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B.G. Konopelchenko

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TWO-DIMENSIONAL SECOND ORDER DIFFERENTIAL SPECTRAL PROBLEM: COMPATIBILITY CONDITIONS, GENERAL BTs AND INTEGRABLE EQUATIONS

B.G.Konopelchenko

Institute of Nuclear Physics,
Novosibirsk-90, 630090, USSR

Abstract

Nonlinear evolution systems in two spatial dimensions integrable by the spectral problem $(\partial_x^2 - \sigma^2\partial_y^2 + \varphi_1(x,y)\partial_x + \varphi_2(x,y)\partial_y + U(x,y))\psi = 0$ are considered. It is shown that such systems possess the matrix commutativity representation $[T_1^M, T_2^M] = 0$ which is equivalent to the usual "L-A-B triad" representation of the compatibility condition. General Backlund transformations (BTs) and general form of integrable equations are found by the recursion operator method.

1. Nonlinear differential equations integrable by the inverse spectral transform (IST) method are the compatibility condition of the linear system of equations $T_i\psi = 0$ where T_i are certain operators [1-3]. For the first examples of integrable equations in 1 + 2 dimensions (t,x,y) (Kadomtsev-Petviashvili equation, Devay-Stewartson equation, and resonantly interacting waves equations) the compatibility condition has been equivalent to the usual commutativity condition

$$[T_1, T_2] \equiv [\partial_y + U(\partial_x), \partial_t + V(\partial_x)] = 0 \tag{1}$$

where $U(\partial_x)$ and $V(\partial_x)$ are differential operators over variable x [4].

Then Manakov [5] has demonstrated that the algebraic form of the compatibility condition which is more adequate to the two-dimensional case is the following L-A-B triad representation

$$[T_1, T_2] = BT_1 \quad \text{or} \quad \frac{\partial L}{\partial t} = [L, A] - BL \tag{2}$$

where $L \equiv T_1$, $T_2 \equiv \partial_t + A$ and B is a certain operator. The examples of the nonlinear systems in 1 + 2 dimensions, connected with the operator $T_1 = \partial_x^2 - \sigma^2\partial_y^2 + \varphi_1(x,y)\partial_x + \varphi_2(x,y)\partial_y + U(x,y)$ and which possess the representation (2) have been constructed in [5-9]. Among these equations is the following two-dimensional generalization of the KdV equation

$$U_t = K_1 U_{\xi\xi\xi} + K_2 U_{\eta\eta\eta} + 3K_1(U\partial_\xi^{-1}U_\eta)_\eta + 3K_2(U\partial_\eta^{-1}U_\xi)_\xi \tag{3}$$

where $\partial_\eta \equiv \partial_x + \sigma\partial_y$, $\partial_\xi \equiv \partial_x - \sigma\partial_y$, $f_\xi \equiv \frac{\partial f}{\partial \xi}$, $f_\eta \equiv \frac{\partial f}{\partial \eta}$, $\sigma^2 = \pm 1$ and K_1, K_2 are arbitrary constants. The case $\sigma = 1$ has been considered in [6]. The case $\sigma^2 = -1$, $K_1 = K_2$ (Veselov-Novikov equation) has been studied in [8]. The particular cases of equation (3) have been considered in [10]. Another interesting example is the two-dimensional integrable generalization of the dispersive long waves equations (BLP system [9]) ($\sigma = 1$, $\varphi_1 = \varphi_2 = \varphi$) :

$$\varphi_t = -\varphi_{\xi\xi} + (\varphi^2)_\xi + 2\partial_\eta^{-1} U_{\xi\xi\xi}, \quad U_t = U_{\xi\xi} + 2(U\varphi)_\xi. \tag{4}$$

In the present paper we will consider the nonlinear integrable systems connected with the generic two-dimensional operator $T_1 = \partial_x^2 - \sigma^2\partial_y^2 + \varphi_1(x,y)\partial_x + \varphi_2(x,y)\partial_y + U(x,y)$ and their properties. It is shown that equations (3) and (4) and other integrable equations connected with the operator under consideration are representable not only in the form (2) but also in the form (1) with the certain matrix operators $U(\partial_x)$ and $V(\partial_x)$. We will also briefly discuss the feature of the Manakov-Zakharov scheme [11,12] in application to equations of the type (3) and (4) and corresponding spectral problems.

It is shown that the two-dimensional version of the recursion operator method is applicable to the two-dimensional second order differential spectral problem $T_1\psi = 0$ under consideration. The general Backlund transformations (BC group) and general form of integrable equations, in particular, the hierarchies of integrable equations associated with (3) and (4), are found.

