 and 149 assuming that the variations of the momentum distribution function of the beam occur adiabatically slower than amplitudes of coherent oscillations. For beams in the storage ring it seems that this assumption can work well near the thresholds of coherent instabilities. The main result is that the overshoot of the threshold momentum spread has been predicted.

Above the threshold due to the blow-up of oscillations besides the fluctuating background, the systematic coherent oscillations can survive in the beam. If, as usual, such oscillations are described by Vlasov's equation, their wake fields perturb the motion of particles in the beam in the same way as the external field. It conserves the total phase space volume of the beam, but changes the symmetry of its stationary state and due to mismatching of particles phase trajectories with new ring acceptance can increase the effective phase space volume as well as momentum spread in the beam. As the increments of coherent instabilities are functionals of the beam stationary state, both they and coherent tune shifts become dependent on the amplitude of coherent oscillations, which is specific for turbulent oscillations. Calculating such dependencies is a difficult mathematical problem and generally needs some perturbation approach. In this section we shall discuss the simplest case, when the beam and its coherent oscillations evolve near the threshold of the negative mass instability. Besides the mathe-

matical simplification, the results of such calculations can be used as heuristic ones to predict behavior of very short bunches, when one may expect that instability rates will significantly exceed the frequency of synchrotron oscillations.

9.2.2. Without the beam cooling the negative mass instability is described by Vlasov's equation

$$\frac{\partial f}{\partial t} + \omega'_0 \Delta p \frac{\partial f}{\partial \varphi} + eE(\varphi, t) \frac{\partial f}{\partial \Delta p} = 0,$$

$$\omega'_0 = \omega_s \alpha_p / p_s, \quad \alpha_p = \gamma^{-2} - \gamma_{tr}^{-2}. \quad (9.2.1)$$

Since for particles E can be considered as the external field, Eq. (9.2.1) describes the evolution of the beam distribution with the constant phase space volume. Thus, if trajectories of particles in this field are known and, for instance, are written using initial conditions $(\varphi_0, \Delta p_0)$,

$$\varphi_0 = \varphi^{(0)}(\varphi, \Delta p, t), \quad \Delta p_0 = p^{(0)}(\varphi, \Delta p, t), \quad (9.2.2)$$

the solution of Eq. (9.2.1) reads

$$f(\varphi, \Delta p, t) = f^{(0)}[\varphi^{(0)}(\varphi, \Delta p, t), p^{(0)}(\varphi, \Delta p, t)],$$

where $f^{(0)}$ is initial distribution function. But in interesting cases the calculation of trajectories Eq. (9.2.2) in the self-consistent field E is very difficult and, hence, some approximations or numerical methods must be used to perform the particular calculations.

Using the periodicity of E and f in φ we can write

$$E(\varphi, t) = \sum_n E_n(t) \exp(in\varphi),$$

$$f(\varphi, \Delta p, t) = f_0(\Delta p, t) + \sum_{n \neq 0} f_n(\Delta p, t) \exp(in\varphi)$$

and replace Eq. (9.2.1) by the system

$$\frac{\partial f_n}{\partial t} + in\omega'_0 \Delta p f_n = -eE_n \frac{\partial f_0}{\partial \Delta p} - \sum_{n' \neq 0} E_{n-n'}(t) \frac{\partial f_{n'}}{\partial \Delta p}, \quad (9.2.3)$$

$$\frac{\partial f_0}{\partial t} = - \sum_{n' \neq 0} E_{-n'}(t) \frac{\partial f_{n'}}{\partial \Delta p}. \quad (9.2.4)$$

This system becomes self-consistent provided harmonics E_n and f_n are coupled. If we adopt the phenomenological description for the interaction of the beam and surroundings in terms of the coupling impedance $Z_n(\omega)$, the necessary equation reads

$$E_{n,\omega} = -\frac{Ne\omega_s Z_n(\omega)}{\Pi} \rho_{n,\omega}, \quad \rho_{n,\omega} = \int d\Delta p f_{n,\omega}(\Delta p), \quad (9.2.5)$$

$$E_{n,\omega} = \int_0^\infty dt \exp(i\omega t) E_n(t), \quad \text{Im}\omega > 0.$$

In this section we again assume that the impedance $Z_n(\omega)$ is a slow function of the frequency, which can be replaced by $Z_n(n\omega_s) = Z_n$. Besides, we shall use $|Z_n''| \gg |Z_n'|$, $(Z_n = Z_n' + iZ_n'')$ and

$$(Z_n''/n) = \begin{cases} \text{const}, & |n| \ll n_0, \\ \propto 1/n^2, & |n| \gg n_0. \end{cases} \quad (9.2.6)$$

If the amplitudes of harmonics E_n are small, we can find solutions of Eq. (9.2.3), Eq. (9.2.4) as a power series of the parameter

$$\chi_n = \frac{eE_n}{|n\omega_0'| \varphi^2} \ll 1. \quad (9.2.7)$$

The value $|\chi_n|^{1/2}$ estimates the ratio of the bucket size, corresponding to the harmonic E_n , to the beam momentum spread φ . If $|\chi_n|$ is small, only a small part of the beam particles is captured by the bucket. In order to avoid the calculation of the whole series, we shall discuss below only the problems, where the possibility of such a capture is not important, and hence, the analysis can be limited by the calculations of a small amount of terms of these series. On the other hand, since the bucket parameters [the bucket size and the frequency of small oscillations in this bucket] vary due to the variation of $E_n(t)$, it may occur that particles, which were initially uncaptured, will be captured by the bucket in some time. For this reason, the solutions, which will be calculated below, will give an adequate description of the beam only for limited time intervals.

9.2.3 The basic properties of solutions of Eqs. (9.2.3) and (9.2.4) in the linear approximation in χ_n were discussed in §3.2. We remind the reader that, if f_0 is the stationary momentum distribution of the beam, the eigenfrequencies of oscillations $f_n(t) = f_{n,\omega} \exp(-i\omega t)$ are given by the roots of the dispersion equation

$$\epsilon(n,\omega) = 1 + \frac{\Omega_n^2}{n\omega_0'} \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0 / \partial \Delta p}{\omega - n\omega_0(\Delta p)} = 0, \quad \text{Im}\omega > 0, \quad (9.2.8)$$

$$\Omega_n^2 = n^2 \frac{Ne^2 \omega_s \omega_0'}{\Pi} \left(\frac{-iZ_n}{n} \right). \quad (9.2.9)$$

For the negative mass instability [$\Omega_n^2 < 0$, and $|Z_n''| \gg |Z_n'|$] Eq. (9.2.8) has unstable solutions only in the region, where $|\Omega_n|$ exceeds the beam frequency spread $\Delta\omega = |\omega_0'| \varphi$. As was shown in §3.2, if f_0 is an even function of Δp such that $f_0(\Delta p) = f_0(-\Delta p)$, the roots of Eq. (9.2.8), corresponding to unstable

solutions, are $\omega = n\omega_0 + i\delta_n$, where δ_n can be found using the equation

$$\int_{-\infty}^{\infty} d\Delta p \frac{\Delta p \partial f_0 / \partial \Delta p}{(n\omega'_0 \Delta p)^2 + \delta_n^2} = 1/\Omega_n^2. \quad (9.2.10)$$

Its solution has simple asymptotes (see in §3.2)

$$\delta_n = |\Omega_n^2|^{1/2} \begin{cases} 1, & N_{th} \ll N, \\ 1 - N_{th}/N, & |N_{th} - N| \ll N, \end{cases} \quad (9.2.11)$$

where,

$$I_{th} = \frac{e\omega_s}{2\pi} N_{th} = |\alpha_p| (\varphi/p_s)^2 \frac{2\pi p_s v_s}{e^2 \omega_s (Z''_n/n)} \quad (9.2.12)$$

is the threshold current of the beam. The increments of the long-wavelength modes $|m| \ll n_0$ increase with an increase in $|n|$. For the short-wavelength modes $|n| > n_0$ the values δ_n generally decrease and change the sign approximately after the harmonic number

$$n_D = \frac{\sqrt{|\Omega_n^2|}}{\Delta\omega}. \quad (9.2.13)$$

Above the threshold ($N > N_{th}$) the amplitudes $E_n(t)$ and $f_n(t)$ vary according to

$$\frac{dE_n}{dt} = \delta_n E_n, \quad \delta_n > 0, \quad (9.2.14)$$

and

$$f_n(\Delta p, t) = f_n^{(0)} e^{-in\omega'_0 \Delta p t} + \frac{eE_n^{(0)} e^{-in\omega'_0 \Delta p t}}{in\omega'_0 \Delta p + \delta_n} \frac{\partial f_0}{\partial \Delta p} - \frac{eE_n(t)}{in\omega'_0 \Delta p + \delta_n} \frac{\partial f_0}{\partial \Delta p}. \quad (9.2.15)$$

