ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

B.G. Konopelchenko

METHOD OF RECURSION OPERATOR FOR PSEUDODIFFERENTIAL SPECTRAL PROBLEMS

1. DISPERSIVE LONG WAVES EQUATIONS

PREPRINT 86-17

НОВОСИБИРСК

METHOD OF RECURSION OPERATOR FOR PSEUDODIFFERENTIAL SPECTRAL PROBLEMS. I. DISPERSIVE LONG WAVES EQUATIONS.

B.G.Konopelchenko

Institute of Nuclear Physics,
630090, Novosibirsk, U S S R

A b s t r a c t

Recursion operator, infinitedimensional group of general Backlund transformations, infinitedimensional group of symmetry and the compact form of hierarchy of equations for dispersive long waves equations are found. The corresponding infinite family of Hamiltonian structures is constructed.

1. Recently A.G.Reyman [1] and B.Kupershmidt [2] shown that the dispersive long waves equations

$$u_t = \frac{1}{2}(u^2 + 2h - u_x)_x,$$
$$h_t = \frac{1}{2}(2uh + h_x)_x \tag{1}$$

are integrable by the inverse scattering transform method with the spectral problem $(\partial \equiv \frac{\partial}{\partial x})$

$$(\partial + u + h\partial^{-1})\psi = \lambda\psi. \tag{2}$$

Equations (1) have been studied earlier in [3-4]. In paper [2] the hierarchy of equations associated to (1) has been constructed and three Hamiltonian structures for this hierarchy has been found.

In the present paper it is shown that the method of recursion operator which has been developed for some matrix and differential spectral problems [5-13] is also applicable to integro-diffewntial problem (2). Recursion operator for spectral problem (2) is calculated. General form of evolution systems integrable by (2) and corresponding infinitedimensional abelian group of general Backlund transformations (BTs) and infinitedimensional abelian symmetry group are found. The infinite family of Hamiltonian structures for equation (1) and whole hierarchy is constructed. The quasiclassical limit of equation (1) and corresponding hierarchy is discussed. The generalization of the problem (2) and the general form of corresponding integrable equations are also considered.

2. Firstly we represent problem (2) in the matrix form

$$\frac{\partial F}{\partial x} = PF = \begin{pmatrix} 0, & 1 \\ -h, & \lambda - u \end{pmatrix} F \tag{3}$$

where $F = (\partial^{-1}\psi, \psi)^T$. Then we introduce the quantity Φ : $(\Phi^{\alpha\beta})_{\gamma\delta} = F'_{\gamma\beta}(F^{-1})_{\alpha\delta}$ $(\alpha,\beta,\gamma,\delta = 1,2)$ where $\partial F' = P'F'$, $\partial F = PF$. This quantity obeys the equation

$$\frac{\partial \Phi^{\alpha\beta}}{\partial x} = P'\Phi^{\alpha\beta} - \Phi^{\alpha\beta}P. \tag{4}$$

Let us multiply (4) by matrix $B(\lambda,t) = \begin{pmatrix} a(\lambda,t) & b(\lambda,t) \\ 0 & a(\lambda,t) + \lambda b(\lambda,t) \end{pmatrix}$ where $a(\lambda,t)$ and $b(\lambda,t)$ are arbitrary functions entire on λ.

Such a matrix B gives the general form of matrix commuting with $A = P_\infty = \begin{pmatrix} 0 & 1 \\ 0 & \lambda \end{pmatrix}$. Taking the trace and integral from the equation obtained we get

$$tr\left(B\left(\Phi^{\alpha\beta}_{(x=+\infty)} - \Phi^{\alpha\beta}_{(x=-\infty)}\right)\right) = \int_{-\infty}^{+\infty} dx\, tr\left(\left(B\tilde{P}' - \tilde{P}B\right)\Phi^{\alpha\beta}_{(x,\lambda)}\right) \tag{5}$$

where $\tilde{P} = P - P_\infty = \begin{pmatrix} 0 & 0 \\ -h & -u \end{pmatrix}$. There always exist the subspace Φ^Δ for which l.h.s. of (5) is equal to zero. As a result

$$\int_{-\infty}^{+\infty} dx\, tr\left(\left(B(\lambda,t)\tilde{P}'(x,t) - \tilde{P}(x,t)B(\lambda,t)\right)\Phi^\Delta(x,\lambda)\right) = 0. \tag{6}$$

The equality (6) is equivalent to the following

$$\int_{-\infty}^{+\infty} dx \Big\{ a(\lambda,t)\left[(h'-h)\Phi_2 + (u'-u)\Phi_4\right] + \tag{7}$$
$$+ b(\lambda,t)\left[h'\Phi_1 + u'\Phi_3 - h\Phi_4 + \lambda\left(h'\Phi_2 + (u'-u)\Phi_4\right)\right]\Big\} = 0$$

where $\Phi^\Delta = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}$.