2. Firstly, we give the two examples of nonlinear systems which are representable in the form (2). The first system is

$$\begin{aligned} \varphi_t &= \Delta\varphi + \alpha(\varphi^2)_\xi - \beta\left((\partial_\xi^{-1}\varphi_\eta)^2\right)_\xi + 2\alpha\,\partial_\eta^{-1} U_{\xi\xi}, \\ U_t &= -\Delta U + \alpha(U\varphi)_\xi - 2\beta(U\,\partial_\xi^{-1}\varphi_\eta)_\eta \end{aligned} \tag{5}$$

or ($\varphi \equiv q_\xi$)

$$\begin{aligned} q_t &= \Delta q + \alpha(q_\xi)^2 - \beta(q_\eta)^2 + 2\alpha\,\partial_\eta^{-1} U_\xi, \\ U_t &= -\Delta U + \alpha(U q_\xi)_\xi - 2\beta(U q_\eta)_\eta \end{aligned}$$

where $\Delta \equiv -\alpha\partial_\xi^2 + \beta\partial_\eta^2$ and α, β are arbitrary constants. The corresponding operators T_1, T_2, B are

$$\begin{aligned} T_1 &\equiv L = \partial_\eta\partial_\xi + \varphi\partial_\eta + U, \\ T_2 &\equiv \partial_t + A = \partial_t + \alpha\partial_\xi^2 + \beta\partial_\eta^2 + 2\beta(\partial_\xi^{-1}\varphi_\eta)\partial_\eta + 2\alpha\partial_\eta^{-1} U_\xi, \\ B &= -2\alpha\varphi_\xi + 2\beta\partial_\xi^{-1} U_{\eta\eta}. \end{aligned} \tag{6}$$

At $\beta = 0$ the system (5) is reduced to the system (4).

The second integrable system is

$$\begin{aligned} q_t &= \Delta q + \alpha(q_\xi)^2 - \beta(q_\eta)^2 - 2\alpha q_\xi p_\xi + 2\beta\partial_\xi^{-1}(q_\xi p_\eta)_\eta, \\ p_t &= -\Delta p - \alpha(p_\xi)^2 + \beta(p_\eta)^2 - 2\beta q_\eta p_\eta + 2\alpha\partial_\eta^{-1}(q_\xi p_\eta)_\xi \end{aligned} \tag{7}$$

where α and β are arbitrary constants. The operators T_1, T_2 and B are

$$\begin{aligned} T_1 &= \partial_\xi\partial_\eta + q_\xi\partial_\eta + p_\eta\partial_\xi, \\ T_2 &= \partial_t + \alpha\partial_\xi^2 + \beta\partial_\eta^2 + 2\alpha p_\xi\partial_\xi + 2\beta q_\eta\partial_\eta, \\ B &= 2\Delta(q-p). \end{aligned} \tag{8}$$

The systems (5) and (7) admit the reductions $U = 0$ and $p = 0$. The equation $q_t = \Delta q + \alpha(q_\xi)^2 - \beta(q_\eta)^2$, which arises under such a reduction, is the two-dimensional Burger's equation in the term of potential q ($q_\xi = \varphi$). An obvious substitution $q = -\ln f$ linearises this equation and $f_t = \Delta f$.

3. Now we will discuss the problem of algebraic formulation of the compatibility condition for integrable equations in 1 + 2 dimensions. For definiteness we firstly consider equations (3) and (4). In the scalar form the compatibility condition of the

system

$$T_1\psi=\left(\partial_x^2-\sigma^2\partial_y^2+(\partial_x+\sigma\partial_y)\varphi+u\right)\psi=0, \tag{9a}$$

$$T_2\psi=\left(\partial_t+A(\partial_x,\partial_y)\right)\psi=0 \tag{9b}$$

for equations (3) and (4) is of the form (2) with $B\neq 0$.

Let us represent the scalar problem (9a) in the 2 x 2 matrix form

$$\left(\partial_x+\sigma\begin{pmatrix}1&0\\0&-1\end{pmatrix}\partial_y+\begin{pmatrix}0&-1\\u&\varphi\end{pmatrix}\right)\chi=0 \tag{10}$$

or, equivalently, in the form

$$T_1^M\chi\equiv\left(\partial_y+U(\partial_x)\right)\chi\equiv\left\{\partial_y+\frac{1}{\sigma}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\partial_x-\frac{1}{\sigma}\begin{pmatrix}0&1\\u&\varphi\end{pmatrix}\right\}\chi=0 \tag{11}$$

where $\chi=(\chi_1,\chi_2)^T=(\psi,\psi_\eta)^T$.

For equation (4) the problem (9b) in the 2 x 2 matrix form is $(\sigma=1,\ \alpha=1)$

$$T_2^M\chi=\left\{\partial_t+(\partial_x-\partial_y)^2-\begin{pmatrix}2\partial_\eta^{-1}u_\xi, & 0\\ 2u_\xi, & 2\partial_\eta^{-1}u_\xi\end{pmatrix}\right\}\chi=0. \tag{12}$$

For equation (4) the operators T_1^M and T_2^M obey equation (2) with $B=2\begin{pmatrix}0, & 0\\ u_\xi, & \varphi_\xi\end{pmatrix}$.

For the common solutions χ of equations (11) and (12) one has $\partial_y\chi=-U(\partial_x)\chi$ and, therefore, one can exclude the derivatives $\partial_y\chi,\ \partial_y^2\chi$. As a result instead (12) one has

$$\overline{T}_2^M\chi\equiv\left\{\partial_t+\begin{pmatrix}4&0\\0&0\end{pmatrix}\partial_x^2-2\begin{pmatrix}0&1\\u&0\end{pmatrix}\partial_x+\right.$$

$$\left.+\begin{pmatrix}-u+2\partial_\eta^{-1}u_\xi, & -\varphi\\ 2u_\xi+u_\eta-\varphi u, & \varphi_\xi-\varphi^2-u+2\partial_\eta^{-1}u_\xi\end{pmatrix}\right\}\chi=0. \tag{13}$$

One can directly check that the system (4) is equivalent to the commutativity condition $[T_1^M,\overline{T}_2^M]=0$.