For particles with momenta $|n\omega'_0 \Delta p| \gg \delta_n$ the first and the second terms in this equation are the fast oscillating functions of time. The third term describes the exponential growth of $f_n(t)$. Below, we shall assume that namely this term determines the behavior of $f_n(\Delta p, t)$ on the large time intervals and therefore, shall write Eq. (9.2.15) in the form

$$f_n(\Delta p, t) = - \frac{eE_n(t)}{in\omega'_0 \Delta p + \delta_n} \frac{\partial f_0}{\partial \Delta p}. \quad (9.2.16)$$

9.2.4. Let us discuss now the influence of the nonlinear terms in Eq. (9.2.3), Eq. (9.2.4) on the variation of f_0 . In the lowest approximation in χ_n we can substitute in the right-hand side of Eq. (9.2.4) f_n from Eq. (9.2.16). This is equivalent to the averaging of Eq. (9.2.16) over the time interval

$$1/\Delta\omega \ll \Delta t \ll 1/\delta_n, \quad \Delta\omega = |\omega'_0 \sigma|.$$

The resulting equation reads

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \Delta p} \left[D(\Delta p) \frac{\partial f_0}{\partial \Delta p} \right],$$

$$D = \sum_n |eE_n|^2 \frac{\delta_n}{(n\omega'_0 \Delta p)^2 + \delta_n^2}, \quad \Delta p \gg \frac{e|E_n|}{n\omega'_0}. \quad (9.2.17)$$

It describes the evolution of f_0 in the so-called quasi-nonlinear approximation. Such an equation was used, for instance, in Ref. 148 for the description of the beam behavior above the threshold of the instability. Although Eq. (9.2.17) has the form of the diffusion equation, similar to initial equations (9.2.3) and (9.2.4) Eq. (9.2.17) describes time-inversive processes: the variation of the effective phase space volume of the beam due to its wakes. Direct calculations show⁸⁴ that this variation of f_0 can be expressed in terms of distortions of particle phase trajectories by the self-consistent fields. We can illustrate this fact calculating the rate of the beam entropy

$$\dot{S} = -\frac{d}{dt} \int d\Delta p f(\Delta p, t) \ln(f).$$

In the first approximation in $|\chi_n|^2$ Eqs. (9.2.16) and (9.2.17) result in

$$\dot{S} = \dot{S}_0 + \delta\dot{S} \equiv 0,$$

where

$$\dot{S}_0 = -\delta\dot{S} = \int d\Delta p D(\Delta p) f_0(\Delta p) \left[\frac{\partial}{\partial \Delta p} \ln(f_0) \right]^2.$$

In analogy to $\dot{S}$, the value $\dot{S}_0$ yields the rise time of the effective phase space volume of the beam. For Gaussian distributions it reads

$$\dot{S}_0 = \frac{1}{\sigma^2} \sum_n \frac{\delta_n}{|\Omega_n^2|} |eE_n|^2.$$

The integration of Eq. (9.2.17) with Δp^2 yields

$$\frac{d\sigma^2}{dt} = -2 \sum_n \delta_n |eE_n|^2 \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0}{\partial \Delta p} \frac{\Delta p}{(n\omega'_0 \Delta p)^2 + \delta_n^2}, \quad (9.2.18)$$

$$\sigma^2 = \int d\Delta p \Delta p^2 f_0(\Delta p, t).$$

Using Eq. (9.2.10) we rewrite Eq. (9.2.18) in the form of the energy conservation law

$$\frac{d}{dt} \left\{ \sigma^2 - \sum_n \frac{e^2 |E_n|^2}{|\Omega_n^2|} \right\} = 0. \quad (9.2.19)$$

If σ_{in} is the initial momentum spread in the beam and $E_n^{(0)}$ is initial value of E_n , Eq. (9.2.19) yields

$$\sigma^2(t) = \sigma_{in}^2 + \sum_n \frac{e^2}{|\Omega_n^2|} (|E_n|^2 - |E_n^{(0)}|^2). \quad (9.2.20)$$

Taking into account that $E_n = -Ne\omega_0 Z_n \rho_n / \Pi$, we can also rewrite Eq. (9.2.20) in the form

$$\sigma^2(t) = \sigma_{in}^2 + \sum_n \sigma_c^2(n) (|\rho_n|^2 - |\rho_n^{(0)}|^2),$$

where $\sigma_c(n)$ is the momentum spread, corresponding to the threshold of the instability [see in Eq. (9.2.12)].

9.2.5. Nonlinear corrections to the increments of coherent oscillations can be calculated from Eq. (9.2.3). We shall find its solution as an iteration series

$$f_n(\Delta p, t) = f_n^{(1)} + f_n^{(2)} + f_n^{(3)} + \dots$$

Here, the term of the first order in χ_n is determined by Eq. (9.2.15) [or Eq. (9.2.16)], while the others can be calculated using

$$\frac{\partial f_n^{(m+1)}}{\partial t} + in\omega'_0 \Delta p f_n^{(m+1)} = -e \sum_{n' \neq 0} E_{n-n'} \frac{\partial f_{n'}^{(m)}}{\partial \Delta p}, \quad m = 1, 2, \dots \quad (9.2.21)$$

The nonlinear correction to the increment δ_n is determined by those terms in Eq. (9.2.3), which are proportional to the harmonic E_n . For the sake of simplicity we shall trace the evolution of the particular harmonic $n=q$. Then the terms of interest are caused by the decay of the harmonic q into harmonics $q-n$ and n and their subsequent merging into harmonic q . Moreover, we shall also assume that initial condition for $f(\Delta p, \varphi, t)$ is

$$f_n(\Delta p, t=0) = \delta_{n,q} f_q(\Delta p, t=0).$$

Then, during the evolution the spectrum of $f(\Delta p, \varphi, t)$ will contain only harmonic numbers proportional to q . Since the nonlinearity in the Eq. (9.2.3) is a quadratic one, the increase of the harmonic number in the beam is associated

with the contribution from processes with increasing number of interacting modes. Therefore, for the higher harmonic numbers the amplitudes will be proportional to higher powers of χ_n . In the lowest approximation in χ_n the nonlinear correction to f_q is determined by the harmonic f_{2q}

$$\frac{\partial f_q}{\partial t} + iq\omega'_0 \Delta p f_q = -eE_q \frac{\partial f_0}{\partial \Delta p} - eE_{-q} \frac{\partial f_{2q}}{\partial \Delta p}, \quad (9.2.22)$$

which in the same approximation satisfies the equation

$$\frac{\partial f_{2q}}{\partial t} + 2iq\omega'_0 \Delta p f_{2q} = -eE_{2q} \frac{\partial f_0}{\partial \Delta p} - eE_q \frac{\partial f_q^{(1)}}{\partial \Delta p}. \quad (9.2.23)$$

Since f_q is the fastest harmonic, and since we consider the oscillations near the threshold, the linear part of the Eq. (9.2.23) yields the function of time either decaying or growing slower than the nonlinear correction. Thus, the fastest part of this solution takes the form

$$f_{2q} = -\frac{e^2 E_q^2}{(q\omega'_0)^2} \frac{1}{\Delta p} \frac{\partial}{\partial \Delta p} \left(\frac{1}{\Delta p} \frac{\partial f_0}{\partial \Delta p} \right), \quad |q\omega'_0 \Delta p| \ll \delta_q \quad (9.2.24)$$