The equality (7) contains the explicit dependence on λ. This explicit dependence on λ can be eliminated by the use of relations

$$\begin{aligned} \Phi_1 &= -\Phi_4 - \partial^{-1}\left[(h'-h)\Phi_2 + (u'-u)\Phi_4\right], \\ \Phi_3 &= -\partial\Phi_4 - h'\Phi_2 - (u'-u)\Phi_4 \end{aligned} \tag{8}$$

and

$$\lambda\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix} = \Lambda\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix} \tag{9}$$

where

$$\Lambda = \begin{pmatrix} I\left(\partial^2 + \partial(u'-u) - (h'-h) + u'(\partial + u'-u)\right), & I\left(\partial h' + h'\partial + h'(u'-u)\right) \\ 2 + \partial^{-1}(u'-u), & -\partial + u + \partial^{-1}(h'-h) \end{pmatrix} \tag{10}$$

where $If \overset{def}{=} \exp(\partial^{-1}(u-u'))\partial^{-1}(\exp\partial^{-1}(u-u')\cdot f)$. These relations follows directly from (4).

The operator Λ is the recursion operator. Using (8) and $\varphi(\lambda)\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix} = \varphi(\Lambda)\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix}$ for entire function $\varphi(\lambda)$, we have

$$\text{l.h.s. (7)} = \int dx \left\{ a(\lambda,t)(u'-u, h'-h)\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix} + (\mathcal{L}_1^+, \mathcal{L}_2^+)b(\lambda)\begin{pmatrix} \Phi_4 \\ \Phi_2 \end{pmatrix}\right\} = \tag{11}$$
$$= \int dx \left\{ (\Phi_4, \Phi_2)\left[a(\Lambda^+,t)\begin{pmatrix} u'-u \\ h'-h \end{pmatrix} + b(\Lambda^+,t)\begin{pmatrix} \mathcal{L}_1^+ \\ \mathcal{L}_2^+ \end{pmatrix}\right]\right\} = 0$$

where

$$\begin{aligned} \mathcal{L}_1^+ &= h'-h + 2u'_x - u_x + \left[h'-h + u_x + 2(u'-u)(\partial - u')\right]\left(1 - \exp\partial^{-1}(u-u')\right), \\ \mathcal{L}_2^+ &= h'_x + \left[\partial h' - 2h'(u'-u)\right]\left(1 - \exp\partial^{-1}(u-u')\right). \end{aligned} \tag{12}$$

and

$$\Lambda^+ = \begin{pmatrix} -\left[\partial^2 - (u'-u)\partial - (h'-h) + (-\partial + u'-u)u'\right]I, & 2 - (u'-u)\partial^{-1} \\ \left(\partial h' + h'\partial - h'(u'-u)\right)I, & \partial + u - (h'-h)\partial^{-1} \end{pmatrix} \tag{13}$$

From (11) one obtain the transformation $(u,h) \to (u',h')$:

$$a(\Lambda^+,t)\begin{pmatrix} u'-u \\ h'-h \end{pmatrix} + b(\Lambda^+,t)\begin{pmatrix} \mathcal{L}_1^+ \\ \mathcal{L}_2^+ \end{pmatrix} = 0 \tag{14}$$

where $a(\Lambda^+,t)$ and $b(\Lambda^+,t)$ are arbitrary functions entire on Λ^+.

The transformations (14) form the abelian infinitedimensional Backlund-Calogero group of general Backlund transformations (BTs) for spectral problem (2). The construction of this group is a main aim of the method of recursion operator [10-13].

Considering the infinitedimensional displacement in time $t \to t' = t + \varepsilon$, $\varepsilon \to 0$, $u' = u + \varepsilon\frac{\partial u}{\partial t}$, $h' = h + \varepsilon\frac{\partial h}{\partial t}$, $a = 1$,

$b = 1 - \varepsilon\Omega(\lambda, t)$ we obtain from (14) the evolution system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ h \end{pmatrix} = \Omega(L^{+}, t)\,\partial\begin{pmatrix} u \\ h \end{pmatrix} \qquad (15)$$

where $\Omega(L^{+}, t)$ is arbitrary function entire on L^{+} and operator $L^{+} = \Lambda^{+}|_{p'=p}$ is

$$L^{+} = \begin{pmatrix} -\partial + \partial u \partial^{-1}, & 2 \\ (\partial h + h\partial)\partial^{-1}, & \partial + u \end{pmatrix}. \qquad (16)$$

Formula (15) gives the general form of nonlinear evolution systems, integrable by the spectral problem (2). The hierarchy (15) coincides with the hierarchy of equations found in [2]. The form (15) of this hierarchy is more explicit and compact. The system (1) corresponds to $\Omega = \frac{1}{2} L^{+}$.