Analogously for the Veselov-Novikov equation (equation (3) with $K_1=K_2=1,\ \sigma^2=-1,\ \partial_\eta=\bar\partial,\ \partial_\xi=\partial$) the direct 2 x 2 matrix representation of the operators $T_1=\partial\bar\partial+u$, $T_2=\partial_t-(\bar\partial^3+\partial^3+3\partial^{-1}\bar\partial u\cdot\bar\partial+3\bar\partial^{-1}\partial u\cdot\partial)$ [8] is

$$T_1^M\chi\equiv\left(\partial_y+\begin{pmatrix}-i&0\\0&i\end{pmatrix}\partial_x+i\begin{pmatrix}0&1\\u&0\end{pmatrix}\right)\chi=0, \tag{14}$$

$$T_2^M\chi\equiv\left(T_2-\begin{pmatrix}0 & 0\\ 3\partial u\cdot\partial; & 3\partial^{-1}\bar\partial^2u\end{pmatrix}\right)\chi=0 \tag{15}$$

and for Veselov-Novikov equation these operators obey equation (2) with $B=\begin{pmatrix}-3\partial^{-1}\bar\partial^2u, & 0\\ -3\bar\partial u\cdot\bar\partial-3\bar\partial^2u, & 3\partial u-3\bar\partial^{-1}\partial^2u\end{pmatrix}$. Excluding $\partial_y\chi$ in (15) with the use of (14), one obtains

$$\overline{T}_2^M=\partial_t-8\partial_x^3-\begin{pmatrix}6u+6(\bar\partial^{-1}\partial u), & 0\\ 6(\partial u)+6(\bar\partial u), & 6u+6(\partial^{-1}\bar\partial u)\end{pmatrix}\partial_x+$$

$$+\begin{pmatrix}6u_x, & 3\bar\partial^{-1}\partial u-3\partial^{-1}\bar\partial u\\ 6(\bar\partial u_x)+3u(\bar\partial^{-1}\partial u-\partial^{-1}\bar\partial u), & 6u_x+3(\partial u-\partial^{-1}\bar\partial^2u)\end{pmatrix}. \tag{16}$$

The Veselov-Novikov equation is equivalent to the equation $[T_1^M,\overline{T}_2^M]=0$.

So the general situation is rather simple. The direct matricezation of the scalar system $T_1(\partial_x,\partial_y)\psi=0$, $T_2(\partial_t,\partial_x,\partial_y)\psi=0$ gives

$$T_1^M \chi \equiv (\partial_y + \mathcal{U}(\partial_x))\chi = 0, \qquad (17a)$$

$$T_2^M \chi \equiv (\partial_t + \mathrm{V}(\partial_x, \partial_y))\chi = 0 \qquad (17b)$$

and integrable equation is equivalent to equation (2). On the common solutions of (17a) and (17b) equation (17b) is equivalent to the equation

$$(\partial_t + \mathrm{V}(\partial_x, -\mathcal{U}(\partial_x)))\chi = 0 \qquad (18)$$

that defines the new operator $\overline{T}_2^M = \partial_t + \overline{\mathrm{V}}(\partial_x) = \partial_t + \mathrm{V}(\partial_x, -\mathcal{U}(\partial_x))$. Now the integrable equation is equivalent to the commutativity condition $[T_1^M, \overline{T}_2^M] = 0$.

Thus equations (3) and (4) (as well the systems (5) and (7)) besides the scalar representation (2) possess also the matrix representation of the type (1).

The existence of the matrix representation (1) for the integrable systems in 1 + 2 dimensions besides the scalar representation (2) is the rather general situation. Indeed, let we have the linear system $T_1(\partial_x, \partial_y)\psi = 0$, $T_2(\partial_t, \partial_x, \partial_y)\psi = 0$. Let this system is representable in the following matrix form

$$T_1^M \chi = (\partial_y + \mathcal{U}(\partial_x))\chi = 0,$$
$$T_2^M \chi = (\partial_t + \mathrm{V}(\partial_x, \partial_y))\chi = 0 \qquad (19)$$

where $\mathcal{U}(\partial_x)$ is the differential operator over X . The algebraic form of the compatibility condition is, in general case, equation (2) with the certain operator B^M, i.e.

$$[T_1^M, T_2^M] = B^M T_1^M . \qquad (20)$$

Note that *if* the operator V does not contain the operator ∂_y at all, then $[T_1^M, T_2^M]$ does not contain the operator ∂_y too and, therefore, in this case $B^M = 0$.

It is well known [13] that the operators T_2 and B are defined up to the transformations

$$T_2 \to T_2' = T_2 + C T_1, \quad B \to B' = B + [T_1, C] \qquad (21)$$

where $C(\partial_x, \partial_y)$ is an arbitrary operator.