After the substitution of Eq. (9.2.24) in Eq. (9.2.22) we obtain the equation describing the evolution of f_q in the cubic approximation in χ_n :

$$\begin{aligned} \frac{\partial f_q}{\partial t} + iq\omega'_0 \Delta p f_q - eE_q \frac{\partial f_0}{\partial \Delta p} = eE_n \frac{e^2 |E_q|^2}{(q\omega'_0)^2} \\ \times \frac{\partial}{\partial \Delta p} \left(\frac{1}{\Delta p} \frac{\partial}{\partial \Delta p} \left(\frac{1}{\Delta p} \frac{\partial f_0}{\partial \Delta p} \right) \right). \end{aligned} \quad (9.2.25)$$

Substituting here

$$f_0 = \frac{1}{\sqrt{2\pi}\sigma} \exp \left(-\frac{\Delta p^2}{2\sigma^2} \right), \quad (9.2.26)$$

we obtain instead of Eq. (9.2.16)

$$f_q(\Delta p, t) = -\frac{eE_q}{iq\omega'_0 \Delta p + \delta_q} (1 - |\chi_n|^2) \frac{\partial f_0}{\partial \Delta p}. \quad (9.2.27)$$

The integration of this equation over Δp yields the dispersion equation

$$\Omega_q^2 (1 - |\chi_q|^2) \int_{-\infty}^{\infty} d\Delta p \frac{\partial f_0}{\partial \Delta p} \frac{\Delta p}{(q\omega'_0 \Delta p)^2 + \delta_q^2 (|\chi_q|^2)} = 1. \quad (9.2.28)$$

It also can be written in the form

$$\epsilon_q(i\delta_q) = \epsilon_q^{(0)} + \delta\epsilon_q = 0,$$

where $\epsilon_q^{(0)}$ is the dielectric constant of the beam calculated in the linear approximation [see Eq. (9.2.8)], while

$$\delta\epsilon_q \simeq |\chi_q|^2 \frac{\Omega_n^2}{(q\omega'_0)^2} \int_{-\infty}^{\infty} \frac{d\Delta p}{\Delta p} \frac{\partial f_0}{\partial \Delta p} = |\chi_q|^2 \frac{\Omega_n^2}{(q\omega'_0\sigma)^2}.$$

Equations (9.2.27) and (9.2.28) show that distortions of phase trajectories by induced fields generally suppress the interaction and thus, stabilizes the oscillations. Using an analogy of Eqs. (9.2.28) and (9.2.10) we can calculate from Eq. (9.2.11) the instantaneous increments of nonlinear coherent oscillations. Taking into account the main powers of $|\chi_q|^2$, we find

$$\delta_q(|\chi_q|^2) = (|\Omega_n^2|)^{1/2} \left\{ \frac{N_{th}}{N} - 1 + |\chi_q|^2 \right\}. \quad (9.2.29)$$

9.2.6. For a given beam temperature, a common evolution of the beam coherent oscillations and of its effective momentum spread will be obtained, if we shall solve Eq. (9.2.19) together with modified Eq. (9.2.14):

$$\frac{d}{dt} |E_q|^2 = -2\delta_q(|\chi_q|^2) |E_q|^2. \quad (9.2.30)$$

These equations have stationary solutions, when $\delta_q(|\chi_q|^2) = 0$, or

$$\frac{\sigma^2}{\sigma_c^2(q)} = 1 - |\chi_q|^2.$$

According to this equation the beam reaches the stationary state above the threshold of the instability and therefore no overshoot phenomena is predicted in the considered approximation.

If the beam is cooled, Eq. (9.2.19) must be modified and will take the form

$$\frac{d}{dt} \left\{ \sigma^2 - \frac{2e^2 |E_q|^2}{|\Omega_q^2|} \right\} = -\lambda(\sigma^2 - \sigma_{st}^2). \quad (9.2.31)$$

Here, σ_{st} is the beam r.m.s. momentum spread without collective effects. If $\sigma_{st} > \sigma_c$, Eqs. (9.2.30) and (9.2.31) have no stationary solutions. On the contrary, above the threshold ($\sigma_{st} < \sigma_c$) the stationary solution of Eqs. (9.2.30) and (9.2.31) exists and has the form

$$\sigma = \sigma_{st} \quad \delta_q(|E_q|_{st}^2) = 0. \quad (9.2.32)$$

If the stationary state is occurred in the close vicinity to the threshold, we find from Eq. (9.2.30)

$$e|E_q| = \frac{qT_{\parallel}}{R_0} \sqrt{1 - T_{\parallel}/T_c}, \quad (9.2.33)$$

$$T_{\parallel} = R_0 |\omega'_0| \sigma^2, \quad T_{\parallel} \leq T_c.$$

Both this stationary state and the found dependence of increments on the amplitude are specific for coherent oscillations in the regime of the weak turbulence.¹⁵⁰

Since Eq. (9.2.30) has no stationary solution below the threshold of the instability, Eq. (9.2.33) determines the value, which exists only in the region $T_{\parallel} \leq T_c$ and vanishes, if $T_{\parallel} \geq T_c$. These properties are specific for the so-called parameter of the order describing the phase transitions of the second kind¹⁵¹ in the beam.

If the beam temperature is not fixed, for a complete description of the beam kinetics near the threshold of collective instability we must add to Eqs. (9.2.19) and (9.2.30) the kinetic equation describing the relaxation of the beam noise. Initial calculations related to this subject, which were performed in §9.1, showed that such a relaxation stops, when the beam temperature becomes slightly above that, corresponding to the threshold of the instability. In this case, the beam momentum spread overshoots the threshold value.

9.3 ON BUNCH LENGTHENING

Let us briefly discuss the observed in many storage rings an increase of the bunch length with the increase in its current. For the definiteness below we discuss the bunch lengthening in e^+e^- storage ring. Experimental studies of this effect have indicated that the observed lengthening of bunches can be caused by two phenomena:

1. the distortion of the synchrotron potential well by the bunch wakes; and
2. the heating of the beam by its off-equilibrium coherent oscillations.

Up to now both cases have been examined in the numerous theoretical and experimental papers (see, for example, in Refs. 76, 152, and 155). Though, it seems that up to now an analytical theory of the bunch lengthening effect, which could enable quantitative predictions, is far from its completion.

Let us first consider the bunch lengthening due to potential well distortion. In this case, except for the external RF-field a particle feels the wake of the head-on particles of the bunch. Assuming the interaction with the wide-band impedance, we can write the induced field in the form:

$$E(\varphi) = -\frac{Ne^2\omega_s}{\Pi} \int_{-\infty}^{\infty} dn Z_n \rho_n e^{in(\varphi - \varphi_s)}. \quad (9.3.1)$$

If the synchrotron radiation power of the particle is $W(\mathcal{E})$, we can write for the synchronous particle

$$\frac{d\mathcal{E}_s}{dt} = \frac{eV\omega_s \cos \varphi_s}{2\pi} - W(\mathcal{E}_s) - v_s \frac{Ne^2\omega_s}{\Pi} \int_{-\infty}^{\infty} dn Z_n \rho_n,$$

and, if $\mathcal{E}_s$ is kept constant, this yields a new synchronous phase:

$$\frac{eV\omega_s \cos \varphi_s}{2\pi} = W + v_s \frac{Ne^2\omega_s}{\Pi} \int_{-\infty}^{\infty} dn Z_n \rho_n. \quad (9.3.2)$$

If $\rho(\varphi)$ is an even function, and since ($Z_n = Z'_n + iZ''_n$)

$$Z'_n = Z'_{-n}, \quad Z''_n = -Z''_{-n}$$

only Z'_n contributes to the variation of the φ_s .