The infinitedimensional group (14) with time-independent $a(\Lambda^{+})$ and $b(\Lambda^{+})$ is the group of general auto BT for equations (15) and, in particular, for system (1). The simplest auto BT is

$$a(u'-u) + b\{h'-h + 2u'_x - u_x + [h'-h + u_x + 2(u'-u)(\partial - u')](1 - \exp\partial^{-1}(u-u'))\} = 0\,, \qquad (17)$$

$$a(h'-h) + b\{h'_x + [\partial h' - 2h'(u'-u)](1 - \exp\partial^{-1}(u-u'))\} = 0$$

where a and b are arbitrary constants. Formula (17) gives the spatial part of BT which is universal.

Backlund-Calogero group (14) contains also the infinite-dimensional symmetry group of equations (15) as the subgroup with $b/a = \exp f(\Lambda^{+})$ where $f(\Lambda^{+})$ is an arbitrary entire function. In the infinitesimal form these symmetry transformations are

$$\begin{pmatrix} \delta u \\ \delta h \end{pmatrix} = f(L^{+})\begin{pmatrix} \partial u \\ \partial h \end{pmatrix}.$$

Using the technique developed in [10-12] one can show that equations (15) are Hamiltonian systems with respect to the infinite family of Hamiltonian structures. The corresponding family of Poisson brackets is of the form

$$\{\mathcal{F}, \mathcal{H}\}_{\varphi} = \int_{-\infty}^{+\infty} dx \left(\frac{\delta\mathcal{F}}{\delta u}, \frac{\delta\mathcal{F}}{\delta h}\right) \varphi(L^{+}) \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta\mathcal{H}}{\delta u} \\ \frac{\delta\mathcal{H}}{\delta h} \end{pmatrix} \qquad (18)$$

where $\varphi(L^{+})$ is arbitrary entire function. Note that $\varphi(L^{+})\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}\varphi(L)$. Three Hamiltonian operators which have found in [2] are $B^{I} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}$, $B^{II} = L^{+}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} L$, $B^{III} = L^{+2}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} L^{2}$.

An important feature of the Poisson brackets (18) is that the three Hamiltonian operators B^{I}, B^{II}, B^{III} are pure differential operators while for previously known examples only first Hamiltonian structure (NLS equation, N-waves equation,..) or first and second Hamiltonian structures (KdV and Gelfand-Dikij equations) are pure differential.

Equations (15) for any $\Omega(L^{+})$ admit the reductions $h = 0$ and $h = u_x$. In both cases the hierarchy (15) is reduced to the Burgers hierarchy. For reduction $h = u_x$ the spectral problem (2) is $\mathcal{L}\psi = \partial(\partial + u)\partial^{-1}\psi = \lambda\psi$. Calculating the recursion operator for this reduction one obtains $\lambda\bar{\psi} = \partial^{-1}(-\partial + u)\partial\bar{\psi} = \mathcal{L}^{+}\bar{\psi}$ where $\bar{\psi} = \varphi_1 - \partial\varphi_2$. For $\Omega = (L^{+})^{2n}$ equations (15) admit also the reduction $u = 0$. In this case the hierarchy (15) is reduced to KdV hierarchy [2].

3. Consider now a quasiclassical limit of equations (15). Follows to Zakharov [14] this means the change $\partial_t \to \varepsilon\partial_t$, $\partial_x \to \varepsilon\partial_x$ with $\varepsilon \to 0$. As a result we obtain the quasiclassical limit of hierarchy (15)

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ h \end{pmatrix} = \Omega(L^{+}_{qc})\begin{pmatrix} \partial u \\ \partial h \end{pmatrix} \qquad (19)$$

where

$$L^{+}_{qc} = \begin{pmatrix} \partial u \partial^{-1}, & 2 \\ (\partial h + h\partial)\partial^{-1}, & u \end{pmatrix}. \qquad (20)$$

In the infinitesimal form the symmetry transformations for (19) are $\begin{pmatrix} \delta u \\ \delta h \end{pmatrix} = f(L^{+}_{qc})\begin{pmatrix} \partial u \\ \partial h \end{pmatrix}$.

Family of equations (19) is Hamiltonian with respect the infinite family of Hamiltonian structures of the form (18) where one should put operator L^{+}_{qc} instead of operator L^{+}.