For given operator T_2^M let us choose the operator C in such a way that the operator $\widetilde{T}_2^M = T_2^M(\partial_x, \partial_y) + \overline{C}(\partial_x, \partial_y)(\partial_y + \mathcal{U}(\partial_x))$ does not contain ∂_y at all. For operators T_2^M which are "polynomials" on ∂_y such an operator $\overline{C}$ always exists: it is the "polynomial" on ∂_y of the degree $(\text{degree}\ \overline{C})_{\partial_y} = (\text{degree}\ \mathrm{V})_{\partial_y} - 1$ and its coefficients are easily calculated via the coefficients of $\mathrm{V}(\partial_x, \partial_y)$ and $\mathcal{U}$. Since $\widetilde{T}_2^M$ is independent on ∂_y then the substitution $\partial_y \to -\mathcal{U}(\partial_x)$ gives $\widetilde{T}_2^M = T_2^M(\partial_x, -\mathcal{U}(\partial_x)) = \overline{T}_2^M(\partial_x)$, i.e.

$$\overline{T}_2^M(\partial_x) - T_2^M(\partial_x, \partial_y) = \overline{C}\, T_1^M . \qquad (22)$$

Thus the operator $\overline{T}_2^M$, which arises from $T_2^M(\partial_x, \partial_y)$ after the exclusion of ∂_y with the use of equation (17a) $(\partial_y \to -\mathcal{U}(\partial_x))$, is obtained from T_2^M by the transformation (21) with the special operator $\overline{C}$.

Multiplying the relation (22) by T_1^M from the left, then by T_1^M from the right, subtracting the equalities obtained and taking into account that $[T_1^M, \overline{T}_2^M] = 0$ on the integrable equation, one gets

$$[T_1^M, T_2^M] = -[T_1^M, \overline{C}]\, T_1^M . \qquad (23)$$

Comparing (23) with (20) we find

$$B^M = -[T_1^M, \overline{C}] . \qquad (24)$$

Now let us perform the transformation (21) with $C = \bar{C}$. As a result $T_2^M \to \bar{T}_2^M(\partial_x)$ and, in virtue of (24), $\hat{B} = 0$.

Thus the nonlinear system which is the compatibility condition of the system (19) and possesses the algebraic compatibility equation (20) possesses also the commutativity representation $[T_1^M, \bar{T}_2^M] = 0$. The assumption that the scalar equation $T_1(\partial_x, \partial_y)\psi = 0$ is representable in the matrix form $T_1^M \chi = (\partial_y + U(\partial_x))\chi = 0$ (or $(\partial_x + \tilde{U}(\partial_y))\chi = 0$) is essential. This is valid for the operators T_1 of the form $T_1 = \sum_{n,m=0}^{n+m=N} U_{nm}(x,y)\,\partial_x^n \partial_y^m$. In this case the problem $T_1\psi = 0$ is equivalent to the N x N matrix problem

$$\left(\partial_y + A\partial_x + \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ P & & \ddots & \vdots \\ & & 1 & 0 \\ & & & 1 \end{array}\right)\right)\chi = 0 \qquad (25)$$

where $A_{ik} = a_i \delta_{ik}$ $(i, k = 1, \ldots, N)$ and $P_{ik} = 0$ at $i < k$ (i, k = 1,...,N). For definiteness we consider the case $U_{0,N} \neq 0$.

4. Here we will discuss the application of the Manakov-Zakharov method [11,12] for the construction of spectral problems of the type $T_1\psi = \sum_{n,m=0}^{n+m=N} U_{nm}(x,y)\,\partial_x^n \partial_y^m \psi = 0$. The starting point in this method is the nonlocal Riemann problem

$$\psi_2(\lambda, x) = \int d\lambda'\, \psi_1(\lambda', x)\, R(\lambda', \lambda; x) \qquad (26)$$

where $R(\lambda', \lambda; x)$ is the certain function. It is assumed that $R(\lambda', \lambda; x)$ obeys the equations

$$\partial_{x_i} R(\lambda', \lambda; x) = I_i' R - R I_i \qquad (27)$$
$$i = 1, 2, 3$$

where x_1, x_2, x_3 are independent variables and I_i, I_i' are some functions. Then the operators $\mathcal{D}_i$ ($\mathcal{D}_i f = \partial_{x_i} f + f I_i$) are introduced and with the use of (26), (27) the set of operators $\mathcal{M}_i$, which

have no singularities on λ, is constructed. The compatibility of the linear system $\mathcal{M}_i \psi = 0$ is equivalent to the nonlinear equation. In the scalar case and the case of one marked variable (for example, x_1) one has $I_1 = \lambda$, $I_2 = I_2(\lambda)$, $I_3 = I_3(\lambda)$ where I_2 and I_3 are some polynomials. This case has been considered in detail in [12].