The oscillations of the off-synchronous particle are described by the following equations:

$$\begin{aligned} \frac{d\Delta\omega}{dt} &= -(\omega_c^2)_0 \varphi - 2\lambda \Delta\omega - \frac{Ne^2\omega_s^2 \alpha_p}{\Pi p} \int_{-\infty}^{\infty} dn Z_n \rho_n (e^{in\varphi} - 1) + X(t), \\ \frac{d\varphi}{dt} &= \Delta\omega, \quad \Delta\omega = \frac{\omega_s \alpha_p}{pv} \Delta\mathcal{E}, \quad \lambda = \frac{1}{2} \frac{\partial W}{\partial \mathcal{E}_s}, \\ (\omega_c^2)_0 &= \omega_s^2 \frac{eV \sin \varphi_s \alpha_p}{2\pi p v}. \end{aligned} \quad (9.3.3)$$

Here, φ is the deviation of the particle's phase from the synchronous value, $X(t)$ is the random force, describing the excitations of synchrotron oscillations due to quantum fluctuations of the synchrotron radiation. Below we assume that

$$\langle X(t) \rangle = 0, \quad \langle X(t)X(t') \rangle = \langle X^2 \rangle \delta(t-t').$$

For the heuristic purposes let us discuss the influence of the bunch wakes on the small synchrotron oscillations of particles. Namely, assuming that wake is slow function of φ and can be therefore expanded in Taylor series, we may write in the linear approximation:

$$\begin{aligned} \ddot{\varphi} + 2\lambda \dot{\varphi} + \omega_c^2 \varphi &= X(t), \\ \psi_c^2 &= (\omega_c^2)_0 \left\{ 1 - \frac{Ne^2\omega_s}{v_s eV \sin \varphi_s} \int_{-\infty}^{\infty} dn \left(\frac{-iZ(n)}{n} \right) n^2 \rho(n) \right\}. \end{aligned} \quad (9.3.4)$$

The driven solution of this equation has the form:

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{X(\omega) e^{-i\omega t}}{\omega_c^2 - 2i\lambda\omega - \omega^2},$$

or

$$\varphi(t) = \int_0^t dt_1 X(t_1) e^{-\lambda(t-t_1)} \frac{\sin \omega_c(t-t_1)}{\omega_c}.$$

Its mean square has the asymptote

$$\langle \varphi^2(t \rightarrow \infty) \rangle = \frac{1}{\omega_c^2} \frac{\langle X^2 \rangle}{2\lambda}.$$

Since for Gaussian distributions $\langle \varphi^2 \rangle = \sigma_\phi^2$, and

$$\sigma_0^2 = \frac{1}{(\omega_c^2)_0} \frac{\langle X^2 \rangle}{2\lambda}$$

is the unperturbed bunch length, one can write:

$$\frac{\sigma_\phi^2}{\sigma_0^2} = \frac{(\omega_c^2)_0}{\omega_c^2} = \frac{1}{\left[1 - \frac{Ne^2 \omega_s}{v_s e V \sin \varphi_s} \int dn \left(\frac{-iZ(n)}{n} \right) n^2 \rho(n) \right]}. \quad (9.3.5)$$

For more detailed calculations we need some particular expressions for Z/N and $\rho(n)$. Taking for instance,

$$\frac{-iZ}{n} = \left(\frac{iZ}{n} \right)_0 \frac{n_0^2}{n^2 + n_0^2}, \quad \rho(n) = \exp\left(-\frac{n^2 \sigma_\phi^2}{2}\right),$$

we can estimate the integral in Eq. (9.3.5) for short bunches ($n_0 \sigma_\phi \ll 1$):

$$\int_{-\infty}^{\infty} dn \left(\frac{-iZ(n)}{n} \right) n^2 \rho(n) \simeq \left(\frac{-iZ}{n} \right)_0 n_0^2 \frac{\sqrt{2\pi}}{\sigma_\phi}.$$

Substituting this expression in Eq. (9.3.5), we can write

$$\frac{\sigma_\phi^2}{(\sigma_\phi^2)_0} = \frac{1}{\left[1 - \frac{Ne^2 \omega_s}{v_s e V \sin \varphi_s} \left(\frac{-iZ}{n} \right)_0 n_0^2 \frac{\sqrt{2\pi}}{\sigma_\phi} \right]}. \quad (9.3.6)$$

Since for relativistic electrons $\sin \varphi_s < 0$, Eq. (9.3.6) predicts the bunch lengthening for the case, when

$$\left(\frac{-iZ}{n} \right)_0 < 0,$$

i.e., when the bunch interacts with inductive-type impedance; and vice versa, the bunch shortening, when the bunch interacts with capacitive-type impedance. This result is not very surprising, since the inductive-type coupling decreases and capacitive increases the rigidity of potential well due to driving RF. If the bunch

lengthening is small, Eq. (9.3.6) can be also rewritten in the form

$$\frac{\sigma_\phi^2}{\sigma_0^2} = 1 + \frac{Ne^2\omega_s}{v_s e V \sin \varphi_s} \left(\frac{-iZ}{n} \right)_0 n_0^2 \frac{\sqrt{2\pi}}{\sigma_\phi}. \quad (9.3.7)$$

Taking into account that at relativistic energies $V \propto \gamma^4$, we can rewrite Eq. (9.3.7) in the form, which was used to scale the bunch lengthening experimental data in early papers:⁷⁶

$$\frac{\sigma_\phi^2}{(\sigma_\phi^2)_0} = 1 + C \frac{N}{\gamma^4 \sigma_\phi}. \quad (9.3.8)$$

Here, C is a dimensionless form-factor depending on ring parameters.

The linear density of a bunch, moving in the storage ring, can be calculated, assuming that due to the balance between cooling and diffusion its distribution function f reaches the Boltzman's equilibrium:

$$f \propto \exp[-h(\Delta p, \varphi)/T]. \quad (9.3.9)$$

Here, h is the Hamiltonian, describing the particle synchrotron oscillations neglecting both synchrotron radiation damping and quantum excitation of oscillations. This approach is strongly based on the assumption that collective effects do not perturb the collision integral and therefore, the evolution of the bunch distribution function is governed by the Fokker-Planck equation with the cooling and diffusion coefficients defined by the synchrotron radiation only.

The Hamiltonian part in Eq. (9.3.3) is generated by the Hamiltonian:

$$h = \frac{\Delta\omega^2}{2} - \frac{(\omega_c^2)_0 \varphi^2}{2} + \omega_s \frac{Ne^2 \alpha_p}{\Pi p} \int_{-\infty}^{\infty} dn \frac{-iZ_n}{n} \rho_n e^{in\varphi}. \quad (9.3.10)$$

Hence, we may expect the distribution function in the form:

$$f(\Delta\omega, \varphi) \propto \exp\left(-\frac{h}{2\sigma_\Delta^2}\right) \propto \exp\left(-\frac{\Delta\omega^2}{2\sigma_\Delta^2} - \frac{\varphi^2}{2\sigma_0^2} - \frac{Ne^2\omega_s}{eV \sin \varphi_s \sigma_0^2} \int dn \frac{-iZ_n}{n} \rho_n e^{in\varphi}\right), \quad \sigma_\Delta = \omega_s \sigma_0. \quad (9.3.11)$$

Here, σ_Δ is the equilibrium revolution frequency spread of the bunch, which in this case, does not depend on the beam intensity N . The integration of the distribution function in Eq. (9.3.11) over $\Delta\omega$ yields the linear density of the bunch in the form of the so-called Haissinski equation:¹⁵³

$$\rho(\varphi) = \rho_0 \exp \left(-\frac{\varphi^2}{2\sigma_0^2} - \int_{-\infty}^{\infty} d\varphi' w(\varphi' - \varphi) \rho(\varphi') \right), \quad (9.3.12)$$

$$w(\varphi) = \frac{Ne^2\omega_s}{eV \sin \varphi_s \sigma_0^2} \int_{-\infty}^{\infty} dn \frac{-iZ_n}{n} e^{-in\varphi}.$$

Due to the causality condition $w(\varphi)=0$, when $\varphi < 0$, and thus, Eq. (9.3.12) indeed presents a nonlinear integral equation of the Volterra type. For reasonable wakes:

$$\overline{w^2} = \int_0^{\infty} d\varphi w^2(\varphi) < \infty, \quad (9.3.13)$$

the solution of Eq. (9.3.12) can be found using the iterations:⁴²

$$\frac{\rho(x)}{\rho_0} = \exp [-x^2/2 - u(x)], \quad x = \varphi/\sigma_0,$$

$$u_0 = 0,$$

$$u_1(x) = \int_x^{\infty} dx_1 w(x_1 - x) e^{-x_1^2/2}, \quad (9.3.14)$$

⋮

$$u_{(n+1)}(x) = \int_x^{\infty} dx_1 w(x_1 - x) \exp [-x_1^2/2 - u_{(n)}(x_1)].$$