Follows to [14] the quasiclassical limit of spectral problem (1) can be obtained by change $\partial_x \to \varepsilon\partial_x$ and $\psi = \exp(\frac{1}{\varepsilon}\partial^{-1}\chi)$.

As a result one has

$$\chi^2 + u\chi + h = \lambda\chi . \tag{21}$$

Evolution in time of χ is given by equation [14]

$$\frac{\partial \chi}{\partial t} = \frac{\partial}{\partial x} N(\chi) \tag{22}$$

where $N(\chi)$ is a certain function. In particular for the dispersiveless long waves equations

$$u_t = \left(\frac{u^2}{2} + 2h\right)_x , \quad h_t = (uh)_x \tag{23}$$

one has $N = \frac{1}{2}\chi^2 + u$. In virtue of (22) $C(\lambda) = \int_{-\infty}^{+\infty} dx\, \chi(x,\lambda)$ is the integral of motion for any λ . As a result for equations (19) we have two infinite series of integrals of motion $C_\pm(\lambda) = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left(\lambda - u(x) \pm \sqrt{(\lambda - u(x))^2 - 4h(x)}\right)$. Expanding $C_\pm(\lambda)$ into asymptotic series on λ^{-1} $\left(C_\pm(\lambda) = \sum_{n=0} \lambda^{-n} C_\pm^n\right)$ one can easily find the explicit form of the integrals of motion $C_\pm^n$. One can prove that $\{C_\pm(\lambda), C_\pm(\lambda')\}_\varphi = 0$ for any function φ. So all integrals of motion $C_\pm^n$ are in involution with respect to any quasiclassical Poisson bracket (18).

Zakharov [14] obtained the system (23) as a quasiclassical limit nonlinear Schroedinger (NLS) equation. The corresponding auxiliary problem differs from (21)-(22). But it is easy to show that this difference is trivial: $\chi(u,h,\lambda) = \chi^{Zakharov}(-u,h,\lambda) + \lambda$. In paper [14] it was also shown that the integrals $C_\pm^{n\,Zakh}$ are in involution with respect the simplest Poisson bracket $\{,\}_{\varphi=1}$.

4. Spectral problem (2) is equivalent to differential spectral problem $(\partial^2 + u\partial + h)\bar{\varphi} = \lambda\partial\bar{\varphi}$. This spectral problem has a natural generalization

$$(\partial^2 + u\partial + h)\psi = \lambda\rho(x)\partial\psi \tag{24}$$

where $\rho(x,t)$ is a scalar function and $\rho \xrightarrow[|x|\to\infty]{} 1$. Nonessentially modifying the construction of section 2 one can find the general form of evolution systems integrable by (24). This systems are of the form

$$\begin{pmatrix} u_t \\ h_t \end{pmatrix} - L^+ \begin{pmatrix} \rho_t \\ 0 \end{pmatrix} - \Omega(L^+,t) \begin{pmatrix} \partial u + (u-\partial)\partial\left(\frac{1}{\rho}\right) \\ \partial\left(\frac{h}{\rho}\right) + h\partial\left(\frac{1}{\rho}\right) \end{pmatrix} = 0 \tag{25}$$

where $\Omega(L^+,t)$ is an arbitrary function entire on L^+ , and

$$L^+ = \begin{pmatrix} \partial(-\partial + u)\frac{1}{\rho}\partial^{-1} , & \frac{2}{\rho} \\ (\partial h + h\partial)\frac{1}{\rho}\partial^{-1} , & \frac{1}{\rho}(\partial + u) \end{pmatrix} . \tag{26}$$

System (25) admit the reductions $\alpha)\, h = 0$, $\beta)\, u = 0$ and $\gamma)\, u = h = 0$. For $h = 0$ system (25) is reduced to equation

$$u_t - \partial(\partial - u)\frac{1}{\rho}\partial^{-1}\rho_t + \Omega(L_{11}^+)\left(\partial u + (u-\partial)\partial\left(\frac{1}{\rho}\right)\right) = 0 .$$

Under the reduction $u = h = 0$ and for $\Omega = -L^+\omega(L^+)$ system (25) is equivalent to equation

$$\rho_t = \omega(\tilde{L},t)\partial^2\left(\frac{1}{\rho}\right) \tag{27}$$

where $\omega(\tilde{L},t)$ is an arbitrary function entire on $\hat{L}$ and $\tilde{L} = -\partial^2\frac{1}{\rho}\partial^{-1}$. The hierarchy (27) is representable also in the form

$$\rho_t = \partial^2\omega(-\rho^{-1}\partial)\frac{1}{\rho} . \tag{28}$$

Equation (28) with ω = const is linearizable by the changing of variable $x \to \bar{x}$ given by $\partial_x = \rho(x)\partial_{\bar{x}}$. For $\omega = 2\rho^{-1}\partial$ one has the equation $\rho_t = (\rho^{-2})_{xxx}$ which has the following Hamiltonian form $\rho_t = \partial^3\frac{\delta}{\delta\rho}\int_{-\infty}^{+\infty} dx\, \rho^{-1}(x)$.