If two variables x_1 and x_2 appear on the equal footing and they are contained in the problem symmetrically, then one should put $I_1 = \lambda_1$, $I_2 = \lambda_2$ where the complex parameters λ_1, λ_2 obey the constraint $\sum_{n,m=0}^{n+m=N} f_{nm} \lambda_1^n \lambda_2^m = \mathrm{const} \neq 0$ where f_{nm} are constants. The operator T constructed by the method of the papers [11,12] will be of the form $T = \sum_{n,m=0}^{n+m=N} U_{nm}(x_1, x_2)\,\partial_{x_1}^n \partial_{x_2}^m$ where $\lim_{\sqrt{x_1^2+x_2^2}\to\infty} U_{nm} = f_{nm}$. For example, in the case $\lambda_1^2 - \sigma^2\lambda_2^2 = 1$ one obtains the operator $T = \partial_{x_1}^2 - \sigma^2\partial_{x_2}^2 + \varphi_1\partial_{x_1} + \varphi_2\partial_{x_2} + U$. The functions I_3 ($x_3 \equiv t$) are the polynomials on the variables $\lambda_+ = \lambda_1 + \sigma\lambda_2$ and $\lambda_- = \lambda_1 - \sigma\lambda_2$ in this case. The variable which parametrizes (uniformizes) the curve $\sum_{n,m} f_{nm}\lambda_1^n\lambda_2^m = \mathrm{const}$ could be considered as the parameter λ in the nonlocal Riemann problem (26).

Similar situation takes place for the multidimensional operators T. For example, to construct the operator T of the form $T = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 + U$ one should consider three parameters $\lambda_1, \lambda_2, \lambda_3$ which obey the constraint $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$.

5. Now we will construct an infinite-dimensional group of general BTs (BC-group) and hierarchy of the integrable equations connected with the two-dimensional spectral problem under consideration. For some spectral problems linear on ∂_y such results have been obtained in [14-16]. Here we will demonstrate that the recursion operator method effectively works for the spectral problem

$$\left(\partial_x^2 - \sigma^2\partial_y^2 + \varphi(x,y,t)(\partial_x + \sigma\partial_y) + U(x,y,t)\right)\psi = 0 \qquad (28)$$

where $\varphi(x,y,t)$, $U(x,y,t)$ are scalar functions such that $\varphi, U \xrightarrow[\sqrt{x^2+y^2}\to\infty]{} 0$ and $\sigma^2 = \pm 1$. Generic problem

$(\partial_x^2 - \sigma^2 \partial_y^2 + \varphi_1 \partial_x + \varphi_2 \partial_y + u)\psi = 0$ is reduced to (28) by the gauge transformation $\psi \to g\psi$.

Firstly we represent (28) in the matrix form (10). Then we introduce the solutions $\hat{F}_\lambda^{\pm}$ of (10) such that $\hat{F}_\lambda^{\pm}(x,y) \xrightarrow[x \to \pm\infty]{} \mathcal{D} \exp(\lambda y - \bar{A} x)$ where $\lambda \in \mathbb{C}$, $\bar{A} = \sigma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathcal{D} = \begin{pmatrix} 1 & 1 \\ 0 & 2\sigma\lambda \end{pmatrix}$. Similar to [14-16] we introduce the scattering matrix $\hat{S}$: $\hat{F}_\lambda^{+}(x,y,t) = \int d\tilde{\lambda}\, \hat{F}_{\tilde{\lambda}}^{-}(x,y,t)\, \hat{S}(\tilde{\lambda},\lambda,t)$. Then we consider (see [14-16] and the review [17]) the transformations of $\varphi, u \to \varphi', u'$ such that

$$\hat{S}(\tilde{\lambda},\lambda,t) \to \hat{S}'(\tilde{\lambda},\lambda,t) = \bar{B}^{-1}(\tilde{\lambda},t)\, \hat{S}(\tilde{\lambda},\lambda,t)\, \bar{C}(\lambda,t) \qquad (29)$$

where $\bar{B}$ and $\bar{C}$ are diagonal matrices. An arbitrary matrix $\bar{B}$ can be represented in the form $\bar{B} = B_0(\lambda,t) + B_1(\lambda,t)\begin{pmatrix} \sigma\lambda & 0 \\ 0 & -\sigma\lambda \end{pmatrix}$ where $B_0(\lambda,t)$ and $B_1(\lambda,t)$ are arbitrary scalar functions. Similar to [14-16] one obtains the relation

$$\left(\hat{S}'(\tilde{\lambda},\lambda) - \hat{S}(\tilde{\lambda},\lambda)\right)_{in} =$$

$$= -\int dx\,dy\,d\tilde{y}\,\delta(\tilde{y}-y)\, tr\left(\left(P'(x,\tilde{y}) - P(x,y)\right) \overset{-+}{\Phi}{}^{in}(x,\tilde{y},y)\right) \qquad (30)$$

and the fundamental relation

$$\int dx\,dy\,d\tilde{y}\,\delta(\tilde{y}-y)\, tr\Big\{ B(-\partial_y,t)\, P'(x,\tilde{y},t)\, \overset{++}{\Phi}{}^{F}(x,\tilde{y},y) -$$

$$- P(x,y,t)\, B(\partial_{\tilde{y}},t)\, \overset{++}{\Phi}{}^{F}(x,\tilde{y},y)\Big\} = 0 \qquad (31)$$

which follows from (29), where $P = \begin{pmatrix} 0 & -1 \\ u & \varphi \end{pmatrix}$, $B(\partial_y,t) = B_0(\partial_y,t) + B_1(\partial_y,t)\begin{pmatrix} \sigma\partial_y & -1 \\ 0 & -\sigma\partial_y \end{pmatrix}$ and $\left(\overset{\pm\mp}{\Phi}{}^{in}(x,\hat{y},y)\right)_{kl} = \hat{F}^{+\prime}_{kn}(x,\tilde{y})\, \check{F}^{\pm}_{il}(x,y)$. where $\check{F}^{\pm}$ are the solutions of the spectral problem adjoint to (10). The adjoint representation