Due to condition (9.3.13) the function $u(x)$ has the following properties. It is limited:

$$u^2(x) < \int_x^{\infty} dx_1 w^2(x_1 - x) \int_x^{\infty} dx_1 \frac{\rho^2(x_1)}{\rho_0^2} \leq \mathcal{N} \overline{w^2} = u_{\max}^2,$$

$$\mathcal{N} = \int_{-\infty}^{\infty} dx \frac{\rho^2(x)}{\rho_0^2},$$

and it can be normalized:

$$\begin{aligned} \int_{-\infty}^{\infty} dx u^2(x) &\leq \exp(2u_{\max}) \int_{-\infty}^{\infty} dx u_1^2(x) \\ &\leq \exp(2u_{\max}) \int_{-\infty}^{\infty} \frac{dk}{2\pi} |w(k)|^2 e^{-k^2}, \end{aligned}$$

$$w(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} w(k).$$

This means that the potential well distortion $u(x)$ is distributed within a finite interval of x , whose length by the order of the value can be estimated by

$$\Delta x \sim (\overline{w^2})^{1/4}.$$

On its turn, if $|\Delta x| \gg 1$, this relation estimates the lengthening of the bunch as follows:

$$\frac{\sigma}{\sigma_0} \sim \sqrt{\frac{Ne^2 \omega_s}{eV \sin \varphi_s \sigma_0^2} \left| \frac{Z_n}{n} \right|_{n \rightarrow 0}}$$

Since the function $|u(x)|$ is limited, we can write

$$|e^{-u_1} - e^{-u_2}| \leq e^{2u_{\max}} |u_1 - u_2|.$$

Then, the convergency of iterations in Eq. (9.3.14) can be proved using the standard calculations,⁴² which result in:

$$|u_{n+1} - u_n| \propto \frac{(\sqrt{\overline{w^2}})^n}{n!}, \quad n \gg 1.$$

This expression shows that the iterations in Eq. (9.3.14) converge and thus, Eq. (9.3.12) has an unique solution,⁴² if the condition in Eq. (9.3.13) holds. On the other hand, this estimation indicates that, if the potential well distortion is strong ($\overline{w^2} \gg 1$), higher order corrections in the iteration row:

$$u(x) = u^{(1)} + (u^{(2)} - u^{(1)}) + \dots + (u^{(n+1)} - u^{(n)}) + \dots$$

will decay after:

$$n > n_0 \sim \sqrt{\overline{w^2}}.$$

We also note that the discussed here estimation of the convergency of the iterations in Eq. (9.3.14) is based on rather rough estimations of integrals, entering into these equations. Therefore, Eq. (9.3.13) specifies the condition, when Eq. (9.3.12) has the solution beforehand. Its violation does not mean necessarily that Eq. (9.3.12) has no solutions.

As an example, we consider the model, where the coupling impedance is a pure resistive one:

$$Z(n) = Z_0.$$

In this case, the Haissinski equation can be solved directly¹⁵⁴ (see also in reviews 155 and 156), which enables one to trace the main features of its solutions. Such

a possibility indicates at least a heuristic worth of this and similar solvable models.

Using

$$w(\varphi) = \pi Z_0 \operatorname{sgn}(\varphi),$$

and $x = \varphi/\sigma_0$, we rewrite the Haissinski equation in the following form:

$$\ln \rho = -\frac{x^2}{2} - \frac{Ne^2 \omega_s Z_0 \alpha_p}{2v_d^2 p v \sigma_0} \left[\int_{-\infty}^x - \int_x^{\infty} \right] dx' \rho(x'),$$

Transforming this integral equation into the differential one:

$$\frac{\rho'}{\rho} = -x - \xi \rho, \quad \xi = \frac{Ne^2 \omega_s Z_0 \alpha_p}{2v_d^2 p v \sigma_0}, \quad (9.3.15)$$

and taking $\rho = \exp(-x^2/2)\chi(x)$, we find

$$\frac{1}{\chi(0)} - \frac{1}{\chi(x)} = \xi \int_0^x dx' e^{-x'^2/2}$$

or,

$$\rho(\varphi) = \frac{\rho(0) \exp(-x^2/2)}{1 + \xi \rho(0) \int_0^x dx' \exp(-x'^2/2)}. \quad (9.3.16)$$

The constant $\rho(0)$ is calculated using the normalization condition

$$\int_{-\infty}^{\infty} d\varphi \rho(\varphi) = 1,$$

which results in

$$\rho(x) = \frac{1}{\xi'} \frac{\exp(-x^2/2)/(\sqrt{2\pi} \sigma_0)}{\coth(\xi') + \operatorname{erf}(x/\sqrt{2})}, \quad (9.3.17)$$

$$\xi = 2\sigma_0 \xi'.$$

Obtained solution has simple asymptotes. In the region $\xi' \ll 1$ it is close to a Gaussian distribution:

$$\rho(x) \simeq \exp(-x^2/2)/(\sqrt{2\pi} \sigma_0) [1 - \xi' \operatorname{erf}(x/\sqrt{2})],$$

and therefore, the distortion of the bunch length is small.

According to Eq. (9.3.17), the bunch lengthening can be strong in the region $\xi' \gg 1$, where the linear density $\rho(x)$ gets a specific triangle shape (see, for instance, in Refs. 155 and 157):

$$\rho(x) = \begin{cases} 0, & x > 0, \\ \frac{|x|}{\xi} \frac{g(x)}{1 + g(x)(1 - 1/x^2)}, & x < 0, \end{cases} \quad (9.3.18)$$

where

$$g(x) = \frac{\exp(-x^2/2 + 2\xi')}{\sqrt{2\pi}|x|}.$$

In this case, the width of $\rho(x)$ obviously is

$$-2\sqrt{\xi'} \leq x \leq 0,$$

and therefore, the bunch length can be evaluated using the following expression:

$$\sigma_\phi(N) \simeq \sqrt{\frac{Ne^2\omega_s Z_0 \alpha_p}{v_c^2 p v}} \gg \sigma_0. \quad (9.3.19)$$

On the other hand, a strong asymmetry of the linear density in Eq. (9.3.18) indicate that, when starting from Gaussian-like distributions, it can be reached only via strong instability of synchrotron coherent oscillations. Moreover, from Eqs. (9.3.18) and (9.3.15) we may conclude that in case of interest, a strong bunch lengthening is accompanied by significant potential well flattening along the beam. In the last case (see in §5.6), the bunch becomes unstable relative the microwave (or, mode-coupling) instability*. According to Eq. (9.3.17) this instability roughly stops in the region, where the amplitudes $\varphi_{\text{coh}} \sim \sigma_\phi(N)/2$, while the phases ψ_c are concentrated around π . As we saw in previous sections, unstable coherent oscillations can modify the collision integral, so that the effects of the second class should be taken into account anyway.