5. It is not difficult to show that the problem (24) by the gauge transformation $\psi \to \psi' = \psi\exp\left(\frac{1}{2}\partial^{-1}(u - \lambda\rho)\right)$ can be convert into the spectral problem

$$\partial^2\psi' + \left(\mu^2\rho^2(x) + \mu R(x) + Q(x)\right)\psi' = 0 \tag{29}$$

where $\mu = \frac{i\lambda}{2}$, $R = -i(\rho_x + \rho u)$, $Q = h - \frac{1}{2}u_x - \frac{1}{4}u^2$.

In the case $\rho \equiv 1$ the general form of evolution equations integrable by (29), their Hamiltonian structures and $BT's$ have been studied in [15-18]. So equations (15), $BT's$ (14) and Poisson brackets (18) are gauge equivalent to equations, $BT's$ and Poisson brackets associated with the problem (29). Nevertheless it is of interest to study the problem (2) itself. The results of section 2 show that the method of recursion operator can be generalized to the general pseudo-differential spectral problem $\sum_{k=-M} u_k\partial^k\psi = \lambda\psi$.

R e f e r e n c e s

1. A.G.Reyman. Notes of LOMI scient. sem., 95, 3 (1980), Leningrad.
2. B.A.Kupershmidt, Commun. Math. Phys., 99, 51 (1985).
3. L.G.F.Broer, Appl. Sci. Res., 31, 377 (1975).
4. D.J.Kaup, Prog. Theor. Phys., 54, 396 (1975).
   V.B.Matveev, M.I.Yavor, Ann. Inst. H.Poincare, Sect. A, XXXi, No 1, 25 (1979).
5. M.J.Ablowitz, D.J.Kaup, A.C.Newell, H.Segur. Stud. Appl. Math., 53, 249 (1974).
6. H.Flaschka, A.C.Newell. Lecture Notes in Phys., 38, 355 (1975).
7. A.C.Newell, in Solitons, Topics in Current Physics, v.17, North-Holland PC, 1980, p.177.
   M.J.Ablowitz, H.Segur, Solitons and inverse scattering transform, SIAM, Philadelphia, 1981.
8. A.C.Newell. Proc. Roy. Soc. London. Ser. A, 365, 284 (1979).
9. V.S.Gerdjikov, P.P.Kulish. Physica, D3, 549 (1981).
10. B.G.Konopelchenko. J. Phys. A: Math. Gen.; 14, 1237 (1981).
11. B.G.Konopelchenko. Fortschr. Physik, 31, 253 (1983).
12. B.G.Konopelchenko, V.G.Dubrovsky. Ann. Phys. (N.Y.) 156, 265 (1984).
    V.G.Dubrovsky, B.G.Konopelchenko. Fortschr. Physik, 32, 25 (1984).
13. F.Calogero, A.Degasperis, Spectral transform and solitons, v.I, North-Holland, P.C., Amsterdam 1982.
14. V.E.Zakharov. Funct. Anal. and its Appl., 14, No 2, 15 (1980).
15. M.Jaulent, I.Miodek. Lett. Math. Phys., 1, 243 (1976).
16. F.G.Guerrero, L.Martiner Alonso. Lett. Nuovo Cimento, 27, 28 (1980).
17. C.Laddomada, G.Z.Tu. Lett. Math. Phys., 6, 453 (1982).
18. M.Boiti, C.Laddomada, F.Pempinelli. Preprint UL-DF-21-83/84, Lecce 1984.

Б.Г.Конопельченко

МЕТОД РЕКУРСИОННОГО ОПЕРАТОРА ДЛЯ ИНТЕГРОДИФФЕРЕНЦИАЛЬНЫХ СПЕКТРАЛЬНЫХ ЗАДАЧ
I. УРАВНЕНИЯ ДЛИННЫХ ВОЛН С ДИСПЕРСИЕЙ

Препринт
№ 86-I7

Работа поступила - 2 декабря 1985 г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати 2I.0I-I986 г. МН II633
Формат бумаги 60x90 I/I6 Усл.0,9 печ.л., 0,8 учетно-изд.л.
Тираж 290 экз. Бесплатно. Заказ № I7.

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90