[16,17] of the problem (10) for the quantity $\Phi^{in}(x,\tilde{y},y)$ is of the form

$$\frac{\partial \Phi^{in}(x,\hat{y},y)}{\partial x} + \sigma A \frac{\partial \Phi^{in}(x,\tilde{y},y)}{\partial \tilde{y}} + \sigma \frac{\partial \Phi^{in}(x,\tilde{y},y)}{\partial y} A +$$

$$+ P'(x,\tilde{y})\, \Phi^{in}(x,\tilde{y},y) - \Phi^{in}(x,\tilde{y},y)\, P(x,y) = 0 \qquad (32)$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It follows from (30) that the dynamical components of $\Phi \equiv \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}$ are Φ_2 and Φ_4. We denote $\Phi_\Delta = \begin{pmatrix} \Phi_2 \\ \Phi_4 \end{pmatrix}$.

We will consider the functions $B_0(\partial_y,t)$ and $B_1(\partial_y,t)$ entire on ∂_y : $B_i(\partial_y,t) = \sum_{n=0}^{\infty} b_{in}(t)\partial_y^n$, $i = 0,1$. The fundamental relation (3.1), rewritten in the components of Φ, contains $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ and their derivatives over y and $\tilde{y}$. It is necessary firstly to express Φ_1, Φ_3 via Φ_2 and Φ_4 (i.e. Φ_Δ) and secondly to express $\partial_{\tilde{y}}^n \Phi_\Delta(\tilde{y},y,x)$ and $\partial_y^m \Phi_\Delta(x,\tilde{y},y)$ via $\Phi_\Delta(x,\tilde{y},y)$. The adjoint representation (32) allows one to do this. Namely, equation (32) gives

$$\Phi_1(x,\hat{y},y) = \partial_+^{-1}\{(-\partial_- + \varphi - \varphi')\Phi_4 + (u - u')\Phi_2\},$$
$$\Phi_3(x,\tilde{y},y) = -(\partial_- + \varphi' - \varphi)\Phi_4 - u'\Phi_2 \qquad (33)$$

and

$$\partial_{\tilde{y}} \overset{++}{\Phi}{}^{F}_{\Delta}(x,\tilde{y},y) = \hat{\Lambda}_{(1)}(\tilde{y},y)\, \overset{++}{\Phi}{}^{F}_{\Delta}(x,\hat{y},y) \qquad (34)$$

where

$$\hat{\Lambda}_{(1)}(\tilde{y},y) = \frac{1}{2\sigma}\begin{pmatrix} -\partial_- + \varphi + \partial_+^{-1}(u'-u), & -\partial_+^{-1}(2\partial_x + \varphi' - \varphi) \\ I(-2\sigma u'_{\tilde{y}} + u'\partial_- + \partial_+ u' + u'(\varphi'-\varphi)), & I(-2\sigma\varphi'_{\tilde{y}} + u' - u + (\partial_+ + \varphi')(\partial_- + \varphi' - \varphi)) \end{pmatrix}. \qquad (35)$$

Here $\partial_\pm \overset{def}{=} \partial_x \pm \sigma(\partial_{\tilde{y}} + \partial_y)$, $If \overset{def}{=} exp(\partial_-^{-1}(\varphi-\varphi'))\cdot \partial_-^{-1}\{exp(\partial_-^{-1}(\varphi'-\varphi))\cdot f\}$

and $u' \equiv u'(x,\tilde{y})$, $\varphi' \equiv \varphi'(x,\tilde{y})$.

It follows from (34) that

$$\partial_{\tilde{y}}^n \overset{++F}{\Phi}_\Delta(x,\tilde{y},y) = \hat{\Lambda}_{(n)}(\tilde{y},y)\overset{++F}{\Phi}_\Delta \qquad (36)$$

where $\hat{\Lambda}_{(n)}$ are calculated by the recurrent relation

$$\hat{\Lambda}_{(n)}(\tilde{y},y) = \hat{\Lambda}_{(n-1)}(\tilde{y},y)\hat{\Lambda}_{(1)}(\tilde{y},y) + \frac{\partial\hat{\Lambda}_{(n-1)}(\tilde{y},y)}{\partial\tilde{y}} ,$$

$n = 2,3,\ldots$. For the adjoint operators one has $\hat{\Lambda}^+_{(n)}(\tilde{y},y) = \hat{\Lambda}^+_{(1)}(\tilde{y},y)\hat{\Lambda}^+_{(n-1)}(\tilde{y},y) + \frac{\partial\hat{\Lambda}^+_{(n-1)}(\tilde{y},y)}{\partial\tilde{y}}$ and, therefore,

$$\hat{\Lambda}^+_{(n)}(\tilde{y},y) = \left(\hat{\Lambda}^+_{(1)}(\tilde{y},y) + \frac{\partial}{\partial\tilde{y}}\right)^{n-1}\hat{\Lambda}^+_{(1)}(\tilde{y},y) \qquad (37)$$

where

$$\hat{\Lambda}^+_{(1)}(\tilde{y},y) = \frac{1}{2\sigma}\begin{pmatrix} \partial_- + \varphi + (u-u')\partial_+^{-1}, & (\partial_- u' + u'\partial_+ + u'(\varphi'-\varphi) + 2\sigma u'_{\tilde{y}})I \\ (2\partial_x + \varphi - \varphi')\partial_+^{-1}, & (u'-u-(\partial_- + \varphi - \varphi')(\partial_+ - \varphi') + 2\sigma\varphi'_{\tilde{y}})I \end{pmatrix} . \qquad (38)$$

Emphasize that $\frac{\partial}{\partial\tilde{y}}$ in (37) acts only on $\hat{\Lambda}^+_{(1)}(\tilde{y},y)$. The operators $\hat{\Lambda}_{(n)}$ are the recursion operators we are interesting in.