Presumably, after Ref. 146, the bunch lengthening due to development of coherent instabilities was called the turbulent one. The calculations given in §9.1 and §9.2 definitely indicate that near the thresholds of coherent instabilities the beam has an increased level of coherent fluctuations. Generally, the increments of mode sin that state more or less strongly depend on amplitudes of oscillations and in this sense are turbulent. Even, if such a dependence is rather weak, due to a large number of oscillations simultaneously existing in the beam and due to a random distribution of their initial phases, such a state can strongly resemble the turbulent state. As we found in §9.1, the interaction of particles and coherent oscillations in such a quasi-turbulent state increases the beam heating with an approach to the instability threshold. In the bunched beam due to the relation

*For longitudinal oscillations, this is especially true for bunches, interacting with resistive elements. The important role of the dissipation in the bunch lengthening was pointed out in Ref. 158.

$$T_{\parallel} = R_0 \frac{\Delta\omega_0^2}{|\omega'_0|} = R_0 \frac{\omega_c^2 \vartheta_b^2}{|\omega'_0|} \quad (9.3.20)$$

such a heating is accompanied with the bunch lengthening $l_b = 2R_0\vartheta_b$. (The values ϑ_b and σ_ϕ generally may differ.) The case of slow instability of synchrotron oscillations requires special consideration, which takes into account of the phase motion of particles. For the fast longitudinal instabilities, an increase in the beam longitudinal temperature can be estimated using equations obtained in §9.1. Such a possibility is due to the fact that, as was mentioned in §5.6, for fast instabilities the spectra of longitudinal coherent oscillations for a bunched beam can be obtained from the spectra of a coasting beam after the substitution of $N \rightarrow N\pi/\vartheta_b$. Accordingly, for the bunched beam we can replace Ω_n^2 for

$$\Omega_n^2 = n^2 \frac{Ne^2\omega_s\omega'_0}{l_b} \left(\frac{-iZ_n}{n} \right), \quad |n| < n_0, \quad (9.3.21)$$

and T_c for $T_c(\pi/\vartheta_b)$. Then, as was found in §9.1, the beam equilibrium temperature generally exceeds the threshold one. The distance to the threshold depends on the resistance of electrodes, which cause the instability, and on the cooling decrements. In the region $\omega'_0 < 0$, which is the most specific for the relativistic energies, this distance is defined by Eq. (9.1.26) and it can be small when holds Eq. (9.1.27). Then, the beam equilibrium temperature is estimated by the expression

$$T_{\parallel} \sim T_c/\vartheta_b, \quad T_c \gg T_0. \quad (9.3.22)$$

Since for the non-resonant elements we can take $Z''_n/n = \text{const}$ for $|n| < n_0$, from Eqs. (9.3.22) and (9.3.20) we obtain that the bunch length in equilibrium

$$\vartheta_b^2 \sim \frac{Ne^2\omega_s|\omega'_0|}{l_b\omega_c^2} \left(\frac{-iZ_n}{n} \right), \quad \vartheta_b \gg \vartheta_0, \quad (9.3.23)$$

increases proportional to $N^{1/3}$. Such a dependence of the bunch length on the current was observed in many installations.^{76,80} In addition, from Eq. (9.3.4) it follows that l_b depends on the beam and machine parameters through the scaling parameter

$$\xi'' = \frac{\alpha N}{v_c^2 \mathcal{E}}, \quad (9.3.24)$$

which also agrees with observations.^{155,159} For the ratio l_b/l_0 Eq. (9.3.23) yields

$$(l_b/l_0)^2 \sim C \frac{N\alpha}{\gamma^3 l_b} \left(\frac{Z''_n}{n} \right). \quad (9.3.25)$$

Here, C is a form-factor depending on the ring parameters.

Bibliography

1. E. Courant, H. Snyder. *Ann. Phys.* **3**, p. 1, 1958.
2. A. A. Kolomensky, A. N. Lebedev. *Theory of Cyclic Particle Accelerators*. Wiley, New York, 1962.
3. H. Bruck. *Accelérateurs Circulaires de Particules*. Presses Universitaires de France, Paris, 1966.
4. M. Sands. *The Physics of Electron Storage Rings, An Introduction*. SLAC-121, 1970.
5. V. I. Smirnov. *Course of Higher Mathematic*. v. 3, Moscow, Fizmatgiz, 1958.
6. H. Goldstein. *Classical Mechanics*. Addison-Wesley, Reading, Mass. 1959.
7. N. N. Bogolubov, U. A. Mitropolsky. *Asymptotical Methods in Theory of Nonlinear Oscillations*. Moscow, Fizmatgiz, 1963.
8. Ya. S. Derbenev. Ph.D. Thesis. Novosibirsk, 1968.
9. J. D. Jackson. *Classical Electrodynamics*. Wiley, New York and London, 1962.
10. L. C. Teng. *Transverse Space Charge Effects*. ANLAD-59, Argonne, 1960.
11. J. L. Laslett. In *Proc. of 1963 Summer Study on Storage Rings*, BNL-7534, p. 324, Brookhaven National Laboratory, 1963.
12. F. Sacherer. *IEEE Trans. on Nucl. Science*, NS-18, p. 1105, 1971.
13. S. Machida Ph.D. Thesis. University of Tokyo, 1990.
14. I. S. Gradshteyn, I. M. Ryjik. *Tables of Integrals, Sums and Products*. Academic Press, New York, 1965.
15. L. D. Landau, E. M. Lifshitz. *Mechanics*. Moscow, Fizmatgiz, 1958.
16. U. L. Klimontovich. *Statistical Theory of Nonequilibrium Processes in Plasma*. Moscow Univ., 1964.
17. N. N. Bogolubov. *Problems of Dynamical Theory in Statistical Physics*. Moscow, Gostechizdat, 1946.
18. Ya. S. Derbenev, S. A. Heifetz. *Sov. Journ. of Tech. Phys.* **49**, p. 352, 1979.
19. L. D. Landau, E. M. Lifshitz. *Statistical Physics*. Moscow, Nauka, 1964.
20. A. Piwinski. In *Proc. of the 9th Intern. Conf. on High Energy Accel.* Stanford SLAC, p. 405, 1974.
21. Ya. S. Derbenev. In *Proc of the 6th All Union Part. Accel. Conf.*, v. 1, p. 119, Dubna, 1979.
22. J. D. Bjorken, S. K. Mtingwa. *Part. Accel.* **13**, p. 115, 1983.
23. N. S. Dikansky, D. V. Pestrikov. *Sov. Journ. of Tech. Phys.* **55**, p. 817, 1985.
24. S. T. Belaev. In *Plasma Physics and Problems of Controlled Fusion*. Acad. of Sci. USSR, v. 3, p. 50, 1958.
25. A. N. Lebedev. In *Proc. of the 6th Intern. Conf. on High Energy Accel.* Cambridge. Mass., p. 284, 1967.
26. Ya. S. Derbenev, N. S. Dikansky. Preprint INP 315. Novosibirsk, 1969.
27. Ya. S. Derbenev, N. S. Dikansky. In *Proc. of the 7th Intern. Conf. on High Energy Accel.*, v. 2, p. 294, Yerevan, 1970.
28. N. S. Dikansky. Ph.D. thesis. Novosibirsk, 1969.
29. F. Sacherer. CERN/SI-BR/72-5.
30. Ya. S. Derbenev, N. S. Dikansky, D. V. Pestrikov. In *Proc. of the 2nd All Union Part. Accel. Conf.*, v. 2, p. 62, Moscow, Nauka, 1970.
31. Ya. S. Derbenev, N. S. Dikansky, D. V. Pestrikov. In *Proc. of the 8th Internat. Conf. on High Energy Accel.*, p. 370, Geneva, 1971.
32. Ya. S. Derbenev, N. S. Dikansky, D. V. Pestrikov. Preprint INP 7-72, Novosibirsk, 1972; *Part. Accel.* **8**, p. 129, 1978.
33. N. S. Dikansky, D. V. Pestrikov. Preprint INP 74-94, Novosibirsk, 1974.
34. N. S. Dikansky, V. V. Parkhomchuk, D. V. Pestrikov. *Sov. Journ. of Tech. Phys.*, **46**, p. 2551, 1976.

35. D. V. Pestrikov. Ph.D. thesis. Novosibirsk, 1975.
36. G. I. Budker, A. N. Skrinsky. *Uspekhi Phys. Nauk*, **124**, p. 561, 1978.
37. N. S. Dikansky, D. V. Pestrikov. In *Proc. of the 6th All Union Part. Accel. Conf.*, v. 1, p. 111, Dubna, 1979.
38. C. Pellegrini. *Il Nuovo Cimento*, **64A**, p.447, 1969.
39. M. Sands. *The Head-Tail Effect: An Instability Mechanism in Storage Rings*. SLAC TN-69-8, Stanford, 1969.
40. E. Keil, W. Schnell. CERN-ISR-TH-RF-69-48, CERN, 1969.
41. A. A. Kolomensky, A. N. Lebedev. *Atomnaya Energia*, **7**, p. 549, 1959.
42. F. G. Tricomi. *Integral Equations*. Interscience Publishers, New York and London, 1957.
43. F. Balescu. *Statistical Mechanics of Charged Particles*. Mir (Moscow), 1967.
44. A. Hofmann, G. Muelhaupt. In *Proc of the 8th Intern. Conf. on High Energy Accel.*, p. 306, CERN, 1971.
45. G. I. Budker et al. *Part Accel.* **7**, p. 197, 1976.
46. Ya. S. Derbenev, N. S. Dikansky. *Transverse Coherent Resonant Effects in Storage Rings*. In *Proc. of All Union Part. Acc. Conf.*, Moscow, 1970, v. 2, p. 391.
47. Von E. Madelung. *Die Mathematischen Hilfsmittel des Physikers*. Springer-Verlag, 1957.
48. Mishnev. Ph-D thesis. Novosibirsk, 1970.
49. A. B. Temnikh, G. M. Tumaikin. *Beam-Beam Effects and Limitations on the Luminosity in Colliders*. In *Proc. of the 13th Intern. Conf. on High Energy Accelerators*, Novosibirsk, Nauka, 1987, v. 2, p. 88.
50. J. T. Seeman. *Observation of the Beam-Beam Interaction*. SLAC PUB-3825, SLAC, 1985.
51. S. Krishnagopal, R. Siemann. *Phys. Rev. D* **41**, 2312, 1990.
52. H. Bateman, A. Erdelyi. *Higher Transcendental Functions*. v. 2, McGraw-Hill, 1953.
53. N. S. Dikansky, P. M. Ivanov, D. V. Pestrikov, E. A. Simonov. *Simulation of Beam-Beam Effects for Round Beams in Φ -meson Factory*. In *Proc. of USA Nat. Particle Accelerators Conf.*, San Francisco, May 1991.
54. A. Gerasimov. *Physica D***41**, 89, 1990.
55. D. H. Rice, *Part. Acc.* **31**, 107, 1990.
56. F. M. Izrailev, I. B. Vasserman. In *Proc. of the 7th All Union Conference on Charged Particle Accelerators*, p. 288, Dubna, USSR, 1981.
57. N. Dikansky, D. Pestrikov. *Nonlinear Coherent Beam-Beam Oscillations in the Rigid Bunch Model*. Preprint INP 90-14, Novosibirsk, 1990; in *Proc. of the 2nd European Part. Accel. Conf.* V. 2, 1644, Editions Frontieres, 1990.
58. Ya. S. Derbenev, S. A. Heifets. *Getting the Kinetic Equation for a Model System with Discrete Spectrum without Hypothesis on Decay of Correlations*. INP 76-57, Novosibirsk, 1976.
59. A. Piwinski. In *Proc. of the 8th Intern. Conf. on High Energy Accelerators*, CERN, Geneva, 1971, p. 357.
60. K. Hirata, *NIM*, A-269, 1988, p. 7.
61. N. S. Dikansky, D. V. Pestrikov. In *Proc of the 3d Advanced ICFA Beam Dynamics Workshop on Beam-Beam Effects in Circular Colliders*. Novosibirsk 1989, p. 76.
62. K. Hirata, E. Keil. *Coherent Beam-Beam Interaction Limit in Asymmetric Ring Colliders*. CERN/LEP-TH/89-54, Geneva, 1989.
63. A. A. Alexandrov, D. V. Pestrikov. *Stability of Coherent Beam-Beam Oscillations in Asymmetric Colliders*. INP 90-8, Novosibirsk, 1990.
64. J.E. Augustin et al. In *Proc. of the 7th Intern. Conf. on High Energy Accel.*, Yerevan, 1970, v.2, p. 113.
65. Ya. S. Derbenev. In *Proc. of the 3rd All-union Conf. on Acceler.*, Moscow, 1972, p. 382.
66. E. Keil. In *Proc of the 3d Advanced ICFA Beam Dynamics Workshop on Beam-Beam Effects in Circular Colliders*. Novosibirsk, 1989, p. 85.
67. A. Chao, E. Keil. CERN-ISR-TH/79-30, 1979.
68. V. I. Balbekov, A. A. Kolomensky. *Sov. Journ. of Tech. Phys.* **37**, p. 2021, 1967.
69. N. Dikansky and D. Pestrikov. *Part. Acc.* **12**, p. 27, 1982.
70. A. W. Chao, R. D. Ruth, *Part. Acc.* **16**, p. 201 1985.
71. T. Ieiri, et al. *NIM*, A-265, 1988, p. 364.
72. D. Pestrikov. SLAC-PUB-5510. SLAC, 1991.
73. D. Pestrikov. SLAC-PUB-5575. SLAC, 1991.
74. *An Asymmetric B Factory Based on PEP*. Conceptual Design Report. LBL PUB-5303, SLAC-372, CALT-68-1715, UCRL-ID-106426, UC-IIRPA-91-01, 1991.

75. V. L. Auslander, N. S. Dikansky, et al. *Atomnaya Energia*, **22**, p. 198, 1967.
76. F. Amman. *IEEE Trans. on Nucl. Sci.*, NS-16, p. 1073, 1969.
77. N. S. Dikansky, M. M. Karliner et al. *Atomnaya Energia*, **22**, p. 188, 1967.
78. M. M. Karliner. Doctor of Science thesis. Novosibirsk, 1973.
79. H. Bateman, A. Erdelyi. *Higher Transcendental Functions*, v.1, McGraw-Hill, 1953.
80. A. Chao. *Coherent Instabilities in Relativistic Beams*. SLAC-PUB-2949. SLAC, 1982.
81. R. D. Ruth, J. M. Wang. *IEEE Trans. on Nucl. Sci.*, NS-28, p. 2405, 1981.
82. W. K. H. Panofski, M. Bander. *Rev. Sci. Instrum.* **39**, p. 206, 1968.
83. D. V. Pestrikov. Preprint INP 85-105. Novosibirsk, 1985; see also Pestrikov in *Proc. of the 4th Advanced ICFA Beam Dynamics Workshop on Collective Effects in Short Bunches*, p. 118, KEK Report 90-21, KEK, 1991.
84. D. V. Pestrikov. *Collective Kinetic Effects in Storage Rings*. Doctor of Science thesis. Novosibirsk, 1986.
85. V. E. Balakin, G. I. Budker, A. N. Skrinsky. In *Proc. of the 6th All Union Part. Accel. Conf.*, v. 1, p. 27, Dubna, 1979.
86. V. E. Balakin, A. V. Novokhatski, V. P. Smirnov. In *Proc. of the 12th Intern. Conf. on High Energy Accel.*, p. 119, Fermilab, 1983.
87. V. I. Kudelainen, V. V. Parkhomchuk, D. V. Pestrikov. *Sov. Journ. of Tech. Phys.* **53**, p. 870, 1983.
88. A. A. Mikhailichenko, V. V. Parkhomchuk. Preprint INP 91-55, Novosibirsk, 1991.
89. R. D. Kohaupt. DESY Report 80/22, 1980.
90. N. S. Dikansky, A. N. Skrinsky. *Atomnaya Energiya*. **21**, p. 176, 1966.
91. Ya. S. Derbenev, N. S. Dikansky. *Ibid.*, **22**, p. 191, 1967.
92. C. Pellegrini. In *Physics with Intersecting Storage Rings*. Acad. Press, New York, p. 221, 1971.
93. M. M. Karliner. *Sov. Journ. of Tech. Phys.*, **41**, p. 1806, 1971.
94. V. L. Auslander. Ph.D. thesis. Novosibirsk, 1965.
95. V. L. Auslander, G. I. Budker, et al. In *Proc. of the 7th Intern. Conf. on High Energy Accel.*, v. 2, p. 26, Yerevan, 1969.
96. T. P. Starodubtseva. Ph.D. thesis. Novosibirsk, 1967.
97. J. Borer, P. Bramham, et al. In *Proc. of the 9th Intern. Conf. on High Energy Accel.*, p. 53, Stanford, SLAC, 1974.
98. V. V. Parkhomchuk, D. V. Pestrikov. *Sov. Journ. of Tech. Phys.*, **50**, p. 1411, 1980.
99. E. N. Dementiev, N. S. Dikansky, et al. *Sov. Journ. of Tech. Phys.*, **50**, p. 1717, 1980.
100. S. Van-der-Meer. *Stochastic Damping of Betatron Oscillations in the ISR*. CERN-ISR-PO-72-31, CERN 1972.
101. F. Sacherer. *Stochastic Cooling Theory*. CERN-ISR-TH 11, CERN, 1978.
102. G. I. Budker, A. F. Bulushev, N. S. Dikansky, et al. In *Proc. of the 10th Internat. Conf. on High Energy Accel.*, v. 1, p. 498, Serpukhov, 1977.
103. D. Möhl. In *CAS: Antiprotons for Collid. Beam Facil.*, p. 97, Geneva, CERN, 1984.
104. V. V. Parkhomchuk, A. N. Skrinsky. In *Phys. of Elem. Part. and Atomic Nucl.(JINR)*, **12**, p. 557, Dubna, 1981.
105. D. Möhl, et al. *Phys. Reports*, **58**, p. 75, 1980.
106. J. J. Bisognano. In *Proc. of the 11th Intern. Conf. on High Energy Accel.*, p. 772, Geneva, CERN, 1980.
107. N. I. Zinevich, M. M. Karliner. Preprint INP 82-43, Novosibirsk, 1982.
108. V. V. Parkhomchuk, D. V. Pestrikov. Preprint INP 80-170, Novosibirsk, 1980.
109. E. N. Dementiev, N. I. Zenevich, et al. *Sov. Journ. of Tech. Phys.*, **52**, p. 1993, 1982.
110. S. Van-der-Meer. CERN-PS-AA 78-22, Geneva, CERN, 1978.
111. E. A. Crosbie. FNAL $\bar{p}$ -Note, Chicago, FNAL, 1979.
112. G. I. Budker. *Atomnaya Energia*, **22**, p. 346, 1967.
113. Ya. S. Derbenev, A. N. Skrinsky. *Kinetics of Electron Cooling of Beams in Storage Rings of Heavy Particles*. Preprint INP-225, Novosibirsk, 1968.
114. G. I. Budker et al. In *Proc of the 4th All Union Part. Accel. Conf.*, v. 2, p. 309, Nauka, 1975.
115. I. N. Meshkov. Doctor of Science thesis. Novosibirsk, 1975.
116. N. S. Dikansky. Doctor of Science thesis. Novosibirsk, 1975.
117. Ya. S. Derbenev. Doctor of Science thesis. Novosibirsk, 1978.
118. G. I. Budker et al. *Part. Accel.*, **7**, p. 197, 1976.
119. N. S. Dikansky et al. *Sov. Journ. of Tech. Phys.*, **46**, p. 2551, 1976.