Analogously one can show that

$$\partial_y^n \overset{++F}{\Phi}_\Delta(x,\tilde{y},y) = \check{\Lambda}_{(n)}(\tilde{y},y)\overset{++F}{\Phi}_\Delta \qquad (39)$$

where $\check{\Lambda}_{(n)}(\tilde{y},y) = \check{\Lambda}_{(n-1)}(\tilde{y},y)\check{\Lambda}_{(1)}(\tilde{y},y) + \frac{\partial\check{\Lambda}_{(n-1)}(\tilde{y},y)}{\partial y}$

and $\check{\Lambda}_{(1)}(\tilde{y},y) = \partial_y + \partial_{\tilde{y}} - \hat{\Lambda}_{(1)}(\tilde{y},y)$. For the adjoint operators $\check{\Lambda}^+_{(n)}$ one has

$$\check{\Lambda}^+_{(n)}(\tilde{y},y) = \left(\check{\Lambda}^+_{(1)}(\tilde{y},y) + \frac{\partial}{\partial y}\right)^{n-1}\check{\Lambda}^+_{(1)}(\tilde{y},y) \qquad (40)$$

$$n = 2,3,\ldots .$$

Using (33), (36), (39) we exclude the explicit dependence on the operators $\partial_y, \partial_{\tilde{y}}$ in (31). Then transfering to the adjoint operators similar to [14-16] we finally obtain from (31) the relation

$$\sum_{n=0}^{\infty} b_{0n}(t)\left\{(-1)^n\check{\Lambda}^+_{(n)}\begin{pmatrix}u'\\ \varphi'\end{pmatrix} - \hat{\Lambda}^+_{(n)}\begin{pmatrix}u\\ \varphi\end{pmatrix}\right\} +$$

$$+\sum_{n=0}^{\infty} b_{1n}(t)\left\{\sigma(-1)^n\check{\Lambda}^+_{(n+1)}\begin{pmatrix}u'\\ \varphi'\end{pmatrix} + \sigma\hat{\Lambda}^+_{(n+1)}\begin{pmatrix}-u\\ \varphi\end{pmatrix} + \right.$$

$$+\hat{\Lambda}^+_{(n)}\begin{pmatrix}0\\ u\end{pmatrix} + \sum_{m=0}^{n-1}(-1)^{n+1}C_n^m\check{\Lambda}^+_{(m)}\begin{pmatrix}\partial_y^{n-m}u\cdot\partial_+^{-1}u'\\ (\partial_+^{-1}u'-\varphi')\cdot\partial_y^{n-m}\varphi\end{pmatrix} + \qquad (41)$$

$$\left. + \begin{pmatrix}(u'-u)\partial_+^{-1}u' + \varphi' u'\\ (\partial_- + \varphi - \varphi')\partial_+^{-1}u' - (\partial_- + \varphi - \varphi')\varphi'\end{pmatrix}\right\} = 0$$

where $C_m^n = \frac{n!}{m!(n-m)!}$ and in all the quantities in (41) one should put $\tilde{y} = y$.

Transformations (41) form an infinite-dimensional abelian group of general BTs ($u,\varphi \to u',\varphi'$) for the spectral problem (28). The corresponding transformation law of the scattering matrix is given by (29).

The consideration of (41) for the infinitesimal displacement in time: $u' = u + \varepsilon u_t$, $\varphi' = \varphi + \varepsilon\varphi_t$, $b_{0n} = 1$,

$b_{1n} = \varepsilon \omega_n(t)$, $\varepsilon \to 0$ gives

$$\begin{pmatrix} u_t \\ \varphi_t \end{pmatrix} = -\sum_{n=0}^{\infty} \omega_n(t) \Big\{ \Big(\sigma(-1)^n \check{L}^{+}_{(n+1)} - \sigma \hat{L}^{+}_{(n+1)} A \Big) \begin{pmatrix} u \\ \varphi \end{pmatrix} +$$

$$+ \hat{L}^{+}_{(n)} \begin{pmatrix} 0 \\ u \end{pmatrix} + \sum_{m=0}^{n-1} (-1)^{n+1} C_n^m \check{L}^{+}_{(m)} \begin{pmatrix} \partial_+^{-1} u \cdot \partial_y^{n-m} u \\ (\partial_+^{-1} u - \varphi) \cdot \partial_y^{n-m} \varphi \end{pmatrix} + \begin{pmatrix} u\varphi \\ \partial_- \partial_+^{-1} u - \partial_- \varphi \end{pmatrix} \Big\} \tag{42}$$