120. G. I. Budker, A. F. Bulushev, Ya. S. Derbenev, et al. In Proc of the 10th Internat. Conf. on High Energy Accel., Serpukhov, 1977, v. 1, p. 498.
121. Ya. S. Derbenev, A. N. Skrinsky. In Proc of the 10th Internat. Conf. on High Energy Accel., v. 1, p. 516, Serpukhov, 1977.
122. V. V. Parkhomchuk, D. V. Pestrikov. Energy Spread of Heavy Particles in Storage Ring with Electron Cooling. Preprint INP 77-37, Novosibirsk, 1977.
123. G. I. Budker, N. S. Dikansky, I. N. Meshkov, et al. In Proc. of the 10th Internat. Conf. on High Energy Accel., Protvino, 1977, v. 2, p. 141.
124. Ya. S. Derbenev, A. N. Skrinsky. Preprint INP 79-87, Novosibirsk, 1979.
125. Ya. S. Derbenev, A. N. Skrinsky. In Physics Reviews, ser. Soviet Physical Reviews, v. 3, p. 165, 1981.
126. V. V. Parkhomchuk. In Proc. of ECOOL 1984, p. 71, Kernforschungszentrum Karlsruhe, 1984.
127. N. S. Dikansky, D. V. Pestrikov. The Influence of the Ordering Effects on Relaxation of Coasting, Cold Beam in Storage Rings. Preprint INP 84-48, Novosibirsk, 1984; see also in Proc. of ECOOL 1984, p. 275, Kernforschungszentrum Karlsruhe, 1984; Sov. Journ. of Tech. Phys. 56, N 2, p. 289 and N 3, p. 505, 1986.
128. V. V. Parkhomchuk. Doctor of Science thesis. Novosibirsk, 1984; Sov. Journ. of Tech. Phys. 56, N2, p. 289 and N3, p. 505, 1986.
129. N. S. Dikansky, V. I. Kononov, V. I. Kudelainen, et al. In Proc of the 6th All Union Part. Accel. Conf., Dubna, 1979, v. 1, p. 99.
130. V. V. Parkhomchuk, D. V. Pestrikov, A. G. Ruggiero. IEEE Trans. on Nucl. Science, 1979, NS-26, p. 3487.
131. N. S. Dikansky, et al. Preprint INP 87-102, Novosibirsk, 1987.
132. N. S. Dikansky, D. V. Pestrikov. Preprint INP 76-40. Novosibirsk, 1976.
133. D. V. Pestrikov. NIM, 1989, A274, p. 435.
134. V. I. Kudelainen, V. A. Lebedev, I. N. Meshkov, et al. Sov. JETPh, 1982, 82, p. 2056.
135. M. Bell, J. Chaney, F. Krienen, et al. NIM, 1981, 190, p. 237.
136. R. Foster, T. Hardek, D. E. Johnson, et al. IEEE Trans. on Nucl. Science. 1981, NS-28, p. 2386.
137. L. D. Landau Collection of Papers. Nauka, 1969, v. 1, p. 199.
138. N. S. Dikansky, V. V. Parkhomchuk, D. V. Pestrikov. In Proc. of the 6th All Union Part Accel. Conf. Dubna, 1979, v. 1, p. 107.
139. VAPP-NAP Group. In Proc of the 8th Intern. Conf. on High Energy Accel. Geneva, CERN, 1971, p. 72.
140. J. P. Hansen. Phys. Rev., 1973, A8, p. 3096.
141. E. L. Pollock, J. P. Hansen. Phys. Rev., 1973, A8, p. 3110.
142. A. G. Ruggiero. In Proc. of the 10th Intern. Conf. on High Energy Accel., Protvino, 1977, v. 2, p. 260.
143. A. Isihara. Statistical Physics.
144. R. W. Hasse, J. P. Schiffer. In Proc. of the Workshop on Crystalline Ion Beams, Wertheim, 1988, GSI-89-10, GSI, 1989, p. 33.
145. D. V. Pestrikov. Collective and Kinetic Effects in Storage Rings. Doctor of Science thesis. Novosibirsk, 1986.
146. P. J. Channell, A. M. Sessler. NIM, 136, p. 473, 1976.
147. A. G. Ruggiero. In Proc. of the 10th Intern. Conf. on High Energy Acceler., v. 2, p. 260, Serpukhov, 1977.
148. Y. Chin, K. Yokoya. Phys. Rev. D 28, p. 2141, 1983.
149. S. A. Bogacz, K.-Y. Ng. Fermilab-PUB-87/55, Fermilab, 1987.
150. L. D. Landau. DAN USSR, 44, p. 339, 1944.
151. L. D. Landau, E. M. Lifshitz. Statistical Physics. Moscow, Nauka, 1964.
152. A. N. Lebedev. On the Bunch Lengthening Effect in Storage Rings. In Physics with Intersecting Storage Rings. New York, Acad. Press, 1971, p. 184.
153. J. Haissinski. Il Nuovo Cimento, 18B, no. 1, p. 72, 1973.
154. A. Ruggiero et al. IEEE Trans. on Nucl. Sci., NS-24, no. 3, p. 1205, 1977.
155. K. Bane. Bunch Lengthening in the SLC Damping Rings. SLAC-PUB-5177, SLAC, 1990.
156. B. Zotter. In Proc. of the 4th Advanced ICFA Beam Dynamics Workshop on Collective Effects in Short Bunches. KEK Report 90-21, KEK, 1991.
157. A. Burov. Sov. Journ. of Tech. Phys., 56, p. 1941, 1986.
158. M. Month, E. Messerschmidt. IEEE Trans. on Nucl. Sci., NS-24, p. 1208, 1977.
159. P. B. Wilson et al. IEEE Trans. on Nucl. Sci., NS-24, p. 1211, 1977.

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