where $\hat{L}^{+}_{(n)} \stackrel{def}{=} \hat{\Lambda}^{+}_{(n)} |_{u'=u, \varphi'=\varphi}$ and $\omega_n(t)$ are arbitrary functions. The corresponding evolution law of the scattering matrix is

$$\frac{\partial \hat{S}(\tilde{\lambda}, \lambda, t)}{\partial t} = Y(\tilde{\lambda}, t) \hat{S}(\tilde{\lambda}, \lambda, t) - \hat{S}(\tilde{\lambda}, \lambda, t) Y(\lambda, t) \tag{43}$$

where $Y(\lambda, t) = \sigma \lambda \Omega(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Omega(\lambda) = \sum_{n=0}^{\infty} \omega_n(t) \lambda^n$.

Formula (42) gives the general form of nonlinear evolution systems in 1 + 2 dimensions (t, x, y) integrable by the spectral problem (28).

The transformations (41) with the time-independent b_{in} form an infinite-dimensional abelian group of general auto-BTs for equations (42). The simplest auto BT (41) with b_{in} = 0 (i = 0,1, n = 1,2,3,...) is of the form

$$b_{00} \begin{pmatrix} u' - u \\ \varphi' - \varphi \end{pmatrix} + b_{10} \Big\{ -\sigma \hat{\Lambda}^{+}_{(1)} \begin{pmatrix} u' + u \\ \varphi - \varphi' \end{pmatrix} + \begin{pmatrix} (u' - u) \partial_+^{-1} u' + \varphi' u' - \sigma u'_y \\ (\partial_- + \varphi - \varphi')(\partial_+^{-1} u' - \varphi') + u - \sigma \varphi'_y \end{pmatrix} \Big\} = 0 \tag{44}$$

where the operator $\hat{\Lambda}^{+}_{(1)}$ is given by (38) with $\tilde{y} = y$. The group of general auto BTs is generated by the two elementary BTs. The BT constructed in [9] is the particular case of BTs (41).

The simplest equation (42) with ω_1 = 2 and $\omega_0 = \omega_2$ =

= ω_3 = ... = 0 and σ = 1 is the BLP system (4).

For the functions Ω of the form $\Omega = \Omega(\lambda^2)$ the system (42) admits the reduction φ = 0. As a result we obtain the hierarchy of equations the simplest of which (Ω = $4\lambda^2$) is equation (3) and, in particular, the Veselov-Novikov equation (σ^2 = -1).

In the one-dimensional limit $\varphi_y = u_y$ = 0 the BTs (41) and integrable equations (42) are reduced to BTs and integrable equations associated with the spectral problem $(\partial_x + \varphi + u\partial_x^{-1})\psi = \lambda\psi$ (see [18,19]).

Similar results can be obtained for the spectral problem $(\partial_x^2 - \sigma^2 \partial_y^2 + \varphi_1 \partial_x + \varphi_2 \partial_y + u)\psi = 0$ in a gauge free form and also for the two-dimensional spectral problem $\sum_{n,m=0}^{n+m=N} u_{nm}(x,y) \times \partial_x^n \partial_y^m \psi = 0$ ($N \geq 3$).

In conclusion note the following. The formulas (37) and (40) define the actions of the operators $\hat{\Lambda}^{+}_n$ and $\check{\Lambda}^{+}_n$ on the space of general bilocal functions $\varphi(x, \tilde{y}, y, t)$. On the subspace of local functions $Z(x,y,t)$ and $\tilde{Z}(x, \tilde{y}, t)$ the actions of these operators are equivalent to the following

$$\hat{\Lambda}^{+}_n(\tilde{y}, y) Z(y) = (\hat{\Lambda}^{+}_1(\tilde{y}, y) + \partial_{\tilde{y}})^n Z(y) = (-\Lambda^{+}(\tilde{y}, y))^n Z(y)$$

and

$$\check{\Lambda}^{+}_n(\tilde{y}, y) \tilde{Z}(\tilde{y}) = (\check{\Lambda}^{+}_1(\tilde{y}, y) + \partial_y)^n \tilde{Z}(\tilde{y}) = (\Lambda^{+}(\tilde{y}, y))^n \tilde{Z}(\tilde{y})$$

where $\Lambda^{+}(\tilde{y}, y) = -\partial_{\tilde{y}} - \hat{\Lambda}^{+}_1(\tilde{y}, y) = \partial_y + \check{\Lambda}^{+}_1(\tilde{y}, y)$. The use of the bilocal recursion operator $\Lambda^{+}(\tilde{y}, y)$ allows one essentially compactify the formulas (41) and (42).

The recursion operators constructed in [14-16] can be represented in such a form too.

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Б.Г.Конопельченко

ДВУМЕРНАЯ ДИФФЕРЕНЦИАЛЬНАЯ СПЕКТРАЛЬНАЯ ЗАДАЧА ВТОРОГО ПОРЯДКА: АЛГЕБРАИЧЕСКИЕ УСЛОВИЯ СОВМЕСТИМОСТИ, ОБЩИЕ БЭКЛУНД-ПРЕОБРАЗОВАНИЯ И ИНТЕГРИРУЕМЫЕ УРАВНЕНИЯ

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